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ON A COMMON GENERALIZATION OF BORSUK'S AND RADON'S THEOREM

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1. The well-known theorem of RADON [3] says that if $A \subset R^d$ and $|A| \geqq d+2$, then there exist $B, C \subset A$, $B \cap C = \emptyset$ such that $\operatorname{conv} B \cap \operatorname{conv} C$ is not empty. It is clear that for each finite set $A = \{a_1, \ldots, a_n\}$ in R^d with $n \geqq d+2$ one can find a linear map $f: R^{d+1} \to R^d$ and a set $A' = \{a_1', \ldots, a_n'\} \subset R^{d+1}$ such that $f(a_i') = a_i$ $i = 1, 2, \ldots, n$ and $\operatorname{int} \operatorname{conv} A'$ is not empty and $\operatorname{vert} \operatorname{conv} A' = A'$. In view of this fact, Radon's theorem can be stated in the following way.

RADON'S THEOREM. *Let $P \subset R^{d+1}$ be a convex polytope with non-empty interior. Put $A = \operatorname{vert} P$. If $f: R^{d+1} \to R^d$ is a linear map, then there exist two disjoint sets $B, C \subset A$ such that $f(\operatorname{conv} B) \cap f(\operatorname{conv} C)$ is non-empty.*

The surprising fact here is that the word "linear" can be replaced by "continuous", namely, a continuous analogue of Radon's theorem is true;

THEOREM 1. *Let $P \subset R^{d+1}$ be a convex polytope with non-empty interior. Given an $f: \partial P \to R^d$ continuous map, there exist two disjoint faces, B and C, of P such that $f(B) \cap f(C) \neq \emptyset$.*

COROLLARY. *Let T be a $(d+1)$-dimensional simplex. Denote its d-faces by $L_1, L_2, \ldots, L_{d+2}$. If $f: \partial T \to R^d$ is a continuous map, then $\bigcap_{i=1}^{d+2} f(L_i)$ is non-empty.*

If f is a linear map, then this statement is an easy consequence (in fact, equivalent) of Helly's theorem (see [3]). The interesting fact here is that in this particular case a continuous version of Helly's theorem holds true.

Let us now introduce some notions. Given a convex compact set $C \subset R^{d+1}$ with non-empty interior and a vector $a \in R^{d+1}$, $a \neq 0$, we write

$$C(a) = \left\{x \in C: \langle a, x \rangle = \max_{t \in C} \langle a, t \rangle\right\}.$$

Two points, x and y, of C are said to be opposite if for some $a \in R^{d+1}$, $x \in C(a)$ and $y \in C(-a)$. If C happens to be a polytope, then $C(a)$ is a proper face of C. In this case we say that the two faces $C(a)$ and $C(-a)$ are opposite.

THEOREM 2. *Given a polytope $P \subset R^{d+1}$ with non-empty interior and a continuous map $f: \partial P \to R^d$, there exist two opposite faces, B and C, of P such that $f(B) \cap f(C)$ is non-empty.*

It is evident that opposite faces of P are disjoint. Thus Theorem 2 implies Theorem 1.

Speaking about points instead of faces Theorem 2 can be formulated as follows.

THEOREM 2′. *Given a polytope* $P\subset R^{d+1}$ *with non-empty interior and a continuous map* $f\colon \partial P\to R^d$, *there exist two opposite points, x and y, of P with* $f(x)=f(y)$.

We shall prove this Theorem 2′ which yields a generalization of Borsuk's theorem [1]. In order to state Borsuk's theorem put $S^d=\{x\in R^{d+1}\colon \|x\|=1\}$.

BORSUK'S THEOREM. *If* $f\colon S^d\to R^d$ *is a continuous map, then there is a point* $x\in S^d$ *with* $f(x)=f(-x)$.

THEOREM 3. *Let* $C\subset R^{d+1}$ *be a convex compact set with nonempty interior. If* $f\colon \partial C\to R^d$ *is a continuous map, then there exist two opposite points, x and y, of C with* $f(x)=f(y)$.

