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On a Class of Balancing Games

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A balancing game is a perfect information two-person game. Given a set $V \subset R^d$, in the ith round Player I picks a vector $v^i \in V$, and then Player II picks $\epsilon_i = +1$ or -1. Player II tries to minimize $\sup \{\| \sum_{i=1}^{n} \epsilon_i v^i \| : n = 0, 1, 2,...\}$. In this paper we generalize this game and give necessary and sufficient conditions for the existence of a winning strategy for Player I and Player II in the generalized game. Later we give upper and lower bounds to the value of the original game; the bounds in many cases are equal. Further we present simple strategies for both players.

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Recently Joel Spencer has proposed the following two-person game [1]. Given a finite set $V \subset R^d$ and a point $z \in R^d$, Player I selects a vector $v^i \in V$, and then Player II selects $\epsilon_i = +1$ or -1 in the ith round. There are an infinite number of rounds. Player II tries to minimize

$$\sup \left\{ \left\| z + \sum_{i=1}^{n} \epsilon_i v^i \right\| : n = 0, 1, 2,... \right\}.$$

(Here $\| \|$ denotes the max-norm in R^d. The empty sum equals the zero vector.) Spencer put $z = \theta$, his choice for V was also special. This game is a perfect information, zero-sum game. We shall see later that this game has a value, Player II has an optimal pure strategy, and Player I has an ϵ-optimal pure strategy. We denote the value by $f(z, V)$ or $f(z)$.

In [1], J. Spencer and R. Graham gave a strategy for Player II yielding an upper bound to $f(\theta, V)$. Jeff Lagarias determined the exact value of $f(\theta, V)$ in the case when V consists of all vectors with 0, 1 or $-1, 0, +1$ components [private communication]. He, in fact, gave an optimal strategy for Player I. Independently, Z. Füredi also determined the exact value of $f(\theta, V)$

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when V is the set of all vectors with 0, 1 components [private communication].

In this paper we generalize this game, and we prove that the generalized game has pure strategies for both players. Nevertheless, it is hard to find these strategies explicitly, and that is why we are to give upper and lower bounds to $f(z, V)$. To find the upper bound is relatively easy. We present two proofs for the lower bound. One of them gives a simple strategy for Player I. The other one follows from a lemma on the so-called V-closed sets. Sets of this type have an important role throughout the paper.

It is interesting that, in fact, the most important case is $d = 2$. That is why the proofs are of geometrical type.

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Suppose $V \subset R^d$ is a nonempty set.

DEFINITION. A set $T \subseteq R^d$ is said to be V-closed if $t \in T$ and $v \in V$ implies $t + v \in T$ *or* $t - v \in T$.

We denote the set of V-closed sets by $\mathscr{C}(V)$.

DEFINITION. The following two-person game is the game $[V, K, z]$: Suppose $K \subset R^d$, $z \in R^d$. In the ith round, Player I selects a vector $v^i \in V$, and then Player II selects $\epsilon_i = 1$ or -1. There are an infinite number of rounds. Player II wins, if for each $n = 0, 1, 2, \ldots$

$$z + \sum_{i=1}^{n} \epsilon_i v^i \in K,$$

and loses otherwise.

Now, we are able to present our first theorems.

THEOREM 1. *Player* I *has a winning strategy in the game* $[V, K, z]$ *if and only if there is no* $T \in \mathscr{C}(V)$ *such that* $z \in T \subseteq K$.

THEOREM 2. *Player* II *has a winning strategy in the game* $[V, K, z]$ *if and only if there is a* $T \in \mathscr{C}(V)$ *such that* $z \in T \subseteq K$.

These theorems do not imply each other because the game is infinite. Further, they reduce the question "who wins" to the existence of a set T with the prescribed property. Once this set is found an explicit winning strategy is given for Player II. However, if such set does not exist, it is difficult to determine the strategy for Player I explicitly, though it exists and it is well defined. That is why one is tempted to find conditions assuring the nonexistence of a set T with the prescribed property.

Now, with these theorems, we are able to prove the existence of the value $f(z, V)$ of the original game.

