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AN EXTENSION OF THE ERDŐS—SZEKERES THEOREM ON LARGE ANGLES

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The existence of a function $n(\varepsilon)$ $(\varepsilon>0)$ is established such that given a finite set V in the plane there exists a subset $W\subseteqq V$, $|W|<n(\varepsilon)$ with the property that for any $v\in V\setminus W$ there are two points $w_1, w_2\in W$ such that the angle $\sphericalangle(w_1vw_2)>\pi-\varepsilon$.

1. Results

The Erdős—Szekeres theorem [5] mentioned in the title says that if $V\subset\mathbf{R}^2$ and $|V|\geqq 2^n$, then V contains three points $a, b, c\in V$ such that $\sphericalangle(abc)>\pi(1-(1/n))$. Here $\sphericalangle(abc)$ denotes the angle at vertex b of the triangle abc, thus $0\leqq\sphericalangle(abc)\leqq\pi$.

The aim of this paper is to extend this theorem in the following way.

Theorem 1. *For any $\varepsilon>0$ there exists $n(\varepsilon)$ such that every finite set $V\subset\mathbf{R}^2$ contains a subset $W\subseteqq V$, $|W|\leqq n(\varepsilon)$ with the property that for any $v\in V\setminus W$ there are points $w_1, w_2\in W$ such that $\sphericalangle(w_1vw_2)>\pi-\varepsilon$.*

In other words, every finite set $V\subset\mathbf{R}^2$ contains a "small" subset W from which any point of $V\setminus W$ is seen at a "large" angle.

We will show that one can take

$$n(\varepsilon)=\left(\frac{c_1}{\varepsilon}\right)^{(c_2/\varepsilon)}$$

where c_1 and c_2 are contants. On the other hand $n(\varepsilon)\geqq 2^{c/\varepsilon}$ since one can construct a set $V\subset\mathbf{R}^2$ with $|V|=2^{c/\varepsilon}$ such that $\sphericalangle(abc)\leqq\pi-\varepsilon$ for every $a, b, c\in V$ (see [6]).

The proof-method of Theorem 1 works in $\mathbf{R}^d$ $(d>2)$ as well giving

Theorem 2. *For any $\varepsilon>0$ and $d\geqq 2$ there exists a constant $n(d, \varepsilon)$ such that every finite set $V\subset\mathbf{R}^d$ contains a subset $W\subseteqq V$, $|W|\leqq n(d, \varepsilon)$ with the property that for any $v\in V\setminus W$ there are points $w_1, w_2\in W$ with $\sphericalangle(w_1vw_2)>\pi-\varepsilon$.*

Let $n_0(d, \varepsilon)$ be the smallest number $n(d, \varepsilon)$ for which Theorem 2 holds. When

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$\varepsilon<1$, say, we obtain from the proof that

$$2^{(c/\varepsilon)^{d-1}} \leqq n_0(d, \varepsilon) \leqq \left(\frac{c_1\sqrt{d}}{\varepsilon}\right)^{(c_2/\varepsilon)^{d-1}}$$

where for the lower bound see [1] or [6].

In the proof of Theorem 1 and 2 we may assume that $\varepsilon>0$ is small because if the theorem holds with some $\varepsilon>0$ then it holds with every $\varepsilon'\geqq\varepsilon$: one can take simply $n(d, \varepsilon')=n(d, \varepsilon)$. So the above estimation for $n_0(d, \varepsilon)$ holds for small ε. When $\varepsilon=\pi/2$, one can do better. More precisely, let $\varepsilon\geqq\arccos(1/d)$. Then a stronger version of Theorem 2 is true, which was also proved by János Pach independently [7].

Theorem 3. *For every* $d=2, 3, \ldots$ *there exists a number* $n(d)$ *such that the following holds. Every finite set* $V\subset\mathbf{R}^d$ *contains a subset* $W\subseteqq V$, $|W|\leqq n(d)$ *such that* $W\subseteqq$ boundary (conv V) *and for any point* $v\in(\text{conv } V)\setminus W$ *there are points* $w_1, w_2\in W$ *with* $\cos\sphericalangle(w_1vw_2)\leqq-1/d$.

