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A VECTOR-SUM THEOREM IN TWO-DIMENSIONAL SPACE

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Abstract

Given a finite set X of vectors from the unit ball of the max norm in the two-dimensional space whose sum is zero, it is always possible to write $X = \{x_1, \ldots, x_n\}$ in such a way that the first coordinates of each partial sum $\sum_1^k x_i$ lie in $[-1, 1]$ and the second coordinates lie in $[-C, C]$ where C is a universal constant.

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R^d denotes, as usual, the d-dimensional Euclidean space with a fixed basis, x^j is the j-th coordinate of $x \in R^d$. $|X|$ denotes the cardinality of X. For the sake of brevity we write $s(X) = \Sigma\{x: x \in X\}$ if $X \subseteq R^d$, $|X| < \infty$. $C_1, C_2, \ldots$ will denote independent constants.

A well-known theorem of Steinitz [4] asserts that if $X \subset B \subset R^d$, B is convex and compact, $|X| = n$ and $s(X) = 0$, then there exists an indexing $X = \{x_1, x_2, \ldots, x_n\}$ such that

$$x_1 + x_2 + \ldots + x_k \in \lambda B = \{\lambda b \in R^d: b \in B\}$$

for each $k = 1, \ldots, n$, where $\lambda > 0$ depends only on d and B.

This result was significantly improved in [3]: it is shown there that $\lambda = d$ will do for any convex, compact B. Using this theorem we are going to give a coordinate-dependent bound for the partial sums in two-dimensional space. More precisely, we will prove the following

THEOREM. *Let*

$$X \subset \{x \in R^2: |x^1| \leq 1, |x^2| \leq 1\}, |X| = n, s(X) = 0.$$

Then there is an indexing $X = \{x_1, x_2, \ldots, x_n\}$ such that, for each k,

$$|x_1^1 + \ldots + x_k^1| \leq 1 \text{ and } |x_1^2 + \ldots + x_k^2| \leq C$$

where C is a universal constant.

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It is clear that the bound for the first coordinates of the partial sums cannot be improved so in this sense our theorem is best possible.

We mention further that our theorem will be proved with an effective $O(n^2)$ algorithm, so it may turn out to be useful in applications of the Steinitz lemma in scheduling theory (cf. [1], [2]).

We split the proof into four steps. We start with a definition. A set

$$Y = \{y_1, y_2, \ldots, y_m\} \subset R^d$$

is called a *bounded grouping* of $X \subset R^d$ if there is a partition $\bigcup_{i=1}^{m} X_i$ of X ($X_i \cap X_j = \emptyset$ for $i \neq j$) with $|X_i| \leq C_1$ and $y_i = s(X_i)$.

LEMMA 1. *If $Z \subset [-1, 1]$, $|Z| < \infty$, then it has a bounded grouping Y whose diameter is not larger than* 1.

PROOF. Adding zeros to Z if necessary, we may assume that $Z = Z^+ \cup \cup Z^-$, $Z^+ \cap Z^- = \emptyset$, $Z^- \subset [-1, 0]$, $Z^+ \subset [0, 1]$ and $|Z^+| = |Z^-| = n$. Let us order $Z^- = \{z_1, z_2, \ldots, z_n\}$ with $z_1 \leq z_2 \leq \ldots \leq z_n$ and $Z^+ = \{u_1, u_2, \ldots, u_n\}$ with $u_1 \geq u_2 \geq \ldots \geq u_n$. Consider the partition

$$\{z_1, u_1\} \cup \{z_2, u_2\} \cup \ldots \cup \{z_n, u_n\}$$

of Z. This gives a bounded grouping $Y = \{y_1, \ldots, y_n\}$ with $y_i = z_i + u_i$. The diameter of Y does not exceed 1 because

$$|y_i - y_j| = |z_i - z_j + u_i - u_j| \leq 1. \qquad \square$$

LEMMA 2. *If $Z \subset [-2\delta, 1 - 2\delta]$ with $\delta \in [0, 1/4]$ and $|Z| < \infty$, then Z has a bounded grouping $Y \subset [-2\varepsilon, 1 - 3\varepsilon]$ with some $\varepsilon \in [0, 1/4]$.*

PROOF. Define

$$Z^- = \{z \in Z: -2\delta \leq z \leq -\delta\} \text{ and } Z^+ = \{z \in Z: 1 - 3\delta \leq z \leq 1 - 2\delta\}.$$