COROLLARY 1. *Suppose V is finite and nonempty. Then there is a number $f(z)$ with the property that in the original game Player* II *has a strategy assuring he never loses more than $f(z)$, and further, to any $\epsilon > 0$ Player* I *has a strategy with which he wins no less than $f(z) - \epsilon$.*

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In this section we prove Theorems 1 and 2 and Corollary 1. The following simple facts about V-closed sets are useful.

(i) The union of any collection of V-closed sets is V-closed.

(ii) $\mathscr{C}(V) = \mathscr{C}(V')$, where one gets V' from V by multiplying some vectors from V by -1.

(iii) If $T \in \mathscr{C}(V)$, then $L(T) \in \mathscr{C}(L(V))$ for any linear transformation $L : R^d \to R^e$.

(iv) If $T \in \mathscr{C}(V)$ and A is a subspace of R^d and $a \in R^d$, then $[a + A] \cap T \in \mathscr{C}(A \cap V)$.

(v) If $T \in \mathscr{C}(V)$, then clos $T \in \mathscr{C}(V)$, too.

(vi) The set

$$P(V) = \left\{ \sum_{v \in W} v \colon W \subseteq V,\ W \text{ is finite} \right\}$$

is V-closed.

(vii) $P(V) = P(V') + a$ with a suitable $a \in R^d$ [and with the set V' defined in (ii)].

(viii) If V is finite, then $P(V)$ is centrally symmetric with respect to the point $\frac{1}{2} \sum_{v \in V} v \in R^d$.

Proof of Theorem 2. First, let us suppose that there is a set $T \in \mathscr{C}(V)$ with $z \in T \subseteq K$. Then Player II has a strategy assuring $z + \sum_{i=1}^{n} \epsilon_i v^i \in T$ for $n = 0, 1, 2, \ldots$. Indeed, this is true for $n = 0$, and if it is true for $n - 1$, i.e., $t = z + \sum_{i=1}^{n-1} \epsilon_i v^i \in T$, then for any $v^n \in V$ Player II may select $\epsilon_n = 1$ if $t + v^n \in T$ or $\epsilon_n = -1$ if $t - v^n \in T$. Because $T \subseteq K$ this strategy is a winning strategy for Player I.

Suppose now, that Player II has a winning strategy. Put

$T = \{t \in R^d$: Player II has a winning strategy in the game $[V, K, t]\}$.

Clearly $z \in T \subseteq K$. We claim that T is V-closed. Assume, on the contrary, that there is a $t \in T$ and $v \in V$ such that neither $t + v$ nor $t - v$ belong to T.

But then Player II could not have a winning strategy in the game $[V, K, t]$. A contradiction.

Proof of Theorem 1. If Player I has a winning strategy, then Player II cannot have a winning strategy, i.e., there cannot be a set $T \in \mathscr{C}(V)$ with $z \in T \subseteq K$.

To complete the proof of Theorem 1 we need the following map H: if $S \subseteq R^d$, then

$$H(S) = \{s \in S : \text{for each } v \in V \text{ either } s + v \in S \text{ or } s - v \in S\}.$$

Clearly $H(S) \subseteq S$. Let $H^1(S) = H(S), H^2(S) = H(H(S)),\ldots,$ and $H^\omega(S) = \bigcap_{i+1}^{\infty} H^i(S)$. Define $H^0(S) = S$ and $H^{-1}(S) = R^d$. Clearly

$$H^{-1}(S) \supseteq H^0(S) \supseteq H^1(S) \supseteq H^2(S) \supseteq \cdots \supseteq H^\omega(S)$$

and

$$H^\omega(S) \in \mathscr{C}(V).$$

Suppose, now, that there is no set $T \in \mathscr{C}(V)$ with $z \in T \subseteq K$. This means that $z \notin H^\omega(K)$, i.e., there is an index n such that $z \in H^{n-1}(K)$ but $z \notin H^n(K)$. But, then, there is a $v = v^1 \in V$ with $z + v \notin H^{n-1}(K)$ and $z - v \notin H^{n-1}(K)$. Now, Player I's strategy runs like this: On the first round, he selects $v^1 \in V$. For any choice of Player II $z + \epsilon_1 v^1 \notin H^{n-1}(K)$, so there is an index $n_1 \leqslant n - 1$ such that $z + \epsilon_1 v^1 \in H^{n_1 - 1}(K)$ but $z + \epsilon_1 v^1 \notin H^{n_1}(K)$. Player I, on the second round, selects $v^2 \in V$ such that for $\epsilon_2 = 1$ and -1