From the proof of this theorem we get that for $\arccos(1/d)<\varepsilon<2\pi$, $n_0(d, \varepsilon)\leqq 2^{d(d-1)}\cdot d^{d(d+1)}$. Erdős and Füredi [4] gave an example $V\subset\mathbf{R}^d$ with $|V|\geqq c_\alpha^d$ points such that $\sphericalangle(abc)<\alpha$ for every $a, b, c\in V$, where $\alpha>\pi/3$ is fixed and $c_\alpha>1$. This shows that for $\arccos(1/d)<\varepsilon<2\pi$, $c_{\pi-\varepsilon}^d\leqq n_0(d, \varepsilon)$.

The reason for the bound $\arccos(1/d)\leqq\varepsilon$ in Theorem 3 is the following fact which is certainly well-known. If the point a belongs to a d-dimensional simplex, then the simplex has two vertices b and c with $\sphericalangle(bac)\geqq\pi-\arccos(1/d)$.

I mention that our theorems hold for compact sets $V\subset\mathbf{R}^d$ (instead of finite). This can be seen using a simple continuity argument.

In [5] Erdős and Szekeres dealt with a related problem. Having fixed a basis in $\mathbf{R}^d$, a box is defined as $\{x\in\mathbf{R}^d\colon a_i\leqq x_i\leqq b_i \ i=1, \ldots, d\}$. Now Erdős and Szekeres show in [5] that if $V\subset\mathbf{R}^d$, $|V|>2^{2^{d-1}}$, then there are points $a, b, c\in\mathbf{R}^d$ such that b is contained in the smallest box containing a and c. This result is extended in a paper by Bárány and Lehel [2] in the same way as Theorem 1 here extends the Erdős—Szekeres theorem on large angles. In particular, Theorem 2 follows from the results of [2] but with a much weaker estimation on $n_0(d, \varepsilon)$.

2. Sketch of the proof of Theorem 1

The proof of Theorem 1 will be algorithmic. In this section we give an informal description of the algorithm together with some preliminary definitions and lemmas. We will work with small squares whose sides are parallel to the coordinate axes. $\varrho(x, y)$ denotes the Euclidean distance of $x, y\in\mathbf{R}^2$.

Definition 1. We say that $s_1, s_2\in\mathbf{R}^2$ *cover* the square $C\subseteqq\mathbf{R}^2$ if $\sphericalangle(s_1cs_2)>\pi-\varepsilon$ for every $c\in C$.

Let C^0 be the smallest square containing V. The algorithm will find a subdivision of C^0 into small squares and a set $W\subset V$, $|W|\leqq n(\varepsilon)$ whose pairs cover every small square C in the subdivision provided $C\cap(V\setminus W)\neq\emptyset$.

Definition 2. (C_Δ, S) is called a *good pair* if C_Δ is a square of side length Δ and $S\subset \mathbf{R}^2$ with the properties

(i) $\varrho(c, s) \geqq 10\sqrt{2}\Delta\varepsilon^{-1}$ $(\forall c\in C, \forall s\in S)$,
(ii) $\sphericalangle(s_1 c s_2) \geqq 0{,}5\varepsilon$ $(\forall c\in C, \forall s_1, s_2\in S, s_1 \neq s_2)$.

Sometimes we shall omit the index Δ and simply say that (C, S) is a good pair. A good pair can be thought of as a square C together with a set of directions S because, by condition (i), the points in S are far away from C (almost at infinity with respect to its diameter). As a rule, S will be a subset of V and the algorithm will produce good pairs (C, S) in such a way that some point $s\in S$ together with a suitable point $v\in V$ will cover C. Property (ii) means that the directions in S are not too close to each other. This fact implies at once a simple property of good pairs:

Lemma 1. *If* (C, S) *is a good pair, then* $|S|\leqq 4\pi\varepsilon^{-1}$.

Definition 3. The good pair (C_Δ^1, S) *covers* the square C_Δ^2 if there is an $s\in S$ such that for every $c\in C_\Delta^1$, c and s cover C_Δ^2.

The crucial property of good pairs is given next.

Lemma 2. *If* (C_Δ^1, S) *and* (C_Δ^2, S) *are good pairs and*

$$0.1\varrho(c_2, s) \geqq \varrho(c_1, c_2) \quad (\forall c_1\in C_\Delta^1, \forall c_2\in C_\Delta^2, \forall s\in S),$$

$$\varrho(c_1, c_2) \geqq 10\sqrt{2}\Delta\varepsilon^{-1} \quad (\forall c_1\in C_\Delta^1, \forall c_2\in C_\Delta^2),$$

then either (C_Δ^1, S) *covers* C_Δ^2, *or* $(C_\Delta^1, S\cup\{u\})$ *is a good pair for every* $u\in C_\Delta^2$.