Again, Theorem 3 implies Theorem 2′. However, we shall get Theorem 3 from Theorem 2′ by a simple continuity argument.

Further, our Theorem 3 contains Borsuk's theorem (put simply $C=\operatorname{conv} S^d$). On the other hand, Theorem 2′ is proved using Borsuk's theorem.

2. We need a simple proposition.

PROPOSITION. *If P is a polytope in* R^d *and* $x, y, x_n\in P$ $n=1, 2, \ldots$ *and* $\lim x_n=x$, *then there is an* $\varepsilon>0$ *and N such that* $x_n+\varepsilon\cdot(y-x)\in P$ *for* $n>N$.

PROOF. This proposition is true for any cone C (instead of P) whose vertex is x (with arbitrary $\varepsilon>0$ and n), so it is true for $C\cap B(x,\delta)$ where $B(x,\delta)$ is the ball with center x and radius δ. But $P\cap B(x,\delta)=C\cap B(x,\delta)$ for a sufficiently small $\delta>0$ where

$$C=\{z\in R^d\colon\ z=x+\lambda(w-x),\ \lambda>0,\ w\in P\}$$

is a cone with vertex x.

PROOF OF THEOREM 2′. Put $Q=P-P$. Q is a polytope with non-empty interior. It is centrally symmetric with respect to the origin. For $x\in Q$ write

$$h(x)=\max\{z\colon\ x=z-w,\ z, w\in P\}$$

where max is meant in the lexicographic ordering of R^{d+1}. Clearly $h\colon Q\to P$ is well-defined. An easy computation shows that the vector w corresponding to $z=h(x)$ equals $h(-x)$.

We claim that h is continuous. Indeed, let $x, x_n\in Q$, $x=\lim x_n$ and $x_n=z_n-w_n$ where $z_n=h(x_n)$. We can choose a subsequence n_i so that z_{n_i} and, consequently w_{n_i} converge. Put $z=\lim z_{n_i}$ and $w=\lim w_{n_i}$; clearly $x=z-w$. We claim that $z=h(x)$. If not, then $z<h(x)$ in the lexicographic ordering. By the Proposition, for a sufficiently small positive ε and large i

$$z'=z_{n_i}+\varepsilon\big(h(x)-z\big)\in P \quad\text{and}\quad w'=w_{n_i}+\varepsilon\big(h(-x)-w\big)\in P.$$

Now $z'-w'=x_{n_i}$ and $z'>z_{n_i}$ contradicting $z_{n_i}=h(x_{n_i})$. This means that $z=h(x)$. Thus, every convergent subsequence of z_n tends to $h(x)$. Now by compactness $\lim z_n=h(x)$, i.e., h is continuous.

Next we claim that $x \in Q(a)$ implies $h(x) \in P(a)$ and $h(-x) \in P(-a)$. Indeed, if $x \in Q(a)$ then $\max_{t \in Q} \langle a, t \rangle = \langle a, x \rangle$. Of course, $x = h(x) - h(-x)$ and $h(x), h(-x) \in P$. Whence

$$\langle a, h(x) \rangle + \langle -a, h(-x) \rangle = \langle a, x \rangle = \max_{t \in Q} \langle a, t \rangle =$$

$$= \max_{u, v \in P} \langle a, u - v \rangle = \max_{u \in P} \langle a, u \rangle + \max_{v \in P} \langle -a, v \rangle$$

and so $h(x) \in P(a)$ and $h(-x) \in P(-a)$. This further implies that for $x \in \partial Q$ $h(x)$ and $h(-x)$ belong to ∂P.