$$z + \epsilon_1 v^1 + \epsilon_2 v^2 \notin H^{n_1 - 1}(K).$$

And so on. This strategy (in at most n steps) will clearly yield to the point $z + \epsilon_1 v^1 + \cdots + \epsilon_k v^k$ not in K. And this is what we wanted to prove.

Remarks. Clearly, this strategy is the best for Player I in the sense that, with it, he can win in the minimal possible number of rounds.

In view of (i), for every $S \subseteq R^d$ there is a set $M \subseteq S$, $M \in \mathscr{C}(V)$ with the property that for any $T \subseteq S$, $T \in \mathscr{C}(V)$ $T \subseteq M$ holds true, namely,

$$M = \bigcup \{T : T \subseteq S,\ T \in \mathscr{C}(V)\}.$$

It is easy to see that we have given a new presentation for M:

$$M = H^\omega(S).$$

We remark further that the map H can be written in the following form:

$$H(S) = S \cap \bigcap_{v \in V} ((S + v) \cup (S - v)).$$

Proof of Corollary 1. Let us denote by B the closed unit ball of the max norm in R^d and consider the game $[V, \lambda B, z]$, where λ is a positive real number. Clearly,

$$\sup \{\lambda : \text{Player I wins in the game } [V, \lambda B, z]\}$$
$$= \inf \{\lambda : \text{Player II wins in the game } [V, \lambda B, z]\}.$$

Denote this number by $f(z)$. $f(z)$ is finite because, for finite V, $P(V)$ is a bounded set in R^d and for sufficiently great λ, $z + P(V) \subseteq \lambda B$.

Now we claim that, in the game $[V, f(z)\, B, z]$, Player II has a winning strategy. Suppose, on the contrary, that Player I has. Then, with the strategy of Theorem 1, he wins in at most n rounds, and we conclude that after k ($\leqslant n$) rounds the game is won in the point $z + \sum_{i=1}^{k} \epsilon_i v^i \notin f(z)\, B$. But there is only a finite number of such points and, with a suitable $\epsilon > 0$, none of these points will belong to $(f(z) + \epsilon)\, B$; i.e., Player I has a winning strategy for the game $[V, (f(z) + \epsilon)\, B, z]$, and this is impossible.

In this way we see that Player II has a winning strategy for the game $[V, f(z)\, B, z]$. Following the same strategy in the original game he will never lose more than $f(z)$.

The second part of Corollary 1 is now trivial.

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The theorems we have proven so far do not provide us with information on the value of $f(z)$. Our next aim is to give upper and lower bounds to $f(z)$.

Let diam S denote the diameter of $S \subseteq R^d$, i.e.

$$\operatorname{diam} S = \sup_{s_1, s_2 \in S} \| s_1 - s_2 \|$$

To state our theorems simply we need the following assumption:

(*) $V \subset R^d$ is finite, nonempty, and does not contain collinear vectors.

THEOREM 3. *If V is finite and nonempty, then*

$$f(z, V) \leqslant \min_{t_1 \in P(V)} \max_{t_2 \in P(V)} \| z - t_1 + t_2 \|.$$

THEOREM 4. *Under assumption* (*)

$$f(z, V) \geqslant \max \mid (\| z \|, \tfrac{1}{2} \operatorname{diam} P(V)).$$

These theorems and (viii) yield to

COROLLARY 2. *If* $\frac{1}{2}\sum_{v\in V} v \in P(V)$ *and* $(*)$ *holds, then*

$$f(\theta, V) = \tfrac{1}{2}\operatorname{diam} P(V).$$

We need the following lemma:

