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MAXIMAL VOLUME ENCLOSED BY PLATES AND PROOF OF THE CHESSBOARD CONJECTURE

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The following conjecture of Fejes Tóth is proved: The density of a lattice of convex bodies in $\mathbb{R}^n$ is at least $\frac{1}{2}$, provided that each connected component of the complementary set is bounded. The cases of equality are also determined.

For any polytope P in $\mathbb{R}^n$ one can define a convexification P^* (see Section 1) and it is proved that $\text{Vol}\, P \leqslant \text{Vol}\, P^*$. It is also shown that if P and P' are two convex polytopes with facets F_i and F_j^*, resp., and $\sum \{\text{Vol}_{n-1} F_i \mid F_i \perp v\} \geqslant \sum \{\text{Vol}_{n-1} F_j^* \mid F_j^* \perp v\}$ holds for every $v \in \mathbb{R}^n$, then $\text{Vol}\, P'/\text{Vol}\, P \leqslant 2^{n/(n-1)}$. Some related questions are also considered.

1. Introduction

Given n linearly independent vectors $v_1, \ldots, v_n$ in the n-dimensional Euclidean space ($n \geqslant 2$), they generate a *point lattice* consisting of all points (vectors) of the form $k_1v_1 + \cdots + k_nv_n$ with integer coefficients k_i. A full-dimensional parallelepiped E of minimal volume all of whose vertices are elements of Λ is an *elementary cell* of the lattice.

For any convex body C (=compact convex set with interior points), the system of translates of C by vectors belonging to Λ is said to form a *body lattice* and is denoted by Λ_C. The *density* of Λ_C is defined by

$$d(\Lambda_C) = \text{Vol}\, C/\text{Vol}\, E,$$

where E is an elementary cell of Λ.

We will establish the following assertion, a variant of a conjecture of L. Fejes Tóth [14], which was proved in case $n = 2$ by Fejes Tóth [13], [14] and Groemer [17]:

Theorem 1 (Chessboard Conjecture). *Let Λ_C be a body lattice in $\mathbb{R}^n$ ($n \geqslant 2$) formed by translates of a closed convex set C. Suppose that the removal of all members of Λ_C splits the space into bounded pieces, i.e., all connected components* *of the complementary set of* $C + \Lambda = \{c + \lambda \mid c \in C, \lambda \in \Lambda\}$ *are bounded. Then, we have* $d(\Lambda_C) \geq \frac{1}{2}$, *and this bound cannot be improved.*

The name of the problem is motivated by the fact that, if Λ consists of all integer points $(k_1, \ldots, k_n)$ whose sum of coordinates is even and C is the unit cube $\{(x_1, \ldots, x_n) \mid 0 \leq x_i \leq 1\}$, then Λ_C is a chessboardlike configuration (called *chessboard lattice*) meeting the requirements of the theorem and having density $\frac{1}{2}$. The minimal density of a lattice of *balls* in $\mathbb{R}^3$ with the property in Theorem 1 was determined by Bleicher [3].

In Section 2 we are going to show that Theorem 1 can be deduced from Theorem 2 which is a weaker version of a conjecture of Fáry and Makai Jr. [9] on the maximal volume enclosed by 'plates', which will be settled in Section 3.

To formulate Theorem 2 we need some preparation.

Let P be an arbitrary polytope in $\mathbb{R}^n$ whose facets (i.e., $(n-1)$-dimensional faces) are denoted by $F_1, \ldots, F_m$. Let v_i denote the outward normal vector of F_i whose length is equal to $\mathrm{Vol}_{n-1} F_i$ i.e., the surface area of F_i. We obviously have $\sum_{i=1}^m v_i = 0$ (e.g. by the Gauss–Green formula, see Section 3). Thus, we can apply a well-known theorem of Minkowski [4, 6] which states that, for any system of vectors $v_1, \ldots, v_m$ generating the whole space and satisfying $\sum_{i=1}^m v_i = 0$, there exists a uniquely determined convex polytope P^* having m facets $F_1^*, \ldots, F_m^*$ such that v_i is an outward normal vector of F_i^* and $|v_i| = \mathrm{Vol}_{n-1} F_i^*$ for all $i = 1, \ldots, m$. P^* is called the *convexification* of P.

Theorem 2. *Given a polytope* P *in* $\mathbb{R}^n$ *whose convexification is* P^*, *we have* $\mathrm{Vol}\, P \leq \mathrm{Vol}\, P^*$, *where equality holds if and only if* P *is convex.*

Some 2-dimensional versions and analogoues of this assertion were proved by Fáry–Makai Jr. [9] and Pach [21]. We remark that the proof in [21] yields in fact the following.

Proposition 1. *If* P *is a closed polygon in* $\mathbb{R}^2$, *then* $\mathrm{Vol}\,($*convex hull of* $P) \leq \mathrm{Vol}\, P^*$.

A possible interpretation of Theorem 2 is: Given an arbitrary polytope P, we can cut up its faces F_i into smaller pieces $F_{i1}, F_{i2}, \ldots$ which can be moved separately. Our goal is now to rearrange these pieces so as to maximize the enclosed volume, with the restriction that the *outward normal* vector of each piece F_{ij} should remain unchanged. In these terms, Theorem 2 states (roughly) that, in an optimal rearrangement, the pieces (plates) enclose a convex polytope. Furthermore, if P was originally convex then its volume cannot be increased by any operation satisfying the above condition.

However, this latter assertion does not remain true if we permit every movement of the pieces keeping the *normal* vectors (but not necessarily the

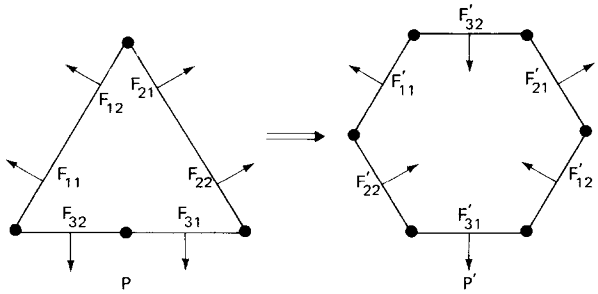

Fig. 1.

outward normals!) fixed. This can be demonstrated by the planar example shown in Fig. 1, where P is convex, but if we translate the segments F_{ij} into the new positions F'_{ij} they will enclose a larger area. This can be explained by the fact that the outward normals of F_{12}, F_{22} and F_{32} change their signs during the transformation.

In Sections 4 and 5 we are going to discuss the question how large the ratio $\mathrm{Vol}\, P'/\mathrm{Vol}\, P$ can be, if P is a convex polytope and P' is obtained from P in a way described above, such that the normal vectors of the corresponding boundary pieces are equal. One of our main results reads as follows.

Theorem 5. *There exists a (minimal) constant $f(n)$ depending only on n such that, if P and P' are two convex polytopes in $\mathbb{R}^n$ whose facets are denoted by F_i $(1 \leq i \leq m)$ and F'_j $(1 \leq j \leq m')$, resp., and they have the property that*

$$\sum_{F_i \perp v} \mathrm{Vol}_{n-1} F_i \geq \sum_{F'_j \perp v} \mathrm{Vol}_{n-1} F'_j$$

holds for every $v \in \mathbb{R}^n$, then

$$\frac{\mathrm{Vol}\, P'}{\mathrm{Vol}\, P} \leq f(n).$$

Furthermore, $f(2) = 1\frac{1}{2}$ and

$$\tfrac{1}{2}\mathrm{e} \leq \liminf_{n\to\infty} f(n) \leq \limsup_{n\to\infty} f(n) \leq 2.$$

A similar theorem for $n = 3$ was announced by Firey [16]. In Section 6 we prove

Theorem 6. *For $n \geq 3$ we have in Theorem 1 equality, i.e., $d(\Lambda_C) = \frac{1}{2}$, if and only if C is a parallelotope, and the body lattice Λ_C is affine equivalent to the chessboard lattice described above.*

Our concluding Section 7 contains some remarks and open problems. Among others we answer a related question of Fejes Tóth [14] (see 7.1), further we prove a generalization of Theorem 2 for affine $(n-1)$-cycles rather than polytopes, which settles a conjecture of [9] (see 7.4).