Now we define a map $g: \partial Q \to R^d$ in the following way: for $x \in \partial Q$ let $g(x) = f(h(x))$. This map is welldefined and continuous. Let us observe now that the conditions of Borsuk's theorem are fulfilled for the map g (instead of S^d we have ∂Q here but this is indifferent). In this case Borsuk's theorem says that there is a point $x \in \partial Q$ with $g(x) = g(-x)$. Now there exists $a \in R^{d+1}$, $a \neq 0$ such that $x \in Q(a)$. Then $z = h(x) \in P(a)$ and $w = h(-x) \in P(-a)$, i.e., z and w are opposite points of P and $f(z) = f(h(x)) = g(x) = g(-x) = f(h(-x)) = f(w)$. And this is what we wanted to prove.

PROOF OF THE COROLLARY. It is easy to check that if B and C are disjoint faces of the simplex T, then for any $i = 1, 2, \dots, d+2$ either $B \subset L_i$ or $C \subset L_i$ (or both). This fact proves the Corollary.

PROOF OF THEOREM 3. Without loss of generality we may suppose that $0 \in \operatorname{int} C$.

Now let P be a polytope inscribed in C, i.e., vert $P \subset \partial C$ and suppose further that $0 \in \operatorname{int} P$. Then a continuous map $f_P: \partial P \to R^d$ can be defined as $f_P(x) = f(\lambda x)$, where λ is the unique positive number with $\lambda x \in \partial C$. By Theorem 2', there are opposite points of P, z_P and w_P with $f_P(z_P) = f_P(w_P)$.

Now choose a sequence of inscribed polytopes $P_1, P_2, \dots$ with $0 \in \operatorname{int} P_n$. Suppose further that vert $P_n \subset$ vert P_{n+1} and $\partial C \cap \bigcup_1^\infty P_n$ is dense in ∂C. Again, for each n there exist opposite (for P_n) points z_n and w_n with $f_n(z_n) = f_n(w_n)$ where $f_n = f_{P_n}$. Since z_n and w_n are opposite points in P_n there is a vector $a_n \in S^d$ such that $z_n \in P_n(a_n)$ and $w_n \in P_n(-a_n)$.

By the compactness of C and S^d we may suppose that z_n, w_n and a_n converge, their limits are $z, w \in \partial C$ and $a \in S^d$ respectively. It is easy to see that z and w are opposite points of C (with normal a) and $f(z) = f(w)$.

3. REMARKS. 1. Theorem 1 can be interpreted in the following way. Let $P \subset R^{d+1}$ be a convex polytope with non-empty interior. Then it is not possible to make a drawing of ∂P in R^d so that disjoint faces of P be disjoint in the drawing.

2. We can give a second proof of Theorem 2 which is more involved than the above one but does not make use of Borsuk's theorem. It relies on a suitably modified version of the main lemma of [2].

3. The following generalization of Theorem 3 holds true.

THEOREM 4. *Let $C \subset R^{d+1}$ be a convex compact set with non-empty interior. Let f be a point to set map from ∂C to the family of all compact convex subsets of a compact set of R^d. If f is upper semi-continuous (i.e., $x_n \to x$, $y_n \in f(x_n)$, and $y_n \to y$*

implies $y \in f(x)$), *then there exist two opposite points,* z *and* w, *of* C *with* $f(z) \cap f(w) \neq \emptyset$.

This theorem follows from Theorem 3 nearly the same way as Kakutani's fixed-point theorem follows from Brouwer's one.

4. We conclude with a conjecture. There is a generalization of Radon's theorem which is due to H. TVERBERG [5]. In the spirit of our formulation of Radon's theorem this generalization runs as follows:

THEOREM. *Let* $P \subset R^n$ *be a convex polytope with non-empty interior. Here* $n=(r-1)(d+1)$. *Given an* $f\colon R^n \to R^d$ *linear map there are disjoint proper faces* $B_1, B_2, \ldots, B_r$ *of* P, *such that* $\bigcap_{i=1}^{r} f(B_i)$ *is non-empty.*

We think (but can neither prove nor disprove) that in this theorem it is enough to assume that $f\colon \partial P \to R^d$ is a continuous map.

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