LEMMA. *If* $(*)$ *holds true and* $T \subseteq R^d$, $T \in \mathscr{C}(V)$, *and* T *is nonempty and bounded, then there is a point* $a \in R^d$ *such that*

$$a + P(V) \subseteq \operatorname{clos}\operatorname{conv} T.$$

This lemma simply means that the "smallest" nonempty and bounded set in $\mathscr{C}(V)$ is just $P(V)$. Moreover, if V happens to consist of linearly independent vectors, then any V-closed, bounded, and nonempty set obviously contains $P(V)$. And what the lemma says is that the vectors from V behave as if they were linearly independent; i.e., nothing can be gained by using their dependences. The author thinks this is the underlying fact that makes the proofs of Theorem 4 work.

We show that while the proof of Theorem 3 is very easy the proof of Theorem 4 is not so simple. The author strongly believes that to prove a lower bound to $f(z, V)$ an appeal to convexity cannot be avoided. In connection with this we mention that the Lemma does not remain true if we put clos T instead of clos conv T in the statement.

Further, we remark that, by the lemma, one is able to describe the set $\operatorname{conv}\{z \in R^d : f(z) \leqslant \alpha\}$ for any $\alpha > 0$. Unfortunately, this result cannot be used determine the sets $\{z \in R^d : f(z) \leqslant \alpha\}$.

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Proof of Theorem 3. Let $t_0 \in P(V)$ be such that

$$\max_{t\in P(V)} \| z - t_0 + t \| = \min_{t_1\in P(V)} \max_{t_2\in P(V)} \| z - t_1 + t_2 \|,$$

and put $T = z - t_0 + P(V)$. Then $z \in T$ and T is V-closed. It follows from Theorem 2 that Player II has a strategy assuring $z + \sum_{i=1}^{n} \epsilon_i v^i \in T$ for $n = 0, 1, 2, \ldots$. But $T \subseteq \sup_{t\in T} \| t \| \cdot B$, and

$$\sup_{t\in T} \| t \| = \min_{t_1\in P(V)} \max_{t_2\in P(V)} \| z - t_1 + t_2 \|,$$

i.e.,

$$f(z, V) \leqslant \min_{t_1\in P(V)} \max_{t_2\in P(V)} \| z - t_1 + t_2 \|.$$

First Proof of Theorem 4. It is clear that $f(z) \geqslant \| z \|$. Since B is convex and centrally symmetric with respect to the origin, diam $(f(z) \cdot B) = 2f(z)$.

Now, in view of Theorem 1 it is enough to prove that there is no set $T \in \mathscr{C}(V)$ with $\operatorname{diam} T < \operatorname{diam} P(V)$ (except for, of course, the empty set). But this fact is an easy consequence of the lemma.

Before proving the lemma we need a proposition. We write $x(u)$ resp. $y(u)$ for the first resp. second component of $u \in R^2$.

PROPOSITION. Suppose $a_1, a_2, \ldots, a_k \in R^2$, $x(a_i) > 0$ $(i = 1, \ldots, k)$, $\sum_{i=1}^{k} x(a_i) = h$, and the slopes of the vectors a_i are strictly increasing. Write D for the convex hull of the two halflines $e_1 = \{(0, y) : y \geqslant 0\}$ and $e_2 = \{(h, y) : y \geqslant \sum_{i=1}^{k} y(a_i)\}$ and the points $a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_k$.

Then for any set of numbers $\alpha_i \in [0, 1]$ $(i = 1, 2, \ldots, k)$ we have