2. Proof of the Chessboard Conjecture (via Theorem 2)

To begin with, let us reformulate Theorem 2 in the (apparently slightly stronger, but, in fact, equivalent) form we shall use.

Theorem 2′. *Let $P = \bigcup_{r=1}^{s} P_r$ be a finite union of some n-dimensional polytopes P_r such that no two of them have an (at least) $(n-1)$-dimensional intersection, $(n \geqslant 2)$. Suppose P' is a convex polytope, and let F_i $(1 \leqslant i \leqslant m)$ and F'_j $(1 \leqslant j \leqslant m')$ denote the facets of P and P', resp. Assume that*

$$\sum \{\mathrm{Vol}_{n-1} F_i \mid v \text{ is an outward normal of } F_i\}$$

$$\leqslant \sum \{\mathrm{Vol}_{n-1} F'_j \mid v \text{ is an outward normal of } F'_j\}$$

holds for every $v \in \mathbb{R}^n$. Then, we have

$$\mathrm{Vol}\, P \leqslant \mathrm{Vol}\, P',$$

with equality if and only if $s = 1$ and $P = P_1$ is convex.

It follows by an easy continuity argument that it is sufficient to prove Theorem 1 in the special case when C is a closed convex *polytope*.

Consider now the collection of all connected components P_i $(i \in I)$ of the complementary set of $C + \Lambda$. Two components P_i and P_j are said to be *equivalent* if there exists a lattice translation (of Λ) carrying P_i onto P_j. (Note that no non-trivial lattice translation will take a component P_i into itself, otherwise P_i would be unbounded.) It is easy to see that there are only finitely many (say s) different equivalence classes. Let us pick one representative from each class, and denote it by P_r, $1 \leqslant r \leqslant s$.

Let F_i be any facet of $P = \bigcup_{r=1}^{s} P_r$ with outward normal vector v_i. Then F_i can obviously be subdivided into finitely many openly disjoint $(n-1)$-dimensional simplices S_{ik} $(1 \leqslant k \leqslant K_i)$ such that each of them is on the boundary of some member of Λ_C. That is, we can choose $\lambda_{ik} \in \Lambda$ such that

$$S_{ik} \subseteq \mathrm{Bd}(C + \lambda_{ik}), \quad 1 \leqslant k \leqslant K_i.$$

It is easy to check now that the simplices $S'_{ik} = S_{ik} - \lambda_{ik}$ are openly disjoint pieces of $\mathrm{Bd}\, C$. To see this, observe only that, if there exists a point x in the interiors of both S'_{ik} and S'_{jl}, then Λ_C looks similar in small neighbourhoods of

$x + \lambda_{ik} \in S_{ik}$ and $x + \lambda_{jl} \in S_{jl}$, showing that $i = j$ and $k = l$. Note further that the outward normal vector of C corresponding to S'_{ik} is equal to $-v_i$.

Thus, we can apply Theorem 2′ (with $P' = -C$) to obtain $\operatorname{Vol} P \leqslant \operatorname{Vol}(-C) = \operatorname{Vol} C$.

If E is an elementary cell of Λ, and E_i $(i = 0, 1, \ldots)$ denotes the set of those points of E which are covered by exactly i members of Λ_C, then we obviously have $\operatorname{Vol} E_0 = \operatorname{Vol} P$, $\sum_{i=1}^{\infty} i \operatorname{Vol} E_i = \operatorname{Vol} C$ and, hence

$$1 = \frac{\sum_{i=0}^{\infty} \operatorname{Vol} E_i}{\operatorname{Vol} E} = \frac{\operatorname{Vol} P}{\operatorname{Vol} E} + \frac{\sum_{i=1}^{\infty} \operatorname{Vol} E_i}{\operatorname{Vol} E} \leqslant \frac{2 \operatorname{Vol} C}{\operatorname{Vol} E},$$

as desired. □

As a matter of fact, we have proved somewhat more than we needed:

Proposition 2. *If Λ_C satisfies the requirements in Theorem* 1, *then $d(\Lambda_C) + d(C + \Lambda) \geqslant 1$, where $d(C + \Lambda) = \sum_{i=1}^{\infty} \operatorname{Vol} E_i / \operatorname{Vol} E$ is the density of the set $C + \Lambda$ in the space.* □

3. Proof of Theorem 2

Recall first that the *surface area function* μ_C of a convex body (=compact convex set with interior points) $C \subseteq \mathbb{R}^n$ is defined as a finite Borel measure on the sphere S^{n-1} such that, for any Borel set $A \subseteq S^{n-1}$, $\mu_C(A)$ is the $(n-1)$-dimensional Lebesque measure of the set of all $x \in \operatorname{Bd} C$ at which there exists a support hyperplane to C whose outward normal vector is in A (cf. [6]).

We will prove the following generalization of Theorem 2 (cf. also the remark at the end of this section).

Theorem 3. *Given a compact set $P = \overline{\operatorname{int} P} \subseteq \mathbb{R}^n$ which is either the union of finitely many (non-degenerate) closed n-simplices or whose boundary is a C^1 submanifold. Define a Borel measure $\mu = \mu_P$ on S^{n-1} by*

$$\mu(A) = \lambda_{n-1}(\{x \in \operatorname{Bd} P \mid \textit{there exists at } x \textit{ a tangent hyperplane to } P \textit{ whose outward normal} \in A\}),$$

where $A \subseteq S^{n-1}$ is a Borel set and λ_{n-1} denotes the Lebesque measure. Let C be the unique convex body with surface area function $\mu_C \equiv \mu$. Then $\operatorname{Vol} P \leqslant \operatorname{Vol} C$, *and equality holds if and only if $P = C$.*

Proof. Note first that there exists a convex body C with the required properties, because μ is a finite Borel measure on S^{n-1} (not concentrated on any great-S^{n-2}

of the sphere) and by the Gauss–Green formula we have

$$\int_{S^{n-1}} s \,\mathrm{d}\mu(s) = \int_{\mathrm{Bd}\,P} \boldsymbol{n}(x)\,\mathrm{d}\lambda_{n-1}(x) = 0,$$

where $\boldsymbol{n}(x)$ denotes the outward normal to $\mathrm{Bd}\,P$ at x. By the theorem of Minkowski–Alexandroff–Fenchel–Jessen [6, 15] quoted in the introduction, this is already sufficient for the existence of a C having the required properties.