We can now explain how the algorithm works. At a certain stage we will have a good pair (C_Δ, S) such that no $s_1, s_2\in S$ cover C_Δ. We then subdivide C_Δ by a set of smaller squares of side length $\Delta'=2^{-k}\Delta$ ($k=k(\varepsilon)$ will be specified later). A small square $C_{\Delta'}$ of this subdivision is called a *cell* if $C_{\Delta'}\cap V\neq\emptyset$. We pick one point $u\in V$ from each cell $C_{\Delta'}$. These points form a set $U\subseteq V$ with $|U|\leqq 4^k$. If $C_{\Delta'}$ is a cell then $(C_{\Delta'}, S)$ is clearly a good pair. Now the main part of the algorithm can be named as "either cover $C_{\Delta'}$ or augment S". More precisely, on applying Lemma 2 to the good pairs $(C_{\Delta'}^1, S)$ and $(C_{\Delta'}^2, S)$ one finds a point $u\in U$ such that either $C_{\Delta'}^1$ is covered by u and a suitable $s\in S$ or $(C_{\Delta'}^1, S\cup\{u\})$ is a good pair, unless a very special case occurs. If this special case does not come up then, in view of Lemma 1, we are finished after at most $4\pi\varepsilon^{-1}$ steps of the type "either cover or augment". In this case the set W will be the union of all the sets U and will have cardinality at most $(4^k)^{4\pi\varepsilon^{-1}}$. The special case which needs special care as well corresponds to, roughly speaking, a point $v\in V$ such that $\sphericalangle(v_1 v v_2)\leqq\pi-\varepsilon$ for every $v_1, v_2\in V$. Such a point v must be contained in W, so we cannot choose $u\in C_{\Delta'}\cap V$ arbitrarily if $v\in C'_\Delta$ as we could in the other case.

To close this section we prove Lemmas 1 and 2.

Proof of Lemma 1. Consider a point $c\in C$ and a unit circle E around it. For each $s\in S$ define the arc A_s as $A_s=\{x\in E\colon \sphericalangle(xcs)<0.25\varepsilon\}$. These arcs are pairwise disjoint by (ii) so the sum of their lengths is at most 2π:

$$2\pi \geqq \sum_{s\in S} \text{length } (A_s) = |S|\cdot 0.5\varepsilon. \quad ■$$

Proof of Lemma 2. Assume first that there is an $s \in S$ such that $\sphericalangle(c_2 c_1 s) \leqq 0.9\varepsilon$ ($\forall c_1 \in C^1$, $\forall c_2 \in C^2$). Then the condition $0.1\varrho(c_2, s) \geqq \varrho(c_1, c_2)$ implies

$$\frac{\sin \sphericalangle (c_2 c_1 s)}{\sin \sphericalangle (c_2 s c_1)} = \frac{\varrho(c_2, s)}{\varrho(c_2, c_1)} \geqq 10,$$

and then

$$\sin \sphericalangle(c_2 s c_1) \leqq 0.1 \sin \sphericalangle(c_2 c_1 s) \leqq 0.1 \sin 0.9\varepsilon < \sin 0.1\varepsilon.$$

This shows that in this case $\sphericalangle(c_1 c_2 s) > \pi - (0.9\varepsilon + 0.1\varepsilon) = \pi - \varepsilon$, i.e., (C^1, S) covers C^2 by definition.

Assume now that for each $s \in S$ there are $c_1^0 \in C^1$ and $c_2^0 \in C^2$ such that $\sphericalangle(c_2^0 c_1^0 s) > 0.9\varepsilon$. Then, using the condition $\varrho(c_1, c_2) \geqq 10\sqrt{2}\,\Delta\varepsilon^{-1}$, an easy argument shows that $\sphericalangle(c_2 c_1 s) \geqq 0.5\varepsilon$ for each $c_1 \in C^1$, $c_2 \in C^2$, $s \in S$, i.e., $(C^1, S \cup \{u\})$ is a good pair for every $u \in C^2$. ∎

3. The proof of Theorem 1

We start with the good pair $(C^0, \varnothing)$ where C^0 is the smallest square containing V.