$$\sum_{i=1}^{k} \alpha_i a_i \in D.$$

Proof. Assume, on the contrary, that there are $\alpha_i \in [0, 1]$ $(i = 1, \ldots, k)$ such that $\sum_{i=1}^{k} \alpha_i a_i \notin D$, and put $I = \{i : 1 \leqslant i \leqslant k, \alpha_i > 0\}$. Clearly I cannot be the empty set. Let $I = \{i_1, i_2, \ldots, i_s\}$ with $i_1 < i_2 < \cdots < i_s$. We may suppose that $\sum_{j=1}^{m} \alpha_{i_j} a_{i_j} \in D$ for $m = 1, 2, \ldots, s-1$. This means that the polygonal path through the points $\theta, \alpha_{i_1} a_{i_1}, \alpha_{i_1} a_{i_1} + \alpha_{i_2} a_{i_2}, \ldots, \alpha_{i_1} a_{i_1} + \cdots + \alpha_{i_s} a_{i_s}$ meets the boundary of D in a point $b = \alpha_{i_1} a_{i_1} + \cdots + \alpha_{i_{s-1}} a_{i_{s-1}} + \lambda a_{i_s}$ with $0 \leqslant \lambda < 1$. On the other hand, $0 < x(b) < h$ and so b has to be on the polygonal path through the points $\theta, a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_k$; i.e., $b = a_1 + a_2 + \cdots + a_{l-1} + \mu a_l$ with $0 \leqslant \mu < 1$. But because $\sum_{j=1}^{s-1} \alpha_{i_j} a_{i_j} \in D$, $\sum_{j=1}^{s} \alpha_{i_j} \alpha_{i_j} \notin D$, the slope of a_{i_s} is less the slope of a_l, and then $i_s < l$. Now, from the two representations of b we conclude that

$$\theta = \sum_{i=1}^{k} \beta_i a_i,$$

where $0 \leqslant \beta_i \leqslant 1$ and $\beta_s \neq 0$. But this is impossible because it implies that

$$0 = x(\theta) = \sum_{i=1}^{k} \beta_i x(a_i) > 0.$$

Proof of the Lemma. In view of the assumption $(*)$ the case in which $d = 1$ is trivial so we begin with $d = 2$ (see Fig. 1). First we choose a support line e_0 to clos conv T in such a way that e_0 and clos conv T have only one point, say z_0, in common. Now put the origin of a coordinate system into z_0, and let the y axis be e_0, and suppose that T is on the right halfplane. We may suppose that $x(v) > 0$ for all $v \in V$ (the choice of z_0 and e_0 implies that $x(v) \neq 0$), and let $V = \{v_1, \ldots, v_k\}$ be indexed in such a way that their slopes increase. Let $e(v)$ be the "lower" support line to conv T, parallel with $v \in V$.

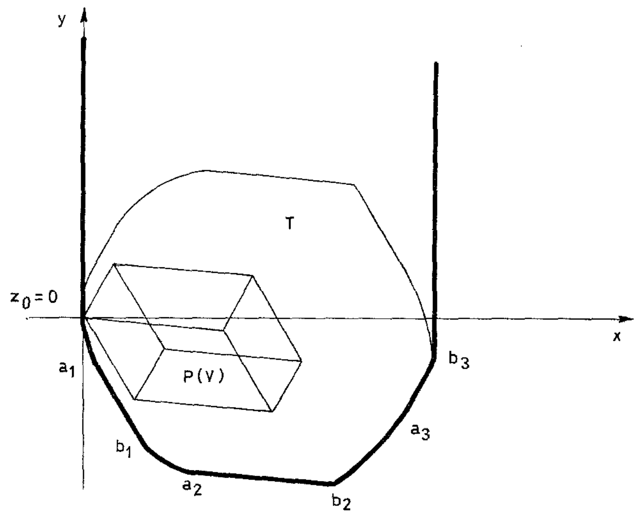

FIG. 1. Illustration to the lemma. The heavy line is the boundary of D_1.