Let $D \subseteq \mathbb{R}^n$ be an arbitrary convex body with support function $h_D(s)$, and let

$$f(D) = \int_{S^{n-1}} h_D(s)\,\mathrm{d}\mu(s)/n(\mathrm{Vol}\,D)^{1/n}.$$

Following Bonnesen–Fenchel [4], observe that

$$\int_{S^{n-1}} h_D(s)\,\mathrm{d}\mu(s) = \int_{S^{n-1}} h_D(s)\,\mathrm{d}\mu_C(s) = nV(D, \underbrace{C, \ldots, C}_{n-1\,\text{times}}).$$

Hence, by the Brunn–Minkowski inequality [4], we have

$$f(D) \geqslant (\mathrm{Vol}\,C)^{1-1/n} \quad (\forall D),$$

with equality if and only if D is homothetic to C. In other words, this means that C can be determined as the (up to homothety) unique convex body for which f attains its minimum. We will apply the following:

Theorem (Busemann[5]). *Given a compact set $P \subseteq \mathbb{R}^n$ with Lebesque measure $\lambda_n(P) > 0$ and a convex body D, we have*

$$\liminf_{\varepsilon\to 0} \frac{\lambda_n(P+\varepsilon D) - \lambda_n(P)}{\varepsilon} \geqslant n\lambda_n(P)^{1-1/n}(\mathrm{Vol}\,D)^{1/n}.$$

Furthermore, equality holds if and only if the set

$$P' = P \cap (\bigcap \{H \mid H \text{ is a closed halfspace}, \lambda_n(P\setminus H) = 0\})$$

is homothetic to D and

$$\liminf_{\varepsilon\to 0} \frac{\lambda_n(P'+\varepsilon D) - \lambda_n(P')}{\varepsilon} = \liminf_{\varepsilon\to 0} \frac{\lambda_n(P+\varepsilon D) - \lambda_n(P)}{\varepsilon}.$$

Thus, we obtain that

$$\begin{aligned} f(D) &= \int_{\mathrm{Bd}\,P} h_D(\boldsymbol{n}(x))\,\mathrm{d}\lambda_{n-1}(x)/n(\mathrm{Vol}\,D)^{1/n} \\ &= \lim_{\varepsilon\to 0} \frac{\lambda_n(P+\varepsilon D) - \lambda_n(P)}{\varepsilon n(\mathrm{Vol}\,D)^{1/n}} \geqslant (\mathrm{Vol}\,P)^{1-1/n} \end{aligned}$$

holds for every convex D. In particular, $f(C) = (\operatorname{Vol} C)^{1-1/n} \geq (\operatorname{Vol} P)^{1-1/n}$, as stated. If $\operatorname{Vol} C = \operatorname{Vol} P$, then $P' = P = C$, completing the proof of Theorem 3. $\square$

In the previous section, for the proof of the Chessboard Conjecture, we used a slightly generalized version of Theorem 2 (Theorem 2′). This can be obtained from Theorem 3 by the application of the following lemma, whose proof is taken from [1].

Lemma 1. *Let C and D be two convex bodies in $\mathbb{R}^n$ with surface area functions satisfying $\mu_C(A) \leq \mu_D(A)$ for every Borel set $A \subseteq S^{n-1}$. Then* $\operatorname{Vol} C \leq \operatorname{Vol} D$, *and equality holds if and only if $C = D$.*

Proof. Let h_D (>0) denote the support function of D. By the Brunn–Minkowski inequality

$$(\operatorname{Vol} D)^{1/n}(\operatorname{Vol} C)^{1-1/n} \leq V(D, \underbrace{C, \dots, C}_{n-1\,\text{times}}) = \frac{1}{n}\int_{S^{n-1}} h_D(s)\, \mathrm{d}\mu_C(s)$$

$$\leq \frac{1}{n}\int_{S^{n-1}} h_D(s)\, \mathrm{d}\mu_D(s) = \operatorname{Vol} D.$$

Hence $\operatorname{Vol} C \leq \operatorname{Vol} D$, and the only case of equality is $C = D$. $\square$

Remark. The proof of Theorem 3 remains valid in any geometric integration theory ([22], [7], [25], [10]) and for any compact set P with finite surface area, whose normal vectors exist almost everywhere, provided

(1) the function μ is a finite Borel measure on S^{n-1}, not concentrated on any great-S^{n-2}, and satisfies $\int_{S^{n-1}} s\, \mathrm{d}\mu(s) = 0$;

(2) $\int_{S^{n-1}} h_D(s)\, \mathrm{d}\mu(s) \geq \int_{\mathrm{Bd}\,P} h_D(\boldsymbol{n}(x))\, \mathrm{d}S$ for any convex body D;

(3) $\int_{\mathrm{Bd}\,P} h_D(\boldsymbol{n}(x))\, \mathrm{d}S \geq \liminf_{\varepsilon \to 0} \dfrac{\lambda_n(P + \varepsilon D) - \lambda_n(P)}{\varepsilon}$ for any convex body D. (h_D denotes the support function of D.)

4. Maximal volume enclosed by plates

Suppose that we are given a finite number of $(n-1)$-dimensional polytopes (plates) $A_1, \dots, A_m$ in the n-dimensional space and we want to translate them independently of one another so as to maximize the volume enclosed by them. That is, we have to find the maximal possible volume of the union of all bounded components of $\mathbb{R}^n \setminus \bigcup_{i=1}^m A_i'$, where A_i' is a translate of A_i. Our Theorem 4 establishes a natural upper bound for this value.

Let C and D be two convex bodies in $\mathbb{R}^n$ with surface area functions μ_C and μ_D, respectively. Recall that the *Blaschke sum* $C \# D$ of C and D is defined as the unique convex body having surface area function $\mu = \mu_C + \mu_D$ ([16]). We will make use of the following classical results (actually proved, but stated in a weaker form, in [20]).

Theorem (Kneser–Süss [20]). *Let $C, D \subseteq \mathbb{R}^n$ be convex bodies. Then*

$$(\mathrm{Vol}(C \# D))^{1-1/n} \geq (\mathrm{Vol}\, C)^{1-1/n} + (\mathrm{Vol}\, D)^{1-1/n}$$

with equality if and only if C is homothetic to D.

Corollary 1. *Let $C \subseteq \mathbb{R}^n$ be a convex body. Then*

$$\mathrm{Vol}(2^{-1/(n-1)}(C \# -C)) \geq \mathrm{Vol}\, C,$$

with equality if and only if C is centrally symmetric, i.e. $C = -C$.

Now we are in a position to prove the following theorem, which was proved for $n = 2$ by Bezdek [26].

Theorem 4. *Let $A_1, \ldots, A_m$ be finitely many subsets of $\mathbb{R}^n$ such that A_i is contained in some hyperplane H_i, and let $\lambda_i < \infty$ denote the $(n-1)$-dimensional outer Lebesgue measure of A_i $(i = 1, \ldots, m)$. Suppose further that P is a centrally symmetric convex polytope whose facets $F_1, \ldots, F_k, F'_1, \ldots, F'_k$ have the property that F_j is parallel to F'_j and*

$$\mathrm{Vol}_{n-1} F_j = \mathrm{Vol}_{n-1} F'_j = \tfrac{1}{2} \sum \{\lambda_i \mid H_i \text{ is parallel to } F_j\}$$

holds for every j $(j = 1, \ldots, k)$, each H_i with $\lambda_i > 0$ being parallel to some F_j.

Then the volume of the region enclosed by $A_1, \ldots, A_m$ is at most $\mathrm{Vol}\, P$, *and equality holds here (for non-degenerate P) if and only if $\bigcup_{i=1}^m A_i \supseteq \mathrm{Bd}\, P$.*

Note that if a (non-degenerate) polytope P meeting the requirements of the theorem exists, then it is necessarily unique. If there is no such P, then $\bigcup_{i=1}^m A_i$ cannot enclose any positive volume.