Assume that at a certain stage of the algorithm we have a good pair (C_Δ^0, S) with the property that no $s_1, s_2 \in S$ cover C_Δ^0. (If this were not so C_Δ^0 would be covered.) Now we subdivide C_Δ^0 by smaller squares of side length $\Delta' = 2^{-k}\Delta$, where $k = k(\varepsilon)$ will be specified later. Let $\mathscr{C}$ denote the set of cells of this subdivision, i.e., $C_{\Delta'} \in \mathscr{C}$ if $C_{\Delta'} \cap V \neq \emptyset$.

Now we try to apply the "either cover or augment" procedure. In order to do so, there are some cases to consider.

Case 1. For each $C \in \mathscr{C}$ there is a $C' \in \mathscr{C}$ such that (C', S) covers C.

In this case we pick one point $u \in V$ from each cell $C \in \mathscr{C}$. The set U of these points satisfies $|U| \leqq |\mathscr{C}| \leqq 4^k$. Moreover, for every $C \in \mathscr{C}$ there are $u \in U$ and $s \in S$ such that C is covered by u and s.

Now we assume that there is a cell $C \in \mathscr{C}$ which is not covered by any pair (C', S) with $C' \in \mathscr{C}$. Such a cell is called *uncovered.*

Case 2. There are two uncovered cells $C_1, C_2 \in \mathscr{C}$ with $\varrho(c_1, c_2) \geqq 1/6\Delta$ for each $c_1 \in C_1$ and $c_2 \in C_2$.

Then we pick one point from each cell as in Case 1. The cells covered by some pair (C, S) with $C \in \mathscr{C}$ will then be covered by a suitable $u \in U$ and $s \in S$.

Consider now a cell $C \in \mathscr{C}$ which is uncovered. Then either

$$\varrho(c, c_1) \geqq \frac{1}{12}\Delta - \frac{1}{\sqrt{2}}\Delta' \quad (\forall c \in C, \forall c_1 \in C_1),$$

or

$$\varrho(c, c_2) \geqq \frac{1}{12}\Delta - \frac{1}{\sqrt{2}}\Delta' \quad (\forall c \in C, \forall c_2 \in C_2).$$

By symmetry we may and do assume that the first inequality holds. Let u be the point picked from C_1.

Claim. $(C, S\cup\{u\})$ *is a good pair provided* $2^k > 300\varepsilon^{-1}$.

Proof. We are going to apply Lemma 2 to the good pairs (C, S) and (C_1, S). The first condition holds because for every $c\in C$, $c_1\in C_1$ and $s\in S$

$$0.1\varrho(c_1, s) \geqq 0.1 \min_{c^0\in C^0_\Delta} (c^0, s) \geqq 0.1\cdot 10\sqrt{2}\Delta\varepsilon^{-1} > \sqrt{2}\Delta \geqq \varrho(c, c_1)$$

(if $\varepsilon < 1$, say). As for the second condition,

$$\varrho(c, c_1) \geqq \frac{1}{12}\Delta - \frac{1}{\sqrt{2}}\Delta' = \left(\frac{1}{12}2^k - \frac{1}{\sqrt{2}}\right)\Delta'$$

and this has to be larger than $10\sqrt{2}\Delta'\varepsilon^{-1}$. This holds, for instance, when we choose $k=\lceil\log_2(300\varepsilon^{-1})\rceil$.

So the conditions of Lemma 2 hold. But $C_1\in\mathscr{C}$ is an uncovered cell and then the second alternative occurs: $(C, S\cup\{u\})$ is a good pair for every $u\in C_1\cap V$. ∎

We can see now that in Case 2 the "either cover or augment" method works.

From now on we assume that there are uncovered cells in $\mathscr{C}$ and for any two uncovered cells $C_1, C_2\in\mathscr{C}$ $\varrho(c_1, c_2)\leqq\Delta/6+2\sqrt{2}\Delta'$ $(\forall c_i\in C_i\ \ i=1, 2)$.