Then (iv) implies that $e(v_i) \cap$ clos conv T is a line segment, parallel to v_i and at least as long as v_i. Going a round on the "lower" boundary of clos conv T from the origin we denote the endpoints of these line segments by $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$; i.e., $\alpha_i (b_i - a_i) = v_i$ with a suitable $0 < \alpha_i \leqslant 1$ for $i = 1, 2, \ldots, k$ ($b_i = a_{i-1}$ may happen). Applying the proposition to the vectors $a_1, b_1 - a_1, a_2 - b_1, b_2 - a_2, \ldots, a_k - b_{k-1}, b_k - a_k$ and to the numbers $0, \epsilon_1\alpha_1, 0, \epsilon_2\alpha_2, 0, \ldots, 0, \epsilon_k\alpha_k$ (with $\epsilon_i = 0$ or 1) we get that $\sum_{i=1}^k \epsilon_i v_i \in D_1$, where D_1 is the convex hull of the two halflines $\{(0, y) : y \geqslant 0\}$ and $\{(\sum_{i=1}^k x(v_i), y) : y \geqslant \sum_{i=1}^k y(v_i)\}$ and the points $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$. Since, in view of the assumption (*), $P(V) = \{\sum_{i=1}^k \epsilon_i v_i : \epsilon_i = 0 \text{ or } 1\}$ we have shown that $P(V) \subseteq D_1$. In a similar way, one can prove that $P(V) \subseteq D_2$, where D_2 is the convex hull of two halflines and certain well-defined points from the "upper" boundary of clos conv T. Clearly, $D_1 \cap D_2 \subseteq$ clos conv T.

In this way, we have proven a bit more than the lemma says. Namely, if e_0, z_0, and the directions in V are chosen as above, then

$$z_0 + P(V) \subseteq \text{clos conv } T.$$

This is what one needs to complete the proof in the case in which $d > 2$. For $d > 2$ one supposes the lemma is not true (in the above form); i.e., choosing a supporting hyperplane e_0 to clos conv T with outer normal n_0 in such a way that $e_0 \cap$ clos conv $T = \{z_0\} \in R^d$, there is a point $z_1 \in z_0 + P(V)$ not in clos conv T. (It is supposed that the vectors from V all satisfy $\langle n_0, v\rangle > 0$), Then z_0 and clos conv T can be strictly separated by a hyperplane e_1 with normal n_1. We write π for the projection from R^d onto the subspace spanned by n_0 and n_1. It is easy to show that e_1 can be chosen so that $\pi(V)$ do not contain collinear vectors. Now, in the two dimensional subspace spanned by n_0 and n_1

$$\pi(z_1) \notin \pi(\text{clos conv } T),$$

but $\pi(z_1) \in \pi(z_0) + \pi(P(V)) = \pi(z_0) + P(\pi(V))$ and $\pi(\text{clos conv } T) = \text{clos conv } \pi(T)$, in contradiction to what we proved for $d = 2$.

We present another proof of Theorem 4. This proof gives a direct strategy for Player I.

Second Proof of Theorem 4. As before, $f(z) \geqslant \| z \|$ is trivial, and we begin with the proof of $f(z) \geqslant \frac{1}{2}$ diam $P(V)$ for $d = 2$. (See Fig. 2.)

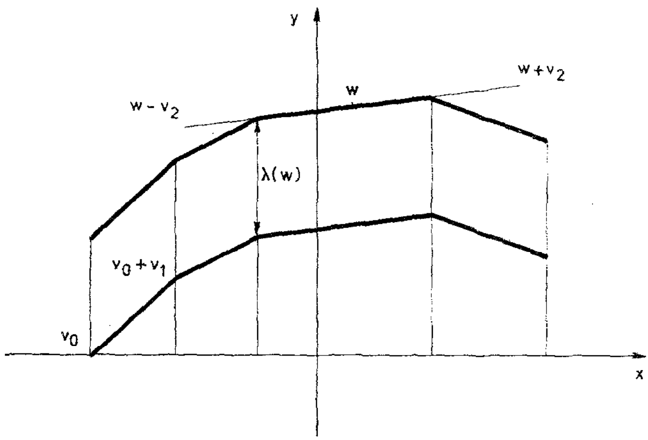

FIG. 2. Illustration to the strategy for Player I.