Proof. The hyperplanes H_i $(i = 1, \ldots, m)$ cut up the space into finitely many open polyhedral cells. Each of these cells is either entirely contained in, or completely disjoint from the region enclosed by $\bigcup_{i=1}^m A_i$. Let B denote the closure of the union of all cells lying in bounded components of $\mathbb{R}^n \setminus \bigcup_{i=1}^m A_i$. Let $B \neq \emptyset$. We obviously have $\mathrm{Bd}\, B \subseteq \bigcup_{i=1}^m A_i$. Moreover, if $F_1^B, F_2^B, \ldots$ denote the facets of B, then for any hyperplane H

$$\sum \{\mathrm{Vol}_{n-1} F_i^B \mid F_i^B \text{ is parallel to } H\} \leq \sum \{\lambda_i \mid H_i \text{ is parallel to } H\}.$$

By Theorem 3 there exists a unique convex polytope C with facets $F_1^C, F_2^C, \ldots$ and with surface area function $\mu_C = \mu_B$. In particular, we have

$$\sum \{\mathrm{Vol}_{n-1} F_i^C \mid F_i^C \text{ is parallel to } H\} = \sum \{\mathrm{Vol}_{n-1} F_i^B \mid F_i^B \text{ is parallel to } H\}$$

for every hyperplane H, and $\mathrm{Vol}\, C \geq \mathrm{Vol}\, B$. Observe that $D = 2^{-1(n-1)}(C \# -C)$ is a centrally symmetric convex polytope. Denoting the facets of D by $F_1^D, F_2^D, \ldots,$

$$\sum \{\mathrm{Vol}_{n-1} F_i^D \mid F_i^D \text{ is parallel to } H\} = \sum \{\mathrm{Vol}_{n-1} F_i^C \mid F_i^C \text{ is parallel to } H\}$$

holds for every hyperplane H. Further, in view of Corollary 1, $\mathrm{Vol}\, D \geq \mathrm{Vol}\, C$.

Putting the above relations together, we obtain that the surface area function μ_P of the polytope P (described in the theorem) satisfies $\mu_P \geq \mu_D$. Hence we can apply Lemma 1 of the previous section to conclude

$$\mathrm{Vol}\, P \geq \mathrm{Vol}\, D \geq \mathrm{Vol}\, C \geq \mathrm{Vol}\, B.$$

If $\mathrm{Vol}\, P = \mathrm{Vol}\, B > 0$, then $P = B$ and $\bigcup_{i=1}^m A_i \supseteq \mathrm{Bd}\, B = \mathrm{Bd}\, P$. On the other hand, $\bigcup_{i=1}^m A_i \supseteq \mathrm{Bd}\, P$ yields that $\mathrm{Vol}\, B \geq \mathrm{Vol}\, P$, i.e., equality holds. □

5. Convex regions enclosed by plates

Let P and P' be two convex polytopes in $\mathbb{R}^n$ whose facets are denoted by F_i $(1 \leq i \leq m)$ and F'_j $(1 \leq j \leq m')$, resp., and suppose that

$$\sum \{\mathrm{Vol}_{n-1} F_i \mid F_i \text{ is parallel to } H\} \geq \sum \{\mathrm{Vol}_{n-1} F'_j \mid F'_j \text{ is parallel to } H\}$$

holds for every hyperplane H. Equivalently, for the surface area functions of the Blaschke sums $P \# -P$ and $P' \# -P'$

$$\mu_{P \# -P} \geq \mu_{P' \# -P'}$$

is valid. Our Fig. 1 however demonstrates that $\mathrm{Vol}\, P'$ can still be larger than $\mathrm{Vol}\, P$.

The aim of this section is to prove Theorem 5 (see Section 1) establishing a fairly good upper bound on the ratio $\mathrm{Vol}\, P'/\mathrm{Vol}\, P$. The following assertion, which is an immediate consequence of Corollary 1 and Lemma 1, reduces this problem to the investigation of a problem with one variable body only.

Corollary 2. *Let C and D be convex bodies in $\mathbb{R}^n$ having the property*

$\mu_{C\#-C} \geq \mu_{D\#-D}$. *Then*

$$\frac{\operatorname{Vol} D}{\operatorname{Vol} C} \leq \frac{\operatorname{Vol}(2^{-1/(n-1)}(C \# -C))}{\operatorname{Vol} C},$$

with equality if and only if $D = 2^{-1/(n-1)}(C \# -C)$. □

Thus the quantity $f(n)$ defined in Theorem 5 can be determined by

$$f(n) = \sup_{C \subseteq \mathbb{R}^n} \frac{\operatorname{Vol}(2^{-1/(n-1)}(C \# -C))}{\operatorname{Vol} C},$$

where the sup is taken over all convex bodies C in $\mathbb{R}^n$. (Note that by [15], Theorem VIII, p. 21 the convergence of convex bodies—with fixed centroids—to a convex body is equivalent to the w^*-convergence of their surface area functions, in $C(S^{n-1})^*$.)

Theorem 5′. *Let* $f(n)$ *be as above. Then*

(i) $$f(n) \geq \left(\sum_{i=0}^{[n/2]} (-1)^i \binom{n+1}{i} \left(1 - \frac{2i}{n+1}\right)^n\right)^{-1/(n-1)},$$

which tends to $\frac{1}{2}\mathrm{e}$ *as* $n \to \infty$.

(ii) $$f(n) \leq \max_C \min_{E=-E} \frac{V(E, C, \ldots, C)^{n/(n-1)}}{(\operatorname{Vol} E)^{1/(n-1)} \operatorname{Vol} C}$$
$$\leq \max_C \min_{\substack{E \subseteq C \\ E=-E}} \left(\frac{\operatorname{Vol} C}{\operatorname{Vol} E}\right)^{1/(n-1)} < 2^{n/(n-1)},$$

where the max *and* min *are taken over all convex bodies* C *and over all centrally symmetric convex bodies* $E \subseteq C$, *respectively.*

(iii) $f(2) = 1\frac{1}{2}$, *and in the plane*

$$\operatorname{Vol}(2^{-1})(C \# -C))/\operatorname{Vol} C = 1\tfrac{1}{2}$$

if and only if C *is a triangle.*

Proof. (i) Let T^n denote an n-dimensional simplex of inradius 1, whose centroid is at 0. Clearly, the surface area $\operatorname{Surf} T^n = n \operatorname{Vol} T^n$. Observe that $T^n \# -T^n$ is homothetic to $T^n \cap -T^n$, i.e., $T^n \# -T^n = \lambda(T^n \cap -T^n)$, where

$$\lambda = \left(\frac{\operatorname{Surf}(T^n \# -T^n)}{\operatorname{Surf}(T^n \cap -T^n)}\right)^{1/(n-1)} = \left(\frac{2 \operatorname{Surf} T^n}{\operatorname{Surf}(T^n \cap -T^n)}\right)^{1/(n-1)}$$
$$= \left(\frac{2 \operatorname{Vol} T^n}{\operatorname{Vol}(T^n \cap -T^n)}\right)^{1/(n-1)}.$$

Hence

$$f(n) \geq \frac{\operatorname{Vol}(2^{-1/(n-1)}(T^n \# -T^n))}{\operatorname{Vol} T^n}$$
$$= \frac{\operatorname{Vol}(2^{-1/(n-1)}\lambda(T^n \cap -T^n))}{\operatorname{Vol} T^n} = \left(\frac{\operatorname{Vol} T^n}{\operatorname{Vol}(T^n \cap -T^n)}\right)^{1/(n-1)},$$

and to obtain (i) we can use the following identity of Fáry-Rédei [8]:

$$\frac{\operatorname{Vol} T^n}{\operatorname{Vol}(T^n \cap -T^n)} = \left(\sum_{i=0}^{[n/2]} (-1)^i \binom{n+1}{i}\left(1 - \frac{2i}{n+1}\right)^n\right)^{-1}$$
$$= \left(\frac{1}{\pi} n! \left(\frac{2}{n+1}\right)^n \int_0^\infty \left(\frac{\sin t}{t}\right)^{n+1} \mathrm{d}t\right)^{-1}.$$

A little calculation shows that this value is asymptotically equal to $(\frac{3}{2}\mathrm{e}^2)^{-\frac{1}{2}}(\frac{1}{2}\mathrm{e})^n$.

