$a_{n+1} + b_{n+1} + c_{n+1}$ for all n or

(5) $$d_n = a_n + b_n + c_n \quad \text{for } n \geqslant 1.$$

Since " $\sim$ " preserves opposition, the elements opposite the equal d_{n+1} and δ must be the same, and we have $b_{n+1} = \beta = c_n - b_n$. Finally, for $n \geqslant 1$ it follows from (5) that $\gamma = d_n - c_n = a_n + b_n > b_n - a_n = \alpha$, and therefore it must be that the smaller of α, γ is a_{n+1} and the larger c_{n+1}: $a_{n+1} = b_n - a_n$ and $c_{n+1} = a_n + b_n$. Thus

$$(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = (b_n - a_n, c_n - b_n, a_n + b_n, d_n - a_n),$$

and we have $h_{n+1} = Th_n$ for $n \geqslant 1$ where $h_n = (a_n, b_n, c_n) \in \mathbb{C}^3$ and

$$T = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

By (4) $d_1 > d_2 > \cdots > d_n > c_n > b_n > a_n$ and therefore

(6) $$\|T^n h_1\|^2 = a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 < 3d_1^2$$

for all n.

Computing the characteristic polynomial of T, we obtain $p(x) = x^3 + 2x^2 - 2$. We saw that p has exactly one real root, λ_0, and $0 < \lambda_0 < 1$. Denote by λ_1, λ_2 the complex roots of p; then $\lambda_2 = \bar{\lambda}_1$ and, since the constant member of p is -2, thus $2 = \lambda_0\lambda_1\lambda_2 = \lambda_0|\lambda_1|^2$, from which it follows that $|\lambda_1| = |\lambda_2| > 1$, for $0 < \lambda_0 < 1$.

Let w_0, w_1, w_2 be the eigenvectors of T in $\mathbb{C}^3$ corresponding to the eigenvalues $\lambda_0, \lambda_1, \lambda_2$, respectively; we can choose $w_0 = (1, 1 + \lambda_0, (1 + \lambda_0)^2)$. Since the λ_i are different, the w_i are independent, and therefore there exist $\alpha_i \in \mathbb{C}$ such that

$$h_1 = \sum_{i=0}^{2} \alpha_i w_i.$$

Since w_i are independent, each of them has positive distance from the subspace generated by the other two. Therefore $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ would imply $\|T^n h_1\| \to \infty$ (recall that $|\lambda_1| = |\lambda_2| > 1$). But by (6) $\|T^n h_1\|$ is bounded. Consequently $\alpha_1 = \alpha_2 = 0$ and $h_1 = \alpha_0 w_0$. Since both h_1 and w_0 have only positive elements, we have $\alpha_0 \in \mathbb{R}$ and $\alpha_0 > 0$.

Finally, from $d_1 = a_1 + b_1 + c_1$, we obtain $d_1 = \alpha_0(1 + \lambda_0)^3$ (using $\lambda_0^3 + 2\lambda_0^2 - 2 = 0$). Thus we have $Pv_0 \sim APv_0 = \alpha_0 z$ and the proof is complete.

HELLY'S THEOREM WITH VOLUMES

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This paper establishes a Helly type result for convex sets in Euclidean d-dimensional space $\mathbb{R}^d$. For many other Helly type results consult [**1**] or [**2**].

Helly's well-known theorem [**3**] states that if $\mathscr{C}$ is a finite family of convex sets in $\mathbb{R}^d$ with the property that any $d + 1$ of them have a nonempty intersection, then $\cap\mathscr{C} \neq \varnothing$. What happens if we assume somewhat more, namely that the intersection of any $d + 1$ sets is of sufficiently large volume? Is it then true that the volume of $\cap\mathscr{C}$ is at least 1? The following example answers this question in the negative. Let $\mathscr{C}$ consist of the $2d$ supporting halfspaces of the d-dimensional cube with sides $1/2$. Then $\mathrm{Vol}(\cap \mathscr{C}) = 2^{-d}$, though the intersection of any $2d - 1$ ($\geq d + 1$) members is of infinite volume. This construction shows that the best result to be expected is the following.

THEOREM. *There exists a constant $V(d) > 0$ such that if $\mathscr{C}$ is a finite family of convex sets in $\mathbb{R}^d$, with the property that if $\mathscr{C}'$ is a $2d$-membered subfamily of $\mathscr{C}$ then $\mathrm{Vol}(\cap \mathscr{C}') \geq V(d)$ is valid, then $\mathrm{Vol}(\cap \mathscr{C}) \geq 1$.*

Proof. We recall Steinitz' lemma ([**4**] or [**1**, Theorem 2.3]) which states that if V is a set of points in $\mathbb{R}^d$ whose convex hull $\mathrm{conv}(V)$ contains a ball around 0, then there is a subset $V' \subseteq V$, $|V'| \leq 2d$, with the same property. This has the immediate consequence that if P is a convex polytope in $\mathbb{R}^d$, then one can choose at most $2d$ of its faces so that the intersection of the corresponding supporting halfspaces is bounded. (Use the lemma for the normal vectors of the faces.)