Observe that $z \in Z^-$ and $u \in Z^+$ imply $-\delta \leq z + u \leq 1 - 3\delta$. Form disjoint pairs $\{z, u\}$ with $z \in Z^-$ and $u \in Z^+$ until possible — this will be the partition for Y plus the non-paired elements of Z taken as singletons. Then Y is a bounded grouping and if $|Z^-| \leq |Z^+|$, then $Y \subset [-\delta, 1 - 2\delta]$ so we may take $\varepsilon = \delta/2$ and if $|Z^-| \geq |Z^+|$, then $Y \subset [-2\delta, 1 - 3\delta]$ so $\varepsilon = \delta$ will do. $\square$

LEMMA 3. *$Z \subset [-1, 1]$, $|Z| = n$, $|y| \leq 1$ and $|s(Z) + y| \leq 1$. Then there is an indexing $Z = \{z_1, z_2, \ldots, z_n\}$ such that $|y + z_1 + z_2 + \ldots + z_k| \leq 1$, ($k = 1, 2, \ldots, n$).*

PROOF. For $n = 1$ this is evident. To use induction for $n > 1$ it is sufficient to find $z_0 \in Z$ with $|z_0 + y| \leq 1$. Assume $y \geq 0$ (the other case is symmetric). Then one can take any non-positive element of Z for z_0, and if each element of Z is positive, then one can take any of them for z_0 because $z_0 + y \leq s(Z) + y \leq 1$. □

LEMMA 4. *Let*

$$X = \{x \in R^2: |x^1| \leq 1, |x^2| \leq 1\}, \ |X| = n$$

and $Y = \{y_1, \ldots, y_m\}$ *be a bounded grouping of* X *with* $|y_1^1 + \ldots + y_k^1| \leq 1$ *and* $|y_1^2 + \ldots + y_k^2| \leq C_1$ $(k = 1, \ldots, m)$. *Then there is an indexing of* $X = \{x_1, x_2, \ldots, x_n\}$ *such that* $|x_1^1 + \ldots + x_k^1| \leq 1$ *and* $|x_1^2 + \ldots + x_k^2| \leq C_2$ $(k = 1, \ldots, n)$.

PROOF. X will be ordered in the same way as Y, so we have to give the ordering inside the groups X_i where $y_i = s(X_i)$. To do so we apply Lemma 3 for the first coordinates. This shows that the first coordinates of each partial sum lie in $[-1, 1]$. Boundedness of the second coordinates of the partial sums follows from the boundedness of the grouping. □

PROOF of the Theorem. First, apply Lemma 1 for the first coordinates of X. For symmetry reasons, we may suppose that the bounded grouping we get lies in $[-2\delta, 1 - 2\delta]$ for some $\delta \in [0, 1/4]$. Applying Lemma 2, we get a bounded grouping

$$Y \subset B = \{x \in R^2: -2\varepsilon \leq x^1 \leq 1 - 3\varepsilon, \ |x^2| \leq C_1\}$$

with some $\varepsilon \in [0, 1/4]$. Using the result from [3] mentioned in the introduction, we find an indexing $Y = \{y_1, y_2, \ldots, y_m\}$ with $y_1 + y_2 + \ldots + y_k \in 2B$ $(k = 1, 2, \ldots, m)$.

Now we are going to find an indexing for Y such that the first coordinates of each partial sum lie in $[-1, 1]$. If this is not so for the indexing $y_1, y_2, \ldots, y_m$, then $\varepsilon < 1/6$ and $y_1^1 + y_2^1 + \ldots + y_p^1 > 1$ for some p. Then $s(Y) = 0$ and $-2\varepsilon < y_i^1$ imply the existence of $q > p$ with $1 - 6\varepsilon \leq y_1^1 + \ldots y_q^1 \leq 1 - 4\varepsilon$. Set now $z_i = y_{i+q}$ ($i + q$ is meant modulo m). Then for the indexing $Y = \{z_1, z_2, \ldots, z_m\}$ we have

$$|z_1^1 + \ldots + z_k^1| \leq 1 \text{ and } |z_1^2 + \ldots + z_k^2| \leq C_2$$

$(k = 1, \ldots, m)$. To finish the proof one has to apply Lemma 4. □

Carrying out the calculations one gets that the Theorem holds with $C = 18$. We think this bound is not sharp. We mention finally that the same method works in R^d $(d \geq 2)$ and gives the following result. If

$$X \subset \{x \in R^d: |x^1| \leq 1, \ldots, |x^d| \leq 1\}, |X| = n \text{ and } s(X) = 0,$$

then there is an indexing $X = \{x_1, \ldots, x_n\}$ such that

$$|x_1^1 + \ldots + x_k^1| \leq \frac{d}{2} \text{ and } |x_1^i + \ldots x_k^i| \leq Cd$$

$(i = 2, \ldots, d;\ k = 1, \ldots, n)$.

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