(ii) Let E be a centrally symmetric convex body in $\mathbb{R}^n$, whose center is at 0. That is, the support function of E satisfies $h_E(s) = h_E(-s)$ for any $s \in S^{n-1}$. Put $D = 2^{-1/(n-1)}(C \# -C)$. Then

$$\int_{S^{n-1}} h_E(s)\,\mathrm{d}\mu_D(s) = \frac{1}{2}\int_{S^{n-1}} h_E(s)\,\mathrm{d}\mu_{C\#-C}(s)$$
$$= \frac{1}{2}\int_{S^{n-1}} h_E(s)(\mathrm{d}\mu_C(s) + \mathrm{d}\mu_C(-s))$$
$$= \frac{1}{2}\int_{S^{n-1}} (h_E(s) + h_E(-s))\,\mathrm{d}\mu_C(s)$$
$$= \int_{S^{n-1}} h_E(s)\,\mathrm{d}\mu_C(s) = nV(E, C, \ldots, C).$$

On the other hand, by the Brunn–Minkowski inequality,

$$\int_{S^{n-1}} h_E(s)\,\mathrm{d}\mu_D(s) = nV(E, D, \ldots, D) \geq n(\operatorname{Vol} E)^{1/n}(\operatorname{Vol} D)^{(n-1)/n}.$$

Comparing the last two relations, we obtain

$$\frac{\operatorname{Vol} D}{\operatorname{Vol} C} \leq \frac{V(E, C, \ldots, C)^{n/(n-1)}}{(\operatorname{Vol} E)^{1/(n-1)} \operatorname{Vol} C},$$

which proves the first inequality in (ii).

To get the second one we only have to observe that if $E \subseteq C$, then by the monotonity of the mixed volumes we have $V(E, C, \ldots, C) \leq V(C, C, \ldots, C) = \operatorname{Vol} C$. The last inequality is due to Stein [24] (see also [18] p. 254 and [2] for stronger results in the 2- and 3-dimensional spaces).

(iii) We proceed similarly to Bezdek [26]. In view of the fact that in the plane the Blaschke sum and the Minkowski sum of convex bodies coincide, it is enough to refer to the following well-known result.

Theorem (Rogers–Shephard [23]). *Let C be any convex body in $\mathbb{R}^n$. Then*

$$\frac{\operatorname{Vol}(\frac{1}{2}(C+(-C)))}{\operatorname{Vol} C} \leqslant \frac{\binom{2n}{n}}{2^n}$$

with equality if and only if C is an n-dimensional simplex. $\square$

6. The case of equality in the Chessboard Conjecture (Theorem 1)

As it was proved by Fejes Tóth [13] and Groemer [17], for $n = 2$ equality holds in Theorem 1 if and only if C is a convex quadrangle and the lattice Λ is generated by its diagonals. (Note that the extreme cases when C degenerates into a triangle are included, too.) Furthermore, in the plane, under the conditions of Theorem 1 the stronger relation $d(C+\Lambda) \geqslant \frac{1}{2}$ is also valid, with equality only for the above described body lattices. This follows from the fact that one can easily construct a convex plate $C' \subseteq C$ satisfying $C' + \Lambda = C + \Lambda$ and $d(\Lambda_{C'}) = d(C' + \Lambda) = d(C + \Lambda)$.

The main objective of this section is to prove the following sharpening of Theorem 6.

Theorem 6′. *Let $n \geqslant 3$. Then, under the conditions of Theorem* 1, *we have $d(\Lambda_C) + d(C+\Lambda) \geqslant 1$, where equality holds if and only if C is a parallelotope and Λ_C is affinely equivalent to the chessboard-lattice.*

The first part of Theorem 6′ has already been proved (Proposition 2), while the second part will be established in three steps (Propositions 3, 4 and 5).

Proposition 3. *Let C be a (closed) convex body and Λ be a point-lattice in $\mathbb{R}^n$ satisfying the conditions of Theorem* 1. *If $d(\Lambda_C) + d(C+\Lambda) = 1$, then*

(i) *C is polytope*;

(ii) *any two bodies of Λ_C have an at most $(n-2)$-dimensional intersection*;

(iii) *the complement of the set $C+\Lambda$ is a translate of $-\operatorname{int} C + \Lambda$ (where* int C *denotes the interior of C).*

Proof. First we shall prove (ii). Suppose, without loss of generality, that $\dim(C \cap (C+\lambda)) \geqslant n-1$ for some $\lambda \in \Lambda$. Let C' be a convex polytope such that $\operatorname{int} C' \supset C$ and no facet (i.e., $(n-1)$-face) of C' is coplanar with any facet of $C' + \lambda$. Further, let P' and S'_{ik} be defined similarly as in the proof of Theorem 1. Then none of the simplices S'_{ik} can have a nonempty intersection with $\operatorname{Bd} C' \cap \operatorname{int}(C' \pm \lambda)$, and by [15], Theorem VIII we have

$$\limsup_{C' \to C} \operatorname{Vol} P' \leqslant \operatorname{Vol} C^*,$$

where C^* is the uniquely determined convex body with surface area function

$$\mu_{C^*} = \lim_{C' \to C} (\mu_{C'} - \tfrac{1}{2}\mu_{C' \cap (C'+\lambda)}) = \mu_C - \tfrac{1}{2}\mu_{C \cap (C+\lambda)}.$$

However, by Lemma 1, $\operatorname{Vol} C^* < \operatorname{Vol} C$. Thus, using the simple calculation at the end of the proof of Theorem 1, we get $d(\Lambda_C) + d(C+\lambda) > 1$, a contradiction.

The body C has common points with only finitely many other bodies $C + \lambda$ of the lattice, and by (ii) each of the non-empty intersection sets $C \cap (C+\lambda)$ is contained in some $(n-2)$-planes. Including these $(n-2)$-planes in $(n-1)$-planes passing through some fixed interior point of C, we obtain a subdivision of C into finitely many convex bodies C_m and a corresponding subdivision of $\operatorname{Bd} C$ into closed parts B_m $(1 \le m \le M)$. From here it easily follows that the number of non-equivalent components P_r of the complementary set of $C + \Lambda$ is also finite. Let $P = \bigcup P_r$. We are going to show next, that the proof of Theorem 3 can be applied to $\bar{P}$ (the closure of P), even if its boundary is not necessarily piecewise smooth. To see this, we have to check only that $\bar{P}$ has an appropriate 'surface area function' μ satisfying conditions (1)–(3) of the remark at the end of Section 3.

Let $\mathrm{d}S$ be the usual surface area (element) on

$$\operatorname{Bd} \bar{P} = \bigcup_r \operatorname{Bd} P_r = \bigcup_r \bigcup_\lambda (\operatorname{Bd} P_r \cap \operatorname{Bd}(C+\lambda))$$

measured on $\operatorname{Bd}(C+\lambda)$. For any Borel set $A \subseteq S^{n-1}$, for any r and λ, let $X_{r,\lambda}(A)$ denote the set of all $x \in \operatorname{Bd} P_r \cap \operatorname{Bd}(C+\lambda)$ with the property that there exists at x a supporting hyperplane to $\operatorname{Bd}(C+\lambda)$ whose outward normal vector (with respect to P_r) is in A. Further, let $\mu(A)$ be defined by

$$\mu(A) := \sum_r \sum_\lambda \lambda_{n-1}(X_{r,\lambda}(A)),$$

where λ_{n-1} is the Lebesgue measure on $\operatorname{Bd}(C+\lambda)$. Thus, μ obviously satisfies condition (2) of the remark, with equality for any convex body $D \subset \mathbb{R}^n$.