Without loss of generality we may suppose that $2h = \text{diam } P(V) = \sum_{v \in V} | x(v)| = \sum_{v \in V} x(v)$ (multiplying some vectors from V by -1, if necessary). Let $v_1, v_2, \ldots, v_k$, the vectors in V with $x(v_i) \neq 0$ ($i = 1, \ldots, k$), be indexed in such a way that their slopes decrease. Let v_0 be defined by $x(v_0) = -h$, $y(v_0) = 0$, and put $w = w_n = z + \sum_{i=1}^{n} \epsilon_i v_i$. Now Player I's strategy runs like this : He selects v_i for the $(n + 1)$th round if

$$\sum_{j=1}^{i-1} x(v_j) \leqslant x(w) < \sum_{j=1}^{i} x(v_j).$$

Note that this strategy depends only on the first component of w, and is not defined for $x(w) < x(v_0) = -h$ and for $x(w) \geqslant \sum_{j=1}^{k} x(v_j) = h$. Now we claim that if $-h \leqslant x(w_n) < h$ for all $n = 0, 1, 2, \ldots$, then $y(w_n)$ tends to infinity. It is clear that if this proposition holds true then we are through in the case in which $d = 2$.

The polygonal path through the points $v_0, v_0 + v_1, v_0 + v_1 + v_2, \ldots,$

$v_0 + v_1 + \cdots + v_k$ is a function $\varphi(x) \cdot \varphi(x)$ is defined on $[-h, h)$ and is concave. Let the level of w with $x(w) \in [-h, h)$ be defined by

$$\lambda(w) = -\varphi(x(w)) + y(w),$$

or less formally, $\lambda(w)$ is the distance through which one should slide the graph of φ upwards so that it contain the point w. In these terms, Player I's strategy is to select a vector which is tangent to the function $\varphi(x) + \lambda(w)$ at the point w. $\lambda(w_n)$ is nondecreasing for φ is concave. We claim that it tends to infinity. Since $x(w_n) = x(z) + \sum_1^n \epsilon_i x(v_i) \in [-h, h)$ for $n = 0, 1, \ldots$, there are an infinite number of indices m with $\epsilon_m = +1$ and $\epsilon_{m+1} = -1$. For such an index $v^m = v_i$ and $v^{m+1} = v_j$ with $i < j$ (here i and j depend on m). Clearly, there is a tangent line to the function $\varphi(x) + \lambda(w_{m-1})$ at the point w_{m-1} which is parallel with v_i. Now $\varphi(x) + \lambda(w_{m-1})$ is concave, and that is why $\lambda(w_{m+1}) - \lambda(w_{m-1})$ is at least as large as the distance through which one should slide this tangent line upwards so that it contain w_{m+1}. This distance depends only on i and j and is positive for each i, j $(i < j)$. But there is only a finite number of such distances, and so there is a minimal among them. This proves that $\lambda(w_n)$ tends to infinity. Now $\varphi(x)$ is bounded so $y(w_n)$ has got to tend to infinity.

We have the case in which $d > 2$ left. One may suppose without loss of generality that diam $P(V) = \sum_{v \in V} x(v)$, where $x(u)$ is the first component of $u \in R^d$. Let $e_1 \in R^d$ be the vector with $x(e_1) = 1$ and zeros in the other components, and let $e \in R^d$ be such that e and e_1 are not collinear. Let π denote the orthogonal projection of R^d onto the two-dimensional subspace spanned by e and e_1. Then it is possible to choose $e \in R^d$ in such a way that $\pi(V)$ does not contain collinear vectors, except possibly for those with $x(v) = 0$.

Now we simply project "the game" from R^d to the subspace spanned by e and e_1. The above strategy in this subspace will do for R^d as well.

Now we clearly have the following result:

COROLLARY 3. *If the assumption* $(*)$ *holds and the upper and lower bounds of Theorems* 3 *and* 4 *agree for* $z = \theta$, *then the strategies given in the proofs are optimal for both players in the game starting with* $z = \theta$.

Thus in this case the strategy for Player I given in the second proof of Theorem 4 is not only ϵ-optimal but optimal as well.

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Remarks. (1) In the proofs of Corollary 1 Theorems 3 and 4 we used only the following properties of B : B is convex, centrally symmetric with respect to the origin, and $\theta \in \operatorname{int} B$. Because these properties are true for the unit ball of any norm, Corollary 1, and Theorems 3 and 4 are also true in case of any norm of R^d.

We mention further that Theorems 1 and 2 also remain true if we replace R^d with any Abelian group.