Let K be (one of) the smallest square(s) $K\subset C^0_\Delta$ which contains all uncovered cells from $\mathscr{C}$. The side length of K is at most $\Delta/6+2\sqrt{2}\Delta'$. Further, let $L\subset C^0_\Delta$ be a square with side length $\Delta/2$ containing K and such that the minimal distance between K and $C^0_\Delta\setminus L$ is at least $10\sqrt{2}\Delta'\varepsilon^{-1}$. We assume further that the sides of L are contained in the lines defining the subdivision of C^0_Δ into the smaller squares $C_{\Delta'}$. Such an L exists if

$$\left(\frac{1}{6}\Delta+2\sqrt{2}\Delta'\right)+10\sqrt{2}\Delta'\varepsilon^{-1} < \frac{1}{2}\Delta,$$

which is again true if $k=\lceil\log_2(300\varepsilon^{-1})\rceil$.

Finally we define a partition of $\mathscr{C}$:
$\mathscr{C}^z=\{C\in\mathscr{C}\colon C$ is covered by some good pair (C', S), $C'\in\mathscr{C}\}$,
$\mathscr{C}^x=\{C\in\mathscr{C}\colon C$ is uncovered and (C, S) covers every cell in $C^0_\Delta\setminus L\}$,
$\mathscr{C}^y=(\mathscr{C}\setminus\mathscr{C}^z\cup\mathscr{C}^x)$.

Case 3. $\mathscr{C}^x=\emptyset$.

Again we pick one point $u\in V$ from each cell, these points form the set $U\subset V$. The cells in $\mathscr{C}^z$ will be covered by some $u\in U$ and $s\in S$ in the same way as in Case 1. Let $C\in\mathscr{C}^y$ and consider a cell $C'\in\mathscr{C}$, $C'\subseteqq C^0_\Delta\setminus L$ which is not covered by the good pair (C, S). Let $u\in C'\cap U$.

Claim. $(C, S\cup\{u\})$ *is a good pair.* ∎

The proof is almost identical with that of Case 2 and is omitted.

We see again that in this case the method of "either cover or augment" works.

Case 4. $\mathscr{C}^x \neq \emptyset$.

In this case we do not pick any point from V but make the promise to pick one point from the set $X_1 = V \cap (\bigcup \mathscr{C}^x)$ at a later stage of the algorithm. For further reference we rename K as K_1, L as L_1, $\mathscr{C}, \mathscr{C}^x, \mathscr{C}^y, \mathscr{C}^z$ as $\mathscr{C}_1, \mathscr{C}_1^x, \mathscr{C}_1^y$ and $\mathscr{C}_1^z$. We observe that if we pick a point $u \in X_1$, then each $C \in \mathscr{C}_1$, $C \subseteqq C_\Delta^0 \setminus L_1$ will be covered by u and a suitable $s \in S$. The algorithm continues by subdividing L_1 into $2^k \times 2^k$ smaller squares and considering L_1 as C_Δ^0 in the preceding step.

If Case 4 occurs, we get two squares K_2 and L_2, sets of cells $\mathscr{C}_2, \mathscr{C}_2^x, \mathscr{C}_2^y, \mathscr{C}_2^z$ (the sides of L_2 are of length $\Delta/4$) and a new set of "promise" $X_2 = V \cap (\bigcup \mathscr{C}_2^x)$. In this case we make the promise to pick a point from the set $X_1 \cap X_2$ (if $X_1 \cap X_2 \neq \emptyset$) and continue in the same way as before. So we go on like that producing the squares K_i, L_i, sets of cells $\mathscr{C}_i, \mathscr{C}_i^x, \mathscr{C}_i^y, \mathscr{C}_i^z$ and sets of promises $X_1 \cap X_2 \cap \ldots \cap X_i$.

We can keep our promises only if $X_1 \cap \ldots \cap X_i \neq \emptyset$. Suppose this is so. Again, we try to use the method of "either cover or augment" on the square L_i (side-length $2^{-i}\Delta$). If it works (i.e. Cases 1, 2 or 3 occur), we can keep our promise by choosing a point from $C \cap (X_1 \cap \ldots \cap X_i)$ for a suitable $C \in \mathscr{C}_i$. If it does not work, then Case 4 occurs again: we get $X_{i+1} \neq \emptyset$, K_{i+1}, L_{i+1}, etc.

Case 4a. $\bigcap_{j=1}^{i} X_i \neq \emptyset$ *but* $\bigcap_{j=1}^{i+1} X_i = \emptyset$ *for some* $i = 1, 2, 3, \ldots$.