On the other hand, for all r and λ, $\operatorname{Bd} P_r \cap \operatorname{Bd}(C+\lambda)$ can be expressed as $\bigcup \{B_m + \lambda \mid m \in I_{r,\lambda}\}$, where $I_{r,\lambda}$ are suitable (pairwise disjoint) subsets of $\{1, 2, \ldots, M\}$. Hence,

$$\operatorname{Bd} \bar{P} = \bigcup_r \bigcup_\lambda \bigcup_{m \in I_{r,\lambda}} \{B_m + \lambda\},$$

and $\lim_{\varepsilon \to 0} [\lambda_n(\bar{P} + \varepsilon D) - \lambda_n(\bar{P})]/\varepsilon$ (which will exist in our case) can be calculated for each m separately. The contribution of any $B_m + \lambda$ to this limit is equal to

$$L_m = \lim_{\varepsilon \to 0} \frac{\lambda_n([C \setminus (C - \varepsilon D)] \cap C_m)}{\varepsilon}$$
$$= \lim_{\varepsilon \to 0} \frac{\lambda_n(C_m \setminus (C_m - \varepsilon D))}{\varepsilon} - \int_{(\operatorname{Bd} C_m) \setminus B_m} h_D(\boldsymbol{n}(x))\, \mathrm{d}S(x).$$

($\setminus$ denotes the set-theoretical difference and $\boldsymbol{n}(x)$ is the outward normal at x). However,

$$\lim_{\varepsilon\to 0}\frac{\lambda_n(C_m\setminus(C_m-\varepsilon D))}{\varepsilon}=\int_{\mathrm{Bd}\,C_m} h_D(\boldsymbol{n}(x))\,\mathrm{d}S(x)$$

as can be seen along the lines of the proof in [19], p. 63. Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\lambda_n(\bar{P}+\varepsilon D)-\lambda_n(\bar{P})}{\varepsilon}&=\sum_{r,\lambda}\sum_{m\in I_{r,\lambda}} L_m\\&=\sum_{r,\lambda}\sum_{m\in I_{r,\lambda}}\int_{B_m} h_D(\boldsymbol{n}(x))\,\mathrm{d}S(x)\\&=\int_{\mathrm{Bd}\,\bar{P}} h_D(\boldsymbol{n}(x))\,\mathrm{d}S(x).\end{aligned}$$

That is, condition (3) holds with equality, too. The validity of (1) can be checked similarly as in the case of convex bodies (cf. [6], Chapter II, Section 8).

Thus, we can apply the proof of Theorem 3, showing that $d(\Lambda_C)+d(C+\lambda)=1$ implies $\mathrm{Vol}\,\bar{P}=\mathrm{Vol}\,C$. (Note that $\mathrm{Vol}\,\bar{P}\leqslant\mathrm{Vol}\,C$ follows by Lemma 1, using $\mu_{\bar{P}}\leqslant\mu_{-C}$.) Furthermore, $\mathrm{Vol}\,\bar{P}=\mathrm{Vol}\,C$ implies that $\bar{P}$ is a translate of $-C$, which proves (iii). Part (i) is now an immediate consequence of the fact that Λ_C and $\Lambda_{\bar{P}}$ together form a (locally finite) tiling of $\mathbb{R}^n$ by convex bodies. □

Proposition 4. *Let C, Λ, P be the same as above, $n\geqslant 3$. Suppose that Λ_C satisfies the conditions of Theorem* 1 *and $d(\Lambda_C)+d(C+\lambda)=1$. Then*

(i) *Λ_C and $\Lambda_{\bar{P}}$ together form a face-to-face tiling of $\mathbb{R}^n$;*

(ii) *C is centrosymmetric.*

Proof. We have seen that Λ_C and $\Lambda_{\bar{P}}$ (a translate of Λ_{-C}) together form a tiling T of $\mathbb{R}^n$. Using this fact, we can easily prove the following

Claim. Let F be any facet (i.e., $(n-1)$-face) of C. Then, for every $(n-2)$-face G of F, there exists another $(n-2)$-face G' in F opposite to and parallel with G.

To see this, observe that any facet F of $C\in\Lambda_C$ is completely covered by a system $\{-F+x_i \mid i\in I\}$ of corresponding facets of some members $-C+x_i\in\Lambda_{\bar{P}}$. Suppose first that for some $i\in I$ the relative interiors of F and $-G+x_i$ have at least one point in common. Then there exist another $j\in I$ and another $(n-2)$-face G' of F such that $\mathrm{rel\,int}(-G+x_i)\cap\mathrm{rel\,int}(-G'+x_j)\neq\emptyset$. Assume next that the relative interiors of F and $-G+x_i$ are disjoint for all $i\in I$. Then there is an $(n-2)$-face G' of F $(G'\neq G)$ such that $G\subseteq\bigcup\{-G'+x_i \mid i\in I\}$. In both cases G' is parallel to G, which proves the Claim.

Let G be an arbitrary $(n-2)$-face of C. Take any facet F of C containing G, and let G' denote the (unique) $(n-2)$-face of F opposite to and parallel with G. G' is obviously contained in another facet F' of C; let G'' denote the $(n-2)$-face of F' opposite to and parallel with G', etc. The set of facets $\{F, F', F'', \ldots\}$ obtained in this way is called the *zone* determined by G. (Cf. [11, I.4.3].) Note that an orthogonal projection to a 2-plane perpendicular to G takes this zone into a convex polygon whose vertices (and edges) are the projections of $G, G', G'', \ldots$ (and $F, F', F'', \ldots$, respectively). As a matter of fact, every $(n-2)$-face of C parallel to G appears in the sequence $G, G', G'', \ldots$.

Suppose now, in order to get a contradiction, that T is not a face-to-face tiling, i.e., there is a facet F of $C \in \Lambda_C$ intersecting both $-F + x_i$ and $-F + x_j$ $(x_i \neq x_j)$, and the $(n-2)$-faces $-G + x_i$, $-G' + x_j$ have a common relative interior point $p \in F$ $(-C + x_i, -C + x_j \in \Lambda_{\bar{P}})$. It is easily seen that there exists a $C + \lambda \in \Lambda_C$ $(0 \neq \lambda \in \Lambda)$ such that $p \in C + \lambda$. Let H denote the hyperplane induced by F, and let G^* be an $(n-2)$-face of C parallel to G, whose distance from H is maximal. Finally, let H^* denote the other supporting hyperplane to C parallel to H. Evidently, $H^* \supseteq G^*$. Observe now that the orthogonal projections of C, $C + \lambda$, $-C + x_i$, $-C + x_j$ on a 2-plane perpendicular to G are openly disjoint convex polygons, hence they are triangles. This yields $\dim(C \cap H^*) = \dim((C + \lambda) \cap H) \leqslant n - 2$. On the other hand, for any other $(n-2)$-face $\tilde{G}$ of F not parallel to G, we can similarly find an $(n-2)$-face $\tilde{G}^*$ of C parallel to $\tilde{G}$ such that $\tilde{G}^* \subseteq H^*$. Thus, $H^* \supseteq G^* \cup \tilde{G}^*$, i.e., $\dim(C \cap H^*) \geqslant n - 1$. This contradiction proves (i).

Since T is a face-to-face tiling, every facet F of C is covered by a facet $-F + x_i$ of some member of $\Lambda_{\bar{P}}$. Consequently, every facet of C is centrosymmetric, and if $n \geqslant 3$ then this implies that C is centrosymmetric itself. (Cf. [11].) □

Proposition 5. *Let C and Λ be as above, $n \geqslant 3$. If Λ_C satisfies the conditions of Theorem* 1 *and $d(\Lambda_C) + d(C + \Lambda) = 1$, then C is a parallelotope and Λ_C is affine equivalent to the chessboard-lattice.*

Proof. Let G be any $(n-2)$-face of $C \in \Lambda_C$, and let q be any point in the relative interior of G. Let $C_1 = C$, $C_2, C_3, \ldots, C_m$ denote those members of the tiling $T = \Lambda_C \cup \Lambda_{\bar{P}}$ which are incident to q (m is even). By Proposition 4(ii), each C_i is a translate of C, hence the orthogonal projections $\pi(C_i)$ of C_i $(1 \leqslant i \leqslant m)$ on a 2-plane perpendicular to G are m non-overlapping translates of a convex polygon, which have a boundary point in common. It is easy to prove (see e.g. [12], Remark 1) that this implies that $m = 4$ and $\pi(C_i)$ is a parallelogram. Consequently, the zone determined by any $(n-2)$-face G has exactly 4 elements.