(2) It is an interesting and useful consequence of the lemma that if the assumption $(*)$ holds true and T is V-closed and $u \in R^d$, then we have for $d(u, T)$, the width of T in the direction u, that $d(u, T) \geqslant \sum_{v \in V} |\langle u, v\rangle|$.

Let V_1 resp. V_2 be the set of all vectors with 0, 1 resp. $-1, 0, +1$ components. V_2 does not satisfy the assumption $(*)$ because $v \in V_2$ implies that $-v \in V_2$. But $V_2' = \{v \in V_2 :$ the first nonzero component of v equals $+1\}$ clearly satisfies $(*)$, and the game is the same with V_2' as with V_2. Now it is easy to show that the hypotheses of Corollaries 2 and 3 are satisfied for V_1 and V_2'. Then we have explicit optimal strategies for both players (in both games) assuring

$$f(\theta, V_1) = 2^{d-2} \qquad \text{for } d \geqslant 2$$

and

$$f(0, V_2) = f(0, V_2') = \tfrac{1}{2}(3^{d-1} + 1) \qquad \text{for } d \geqslant 1.$$

(3) If we drop the condition of noncollinearity from the assumption $(*)$ we can prove similar results. Trying to carry out the proofs of Theorem 4 for this case the following question arises. Suppose $v \in R^d$ is a unit vector in the Euclidean norm and $V = \{\lambda_i v : \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0\}$. What is, then,

$$\delta(V) = \inf\{\operatorname{diam} T : T \in \mathscr{C}(V)\}?$$

Clearly, $\delta(V) = \lambda_1$ if $k = 1$, $\delta(V) = \lambda_1 + \lambda_2$ if $k = 2$. The author could only suggest the formula for $\delta(V)$ in the general case. This formula seems to be rather complicated. However, it is easy to give a simple estimation:

$$\lambda_1 + \lambda_2 \leqslant \delta(V) \leqslant 2\lambda_1 \qquad \text{if } k \geqslant 2.$$

Further, it can be shown easily that there is a set $T \in \mathscr{C}(V)$ with $\theta \in T \subseteq \{\lambda v : \lambda \geqslant 0\}$ and $\operatorname{diam} T = \delta(V)$. Writing $T_0(V)$ for the union of all sets of this type the following theorem can be proven:

THEOREM. *Suppose $V \subset R^d$ is finite and nonempty. The equivalence relation "to be collinear" splits V into the disjoint union of $V_1, \ldots, V_k$. The vectors from V_i may be supposed to have the same direction. Put*

$$P_0(V) = \sum_{i=1}^{k} T_0(V_i) = \left\{\sum_{i=1}^{k} x_i : x_i \in T_0(V_i)\right\}.$$

Then

$$\max(\|z\|, \tfrac{1}{2} \operatorname{diam} P_0(V)) \leqslant f(z, V) \leqslant \min_{t_1 \in P_0(V)} \max_{t_2 \in P_0(V)} \|z - t_1 + t_2\|.$$

We mention further that this theorem remains true (in the obvious form) if we replace the assumption "V is finite" by

$$\sum_{v \in V} \| v \| < \infty.$$

(4) It is possible to generalize the lemma in the following way. Put

$$R(V) = \left\{ \sum_{v \in W} \alpha(v)\, v \colon W \subseteq V \text{ finite}, \alpha(v) = 0, 1, 2, \ldots \right\}$$

THEOREM. *Suppose $V \subseteq R^d$ is nonempty and does not contain collinear vectors, and $u \in R^d$ is such that $\langle u, v \rangle \geqslant 0$ for each $v \in V$. Let $T \in \mathscr{C}(V)$ and e_0 be a supporting hyperplane to* clos conv T *with normal u such that $e_0 \cap$* clos conv $T \neq \varnothing$. *Put $V' = V \cap 0^+$* (clos conv T), *$V'' = V \backslash V'$. Then there is a point $z_0 \in e_0 \cap$* clos conv T *such that*

$$z_0 + R(V') + R(V'') \subseteq \text{clos conv } T,$$

where 0^+ (clos conv T) *denotes the recession cone for* clos conv T (*see* [2]).

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