Case 4b. $\bigcap_{j=1}^{i} X_i \neq \emptyset$ *for* $i = 1, 2, 3, \ldots$.

In Case 4b $\bigcap_{i=1}^{\infty} X_i \subseteqq \bigcap_{i=1} L_i = \{u\} \subseteqq V$ because the squares L_i shrink to a single point. Then we pick the point u and put it in the set W. It is easily checked that for every i and every $C \in \mathscr{C}_i$, $C \subseteqq C_\Delta^0 \setminus L_i$ is covered by u and a suitable point $s \in S$.

In Case 4a we pick a point u from each cell $C \in \mathscr{C}_{i+1}$ according to the following rules:

$$u \in V \cap C, \quad \text{if} \quad C \cap \bigcap_{j=1}^{i} X_j = \emptyset \quad \text{and} \quad C \cap X_{i+1} = \emptyset,$$

$$u \in X_{i+1} \cap C, \quad \text{if} \quad C \cap X_{i+1} \neq \emptyset,$$

$$u \in C \cap \bigcap_{j=1}^{i} X_j \quad \text{if} \quad C \cap \bigcap_{j=1}^{i} X_j \neq \emptyset.$$

The points chosen this way form the set U, clearly $U \subseteqq V$ and $|U| \leqq 4^k$. In addition, the conditions $X_{i+1} \neq \emptyset$ and $X_1 \cap \ldots \cap X_i \neq \emptyset$ imply that U contains a point $u' \in X_{i+1}$ and $u'' \in X_1 \cap \ldots \cap X_i$. Then every cell in $C_\Delta^0 \setminus L_{i+1}$ is covered by the points u', s or u'', s for a suitable $s \in S$.

So we are concerned with the cells $C \in \mathscr{C}_{i+1}$ lying in L_{i+1}.

If $C \in \mathscr{C}_{i+1}^y$ or $C \in \mathscr{C}_{i+1}^z$, then the method of "either cover or augment" works in the same way as in Cases 1, 2 and 3.

Finally, if $C \in \mathscr{C}_{i+1}^x$, then $C \subset C_i \subset C_{i-1} \subset \ldots \subset C_1$ where $C_j \in \mathscr{C}_j$. (This chain of cells exists and is unique because the subdivision of L_{j+1} is a refinement of the subdivision of L_j, restricted to L_{j+1}.) But it cannot be the case that each $C_j\in\mathscr{C}_j^x$ because then $X_{i+1}\cap\ldots\cap X_1\supseteq V\cap(C\cap C_i\cap\ldots\cap C_1)\neq\emptyset$, a contradiction. So $C_j\in\in\mathscr{C}_j^y\cup\mathscr{C}_j^z$ for some $j=1,\ldots,i$.

If $C_j\in\mathscr{C}_j^z$, then C_j is covered by a pair u, $s\in S$ where $u\in V$ is a point from the cell in $\mathscr{C}_j$ that covers C_j. The points u, s cover then the cell C as well.

If $C_j\in\mathscr{C}_j^y$, then the pair $C_j,(S\cup\{u\})$ is a good pair for a suitable $u\in V\cap L_{j-1}$, and, a fortiori, $(C, S\cup\{u\})$ is a good pair as well.

So for each $C\in\mathscr{C}_{i+1}^x$ we pick one more point $u\in V\cap C_\Delta^0$ in such a way that either C is covered by u and some $s\in S$ or $(C, S\cup\{u\})$ is a good pair. The set of these points is U_0, clearly $|U_0|\leqq|\mathscr{C}_{i+1}^x|\leqq 1/4|\mathscr{C}_i|=4^{k-1}$.

What we did in Case 4a is a modification of the method "either cover or augment". This modification uses at most $|U|+|U_0|\leqq 5\cdot 4^{k-1}$ points. So we see that in Cases 1, 2, 3 and 4a our method works using at most $5\cdot 4^{k-1}$ points, while in Case 4b we need only one point to cover every point in $C_\Delta^0\cap V$.