Let F and F' be any pair of parallel (opposite) facets of C. Applying the last remark to all zones determined by $(n-2)$-faces of F, we obtain that C is a prism over F.

Proposition 5 is now an immediate consequence of (Proposition 4 and) the following lemma whose simple inductional proof is left to the reader.

Lemma 2. *Let $C \subseteq \mathbb{R}^n$ be a convex polytope with the property that C is a prism (pyramid, resp.) over each of its facets. Then C is a parallelotope (simplex, resp.).* □

7. Concluding remarks

(7.1) Fejes Tóth [14] originally raised the following problem: determine $d_n = \inf d(\Lambda_C)$ over all body lattices Λ_C in $\mathbb{R}^n$ having the property that *some* connected component of the complement of $C+\Lambda$ is bounded. He proved that $d_2 = \frac{1}{2}$. However, the following assertion is true.

Proposition 6. *$d_n = 0$ for all $n \geqslant 3$.*

Proof. We consider the case $n = 3$ only. Let $0 < \varepsilon \leqslant \frac{1}{3}$ be fixed, and let C be defined as the convex hull of the following eight points:

$$(1+\varepsilon, 1-\varepsilon, 1),\quad (1+\varepsilon, -1+\varepsilon, 1),\quad (-1-\varepsilon, -1+\varepsilon, 1),\quad (-1-\varepsilon, 1-\varepsilon, 1),$$
$$(1-\varepsilon, 1+\varepsilon, -1),\quad (1-\varepsilon, -1-\varepsilon, -1),\quad (-1+\varepsilon, -1-\varepsilon, -1),\quad (-1+\varepsilon, 1+\varepsilon, -1).$$

Further, let Λ' be the 2-dimensional lattice of all points of the form $(k_1(1+\varepsilon), k_2(1+\varepsilon), 0)$, where $k_1, k_2 \in \mathbb{Z}$ and $k_1 + k_2$ is even. It is easily seen that e.g. the tetrahedron S spanned by the points $(\pm 2\varepsilon, 1+\varepsilon, -1)$, $(0, 1+\varepsilon \pm 2\varepsilon, 1)$ is a connected component of $\mathbb{R}^3 \backslash (C + \Lambda')$. Including Λ' in the 3-dimensional lattice $\Lambda^r = \Lambda' + \{(0, 0, k_3 r) \mid k_3 \in \mathbb{Z}\}$, $r \geqslant 2$, S remains an intact region enclosed by Λ^r_C. Evidently, $\lim_{r\to\infty} d(\Lambda^r_C) = 0$, which yields the result. The case $n > 3$ can be treated similarly. □

(7.2) Let $d_n^* = \inf d(\Lambda_C)$, where the infimum is taken over all centrosymmetric convex bodies $C \subseteq \mathbb{R}^n$ and over all lattices Λ with the property that some connected component of $\mathbb{R}^n \backslash (C + \Lambda)$ is bounded. Clearly, $d_2^* = d_2 = \frac{1}{2}$.

Proposition 7. *$d_3^* = \frac{1}{2}$ and $d_n^* = 0$ for $n \geqslant 4$.*

Proof. First we give an example showing $d_4^* = 0$. The construction for $n > 4$ is similar.

As in Proposition 6 it suffices to construct a 3-dimensional lattice Λ' and a 4-dimensional centrosymmetric convex body C such that $\mathbb{R}^4 \backslash (C + \Lambda)$ has a bounded component. Let Λ' be the lattice generated by e_1, e_2 and e_3. Let C_0 denote the cube $\{x \in \mathbb{R}^4 \mid 0 \leqslant x_i \leqslant 1,\ i = 1, 2, 3, 4\}$. Consider the following five vertices of C_0: $a_0 = 0$, $a_1 = e_1$, $a_2 = e_2$, $a_3 = e_3$, $a_4 = e_1 + e_2$ and their centrosymmetric images with respect to the centre of C_0. Denote by C_1 the convex hull of these ten points. Consider the vectors $u_0 = -e_1 - e_2 - e_3 + c\varepsilon e_4$, $u_1 = e_1 - \varepsilon e_4$, $u_2 = e_2 - \varepsilon e_4$, $u_3 = e_3 - \varepsilon e_4$, $u_4 = e_1 + e_2 - \varepsilon e_4$ where $\varepsilon > 0$ is sufficiently small and $c > 0$. For a suitable choice of c the convex hull of these five vectors contains 0 in its interior. Now we replace each u_i by a nearby v_i so that the convex hull of $v_0, \ldots, v_4$ contains 0 in its interior and v_i is an outward normal to C_1 at a_i,

$i = 0, \ldots, 4$, moreover, the support hyperplane H_i to C_1 with outward normal v_i intersects C_1 in a single point a_i. Denote by A_i the halfspace containing C_1 with boundary hyperplane H_i. Let A_i' be the mirror image of A_i with respect to the centre of C_1. Blow up C_1 from its centre by a factor $\alpha > 1$ near to 1, the body obtained this way is $C_1(\alpha)$. Consider now $C_2 = C_1(\alpha) \cap \bigcap_{i=0}^{4} (A_i \cap A_i')$. This is a centrosymmetric convex body. Observe that there are only five bodies, $-a_i + C_2$ $(i = 0, \ldots, 4)$ from the body lattice Λ_{C_2}' that contain 0, in addition, $-a_i + C_2$ contains 0 on its facet with outward normal v_i. So if we replace C_2 by its $(1-\delta)$ times smaller homothetic image C centred at the same point and $\delta > 0$ is small enough, then the body lattice Λ_C' has the desired properties.

The proof of $d_3^* = \frac{1}{2}$ is long and very technical and is therefore omitted. Here we are going to show that d_3^* is positive, more precisely, $d_3^* \geq \frac{1}{6}$. The proof of this is based on the fact that if C is centrosymmetric and $\mathbb{R}^3 \setminus (C + \Lambda)$ has a bounded component, then $C + \Lambda$ is connected. And it is easy to see that if a body lattice in $\mathbb{R}^n$ is connected, then its density is at least $1/n!$ ([13, 17]).

Assume now that C is a centrosymmetric convex body, $\mathbb{R}^3 \setminus (C + \Lambda)$ has a connected component and $C + \Lambda$ is not connected. Then there is a 2-dimensional sublattice Λ' of Λ such that $\mathbb{R}^3 \setminus (C + \Lambda')$ has a connected component.

By an approximation argument we may assume that C is smooth and strictly convex ([4]). Let D denote a bounded connected component of $\mathbb{R}^3 \setminus (C + \Lambda')$. Denote by $r > 1$ the largest real number such that the sets $r \cdot \operatorname{int} C + \lambda$ $(\lambda \in \Lambda')$ still do not cover D, say, $p \in D$ and $p \notin r \cdot \operatorname{int} C + \lambda$ $(\lambda \in \Lambda')$. Then $p \in r \cdot \operatorname{Bd} C + \lambda_i$ for $i \in I$, say, and the outward normals to $r \cdot C$ at $p - \lambda_i$ $(i \in I)$ do not lie in a closed halfspace. Consequently $|I| \geq 4$.

If some $\lambda \in \Lambda$ were a nontrivial convex combination of some of the λ_i's $(i \in I)$, then the same would hold for $p - \lambda$ and the corresponding $p - \lambda_i$'s. So by the strict convexity of C, $p \in r \cdot \operatorname{int} C + \lambda$, a contradiction. Hence any three of the λ_i's $(i \in I)$ form a lattice-point free triangle. Thus $|I| \geq 4$ implies that $|I| = 4$, moreover, $\{\lambda_i \mid i \in I\}$ is the set of vertices of a basic parallelogram of Λ'. Then $p - \lambda_i \in r \cdot \operatorname{Bd} C$ and by symmetry (with respect to 0, say) $\lambda_i - p \in r \cdot \operatorname{Bd} C$. Thus $r \cdot C$ admits an inscribed parallelepiped with vertices $p - \lambda_i$, $\lambda_i - p$ $(i \in I)$. This implies that the normal vectors to $r \cdot C$ at the vertices $p - \lambda_i$ lie in a closed halfspace or else 0 lies in the plane spanned by Λ'. The first of these cases immediately yields a contradiction and so does the second unless 0 is the centre of the parallelogram $\{p - \lambda_i \mid i \in I\}$. In this latter case, however, by the smoothness of C the normal vectors of $r \cdot C$ at $p - \lambda_i$ lie in a plane, a contradiction again.

Hence $C + \Lambda$ is connected and so its density is at least $\frac{1}{6}$.

We mention further that our proof of $d_3^* \geq \frac{1}{2}$ which is not given here implies also that $d_3^* = \frac{1}{2}$ if and only if Λ_C is the 3-dimensional chessboard lattice. □

(7.3) Conjecture. *Under the conditions of Theorem* 1, $d(C + \Lambda) \geq \frac{1}{2}$ *holds, with equality (for* $n \geq 3$*) if and only if* C *is a parallelotope and* Λ_C *is affinely equivalent to the chessboard-lattice.*

(7.4) In [9], Lemma 1, an analogue of Theorem 2 was proved for oriented closed (possibly self-intersecting) polygons. Now we are going to generalize this result to higher dimensional spaces.

Let $h = \sum_{i=1}^{m} \gamma_i S_i$ be an affine $(n-1)$-cycle in $\mathbb{R}^n$, where the S_i's are oriented $(n-1)$-simplices with unit outward normals u_i, and all coefficients γ_i are non-zero integers. Suppose further that the outward normals of the (non-degenerate) simplices S_i induce $\mathbb{R}^n$.

The *signed volume* of h is defined as

$$\sigma(h) := \int_{\mathbb{R}^n} w(h, x)\,\mathrm{d}x,$$

where $w(h, x)$ denotes the winding number of h with respect to x. The *convexification* of h is the unique convex polytope C with outward normals u_i such that, for any $u \in \{u_i \mid 1 \leq i \leq m\}$, there is a facet F of C whose outward normal is u and whose surface area is equal to

$$\sum \{\gamma_i \operatorname{Vol}_{n-1} S_i \mid \gamma_i > 0,\ u_i = u\} - \sum \{\gamma_i \operatorname{Vol}_{n-1} S_i \mid \gamma_i < 0,\ u_i = -u\}.$$

Theorem 2″. *Let $h = \sum \gamma_i S_i$ be an affine $(n-1)$-cycle in $\mathbb{R}^n$ with non-zero integer coefficients γ_i, and assume that the normal vectors of the (non-degenerate) simplices S_i occurring in h generate the whole space. Let C denote the convexification of h. Then, we have*

$$\sigma(h) \leq \operatorname{Vol} C,$$

with equality if and only if $h = \partial C$ (i.e., h gives a 'well-oriented' triangulation of Bd C).

Proof. Let supp h denote the *support* of h, i.e., the union of all simplices S_i occurring in h. We may clearly suppose without loss of generality that

$$w(h) := \max\{w(h, x) \mid x \in \mathbb{R}^n \setminus \operatorname{supp} h\}$$

is positive. Let us define a sequence of affine $(n-1)$-cycles h_i $(i = 0, 1, \ldots, w(h))$, as follows. Put $h_0 = h$. If h_i has already been defined, then let $h_{i+1} = h_i - \partial \bar{D}_i$, where

$$D_i := \{x \in \mathbb{R}^n \setminus \operatorname{supp} h_i \mid w(h_i, x) = w(h_i)\}.$$

For $k = w(h)$, we obviously have $w(h_k) = 0$, hence $\sigma(h_k) \leq 0$. Denoting the convexification of $\bar{D}_i$ by E_i, Theorem 2 implies that $\operatorname{Vol} \bar{D}_i \leq \operatorname{Vol} E_i$, thus

$$\sigma(h) = \sum_{i=0}^{k-1} \operatorname{Vol} \bar{D}_i + \sigma(h_k) \leq \sum_{i=0}^{k-1} \operatorname{Vol} \bar{D}_i \leq \sum_{i=0}^{k-1} \operatorname{Vol} E_i.$$

Let $E = E_0 \# E_1 \# \cdots \# E_{k-1}$. Then, by the Kneser–Süss theorem (cited in Section 4), we have $\sum \operatorname{Vol} E_i \leq \operatorname{Vol} E$. However, E is a convex body with surface area function $\mu_E \leq \mu_C$, hence $\operatorname{Vol} E \leq \operatorname{Vol} C$, according to Lemma 1. Comparing these inequalities we obtain $\sigma(h) \leqslant \operatorname{Vol} C$, and if equality holds here then $C = E$, $k = 1$, $\bar{D}_0 = E_0$, thus $h = \partial C$. $\square$

One can readily generalize Theorems 2″ and 4 for 'nicely self-intersecting' C^1-cycles (instead of affine cycles) and for 'nicely intersecting' C^1-hypersurfaces (instead of hyperplanes), respectively. However, we suspect that these assumptions cannot be essentially weakened (presumably some geometric integration theory should be used).

(7.5) Another (2-dimensional) generalization of Theorem 2 is Proposition 1 in Section 1. It might be worth noting that, besides the trivial cases (i.e., when P is linear of convex), equality holds in Proposition 1 for self-intersecting trapezoids, too. However, a careful analysis of the inductional proof in [21] shows that there are no more cases of equality.

According to [9], Section 3, one can easily extend the concept of convexification and Proposition 1 to any closed rectifiable curve of the plane.

Conjecture. The only non-convex closed rectifiable curves for which in Proposition 1 equality holds are the self-intersecting trapezoids.

It is obvious that Proposition 1 does not remain true in higher dimensional spaces, even if restricted to polytopes homeomorphic to S^{n-1} ($n \geqslant 3$). For $n \geqslant 3$ we are unable to answer the following related question.

Problem. Let P be a (possibly self-intersecting) polytope in $\mathbb{R}^n$, whose convexification is denoted by P^*. Is it then true that the total volume of the regions enclosed by the facets of P is at most $\operatorname{Vol} P^*$? (Note that this value cannot exceed $f(n)\operatorname{Vol} P^*$, as shown by Theorem 5.)

(7.6) Conjecture. *Equality holds in Theorem* 5′(*i*). *Moreover, the only convex body* C, *for which* $\operatorname{Vol}(2^{-1/(n-1)}(C \# -C))/\operatorname{Vol} C = f(n)$, *is the n-dimensional simplex.*

Note that this would follow immediately from the validity of a conjecture of [8], stating that

$$\max_{C} \min_{\substack{E=-E \\ E\subseteq C}} \frac{\operatorname{Vol} C}{\operatorname{Vol} E}$$

can be attained only if C is a simplex. (See Theorem 5′(ii).) Similarly,

$$k_i(C) := \min_{E=-E} \frac{V(\overbrace{E, \ldots, E}^{i \text{ times}}, \overbrace{C, \ldots, C}^{n-i \text{ times}})}{(\operatorname{Vol} E)^{i/n} (\operatorname{Vol} C)^{(n-i)/n}}$$

can be considered as a measure of symmetry of a convex body $C \subset \mathbb{R}^n$, and it seem plausible that $k_i(C)$ is maximal for simplices only ($i = 1, 2, \ldots, n-1$).

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