The number of steps being limited by $|S|\leqq 4\pi\varepsilon^{-1}$ we conclude that Theorem 1 holds with

$$n(\varepsilon)=(5\cdot 4^{k-1})^{4\pi\varepsilon^{-1}}.$$

The proof of Theorem 2 is almost identical with the previous one. The only differences are in Definition 2, Lemma 1 and 2. In Definition 2, (i) is to be modified to $\varrho(c,s)\geqq 10\sqrt{d}\Delta\varepsilon^{-1}$ $(\forall c\in C, \forall s\in S)$, in Lemma 1 we have only $|S|\leqq(c_2\varepsilon^{-1})^{d-1}$, and in Lemma 2, condition $\varrho(c_1,c_2)\geqq 10\sqrt{2}\Delta\varepsilon^{-1}$ has to be replaced by $\varrho(c_1,c_2)\geqq\geqq 10\sqrt{d}\Delta\varepsilon^{-1}$. In the proof, $k=k(\varepsilon)=[\log_2(200\sqrt{d}\varepsilon^{-1})]$ giving

$$n(d,\varepsilon)=\left(\left(1+\frac{1}{2^d}\right)\frac{200\sqrt{d}}{\varepsilon}\right)^{(2\pi d^{3/2})(4/\varepsilon)^{d-1}}.$$

4. Proof of Theorem 3

As mentioned in the introduction, we will use the following

Lemma 3. *If the origin belongs to* $\operatorname{conv}\{a_0,\ldots,a_d\}$ *where* $a_0,\ldots,a_d\in\mathbf{R}^d$, *then there are indices i and j with* $\sphericalangle(a_i 0 a_j)\geqq\pi-\arccos 1/d$.

Proof. We may assume that $\|a_0\|=\|a_1\|=\ldots=\|a_d\|=1$ because we are concerned with angles at 0. Assume indirectly, that $\langle a_i,a_j\rangle>-1/d$ for each $0\leqq i<j\leqq d$. The condition $0\in\operatorname{conv}\{a_0,a_1,\ldots,a_d\}$ implies the existence of $\alpha_0,\ldots,\alpha_d\geqq 0$ with $\sum_{i=0}^{d}\alpha_i=1$ and

$$0=\sum_{i=0}^{d}\alpha_i a_i.$$

Multiplying this by a_j and using $\langle a_i,a_j\rangle>-1/d$ we get

$$-(1-\alpha_j)\frac{1}{d}=\sum_{\substack{i=0\\ i\neq j}}^{d}\alpha_i\left(-\frac{1}{d}\right)<\sum_{\substack{i=0\\ i\neq j}}^{d}\alpha_i\langle a_i,a_j\rangle=-\alpha_j.$$

Summing these inequalities for $j=0, \dots, d$ we have

$$-1 = \sum_{j=0}^{d} -(1-\alpha_j)\frac{1}{d} < -\sum_{j=0}^{d} \alpha_j = -1$$

a contradiction. ∎

The proof of Theorem 3 is by induction on d. The cases $d=1$ and $d=2$ are easy.

Assume the theorem holds in the k-dimensional space for $k<d$. We are going to prove it in the d-dimensional space.

As V is finite, conv V is a convex polytope. Denote the set of its vertices by U, the set of facets by $\mathcal{L}$, and the set of unit outer normals to the facets by N. So $N \subset S^{d-1}$ the unit sphere in $\mathbf{R}^d$. Take an ε-net $M \subseteq S^{d-1}$ with

$$|M| \leqq d\left(\frac{2}{\varepsilon}\right)^{d-1}$$

$\varepsilon>0$ will be specified later. Set $\omega_d = \arccos(-1/d)$.

Consider a point $m \in M$ and the set $N_m = \{n \in N: |n-m| \leqq \varepsilon\}$. N_m is the set of outer normals to the facets $\mathcal{L}_m$ of conv V. Let $H_m \subseteq \mathbf{R}^d$ be a hyperplane with normal m. The projection π_m (or π, for short) to H_m maps the set $K_m = \bigcup(L\colon L \in \mathcal{L}_m)$ to H_m that can be taken for $\mathbf{R}^{d-1}$. The vertex set U_m of the polytope conv K_m is clearly a subset of U, and $\pi(\operatorname{conv} K_m) = \pi(\operatorname{conv} U_m) = \operatorname{conv} \pi(U_m)$. So we may apply the induction hypothesis to the set $\pi(U_m) \subseteq H_m = \mathbf{R}^{d-1}$. Then we get a subset $W_m \subseteq \pi(U_m)$, $|W_m| \leqq n(d-1)$ such that for any point $u \in (\operatorname{conv} \pi(U_m)) \setminus W_m$ there exist two points $w_1, w_2 \in W_m$ with $\sphericalangle(w_1 u w_2) \geqq \omega_{d-1}$. It is clear that for each $u \in \operatorname{conv} \pi(U_m)$ there is exactly one point $\hat{u} \in K_m$ with $\pi(\hat{u}) = u$.

Now we show that $\sphericalangle(\hat{w}_1 \hat{u} \hat{w}_2) \geqq \omega_{d-1} - 2\varepsilon$ if $\hat{u} \in L$ for some $L \in \mathcal{L}_m$. For this end it is enough prove that if $w \in W_m$ then the angle between the lines wu and $\hat{w}\hat{u}$ is at most ε where $u = \pi(\hat{u})$ and $w = \pi(\hat{w})$. This implies the claim because then

$$|\sphericalangle(\hat{w}_1 \hat{u} \hat{w}_2) - \sphericalangle(w_1 u w_2)| \leqq 2\varepsilon.$$

If $w \in W_m$ then $\hat{w} \in L'$ for some $L' \in \mathcal{L}_m$. Pick a point $\hat{z} \in L'$ from the relative interior of L' and very close to $\hat{w}$. Similarly, pick a point $\hat{v} \in L$ from the relative interior of L and very close to $\hat{u}$. We prove that the angle between the lines through $\hat{z}$ and $\hat{v}$ and through z and v is at most ε where $z = \pi(\hat{z})$ and $v = \pi(\hat{v})$. This is clearly sufficient.

Consider the two-dimensional plane P spanned by $\hat{z}, \hat{v}, z$ and v. $P \cap \operatorname{conv} K_m$ is a convex polygon on the plane. The points $\hat{z}$ and $\hat{v}$ lie on its boundary. The angle between the tangent line to the polygon at $\hat{z}$ and the line zv is at most ε, the same is true at $\hat{v}$. This follows from the definition of N_m. Observe that $\hat{z}$ and $\hat{v}$ are not on "opposite" sides of the polygon because the outer normals at them are near to each other. Then the slope of the chord $\hat{z}\hat{v}$ of the polygon is between the slopes of the tangent lines at $\hat{z}$ and $\hat{v}$. And this is what we wanted to prove.

Choose now $\varepsilon>0$ in such a way that $\omega_d = \omega_{d-1} - 2\varepsilon$. Set

$$W = \bigcup_{m \in M} W_m = \{x \in U\colon \pi_m(x) \in W_m,\ x \in K_m \text{ for some } m \in M\}.$$

Then $|W| \leqq n(d-1) \cdot |M| \leqq n(d-1) \cdot 4^{d-1} d^{2d-1}$, and for any point $u \in$ boundary (conv V) there exist two points $w_1, w_2 \in W$ with $\sphericalangle(w_1 u w_2) \geqq \omega_d$.

Moreover, W "covers" also conv W because by Carathéodory's theorem [3] conv W is the union of d-dimensional simplices with vertices from W, and then we can apply Lemma 3.

So far we have proved that the theorem is true for any $u \in$[conv $W \cup$ boundary (conv V)]$\setminus W$. So we pick a point $u \in$ conv V with $u \notin$ conv $W \cup$ boundary (conv V). We are going to find two points $w_1, w_2 \in W$ with $\sphericalangle(w_1 u w_2) \geqq \omega_d$.

Denote by z the point in conv W, nearest to u. The halfline zu (starting at z) meets boundary(conv V) in the point v. The hyperplane H through z with normal $u-z$ is a supporting hyperplane to conv W, so the two points $w_1, w_2 \in W$ with $\sphericalangle(w_1 v w_2) \geqq \omega_d$ lie on that side of H which does not contain v, i.e.,

$$\sphericalangle(w_1 z v) \geqq \frac{\pi}{2}, \quad \sphericalangle(w_2 z v) \geqq \frac{\pi}{2}.$$

Using this property it is easy to show that $\sphericalangle(w_1 u w_2) \geqq \sphericalangle(w_1 v w_2)$. We omit the details.

A simple computation shows now that

$$n(d) \leqq 2^{d(d-1)} d^{d(d+1)}.$$

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