First we prove the theorem in the case when all members of $\mathscr{C}$ are halfspaces. Suppose, indirectly, that there is an infinite sequence $\mathscr{H}_1, \mathscr{H}_2, \ldots$ of finite families of halfspaces satisfying

(i) $\mathrm{Vol}(\cap\mathscr{H}_1) = \mathrm{Vol}(\cap\mathscr{H}_2) = \cdots = 1$,
(ii) the intersection of any $2d$ members of $\mathscr{H}_n$ has volume at least n ($n = 1, 2, \ldots$).

For each n take a simplex S_n contained in $\cap\mathscr{H}_n$ such that S_n has the largest possible volume. Suppose, without loss of generality, that each S_n is regular with center at the origin. (Otherwise, apply to $\mathscr{H}_n$ a suitable volume-preserving affine transformation. Clearly, this will not affect the validity of (i) and (ii).) Let $c_0, c_1, \ldots, c_d$ denote the vertices of S_n. Define another simplex S_n' by

$$S_n' = -dS_n = \mathrm{conv}\{-dc_0, -dc_1, \ldots, -dc_d\},$$

and denote the faces of S_n' by $F_0, F_1, \ldots, F_d$, where $c_i \in F_i$ for all i ($i = 0, 1, \ldots, d$). Write G_i for the open halfspace whose boundary is the hyperplane supporting F_i and for which $0 \notin G_i$. Thus, S_n' is exactly the complement of $\cup_{i=0}^{d} G_i$.

We claim that $\cap\mathscr{H}_n \subseteq S_n'$. Suppose, to the contrary, that there exists a point $c \in \cap\mathscr{H}_n$ which is outside of S_n', that is, $c \in G_i$ for some i. Now the simplex spanned by the vertices $c_0, c_1, \ldots, c_{i-1}, c, c_{i+1}, \ldots, c_d$ is also contained in $\cap\mathscr{H}_n$ and has obviously larger volume than S_n, contradicting the definition. Hence,

$$S_n \subseteq \cap\mathscr{H}_n \subseteq S_n', \tag{1}$$

which implies that

$$\mathrm{Vol}\, S_n' = d^d \mathrm{Vol}\, S_n \leq d^d \mathrm{Vol}(\cap\mathscr{H}_n) = d^d. \tag{2}$$

For each $i = 0, 1, \ldots, d$, there exists a d-membered subfamily of $\mathscr{H}_n$ whose intersection is disjoint from G_i. (This follows from Helly's theorem applied to the family $\mathscr{H}_n \cup \{G_i\}$.) Take the union of these subfamilies for all i, to obtain a system $\mathscr{H}_n' \subseteq \mathscr{H}_n$ consisting of at most $d(d+1)$ halfspaces. Put $P_n = \cap\mathscr{H}_n'$. We clearly have

$$S_n \subseteq P_n \subseteq S_n', \tag{1'}$$

where P_n is a convex polytope with $f(n) \leq d(d+1)$ faces. P_n can be written in the following form

$$P_n = \left\{ x \in \mathbb{R}^d | \langle a_{ni}, x\rangle \leq 1; \quad i = 1, 2, \ldots, f(n) \right\},$$

where $a_{n1}, a_{n2}, \ldots, a_{nf(n)}$ are the so-called *outer normals* to the faces of P_n. Observe that the set of vectors $\cup_{n=1}^{\infty} \{a_{n1}, \ldots, a_{nf(n)}\}$ is necessarily bounded, because each P_n contains a regular simplex S_n with center in the origin, satisfying

$$\mathrm{Vol}\, S_n = d^{-d} \mathrm{Vol}\, S_n' \geq d^{-d} \mathrm{Vol}(\cap\mathscr{H}_n) = d^{-d}, \tag{2'}$$

that is, each P_n contains also a small fixed ball B around 0. This means that $\|a_{ni}\|$ cannot exceed the reciprocal of the radius of B. Since $f(n)$ is also bounded, one can choose a subsequence

$n_j \to \infty$, such that $f(n_1) = f(n_2) = \cdots = f$ and

$$\lim_{j\to\infty} a_{n_j i} = a_i$$

exists for each $i = 1, 2, \ldots, f$. Put

$$P = \{x \in \mathbb{R}^d | \langle a_i, x\rangle \leqslant 1; \quad i = 1, 2, \ldots, f\}.$$

Take a large ball B' which contains all S_n'. (This is possible, by (2).) It is now clear that

$$B \subseteq P \subseteq B',$$

which shows that P is a nonsingular convex polytope.

We are going to show that the intersection of any $2d$ or fewer halfspaces supporting faces of P is unbounded, contradicting the consequence of Steinitz' lemma which we mentioned above. Assume indirectly that, say,

$$Q = \bigcap_{i=1}^{k} \{x \in \mathbb{R}^d | \langle a_i, x\rangle \leqslant 1\} \qquad (k \leqslant 2d)$$

is bounded, i.e., $\max_{q\in Q}\|q\| < R$. Consider now the sets

$$Q_{n_j} = \bigcap_{i=1}^{k} \{x \in \mathbb{R}^d | \langle a_{n_j i}, x\rangle \leqslant 1\}, \quad j = 1, 2, \ldots.$$

By condition (ii) we know that $\operatorname{Vol} Q_{n_j} \geqslant n_j$, hence we can choose points $q_{n_j} \in Q_{n_j}$ satisfying $\|q_{n_j}\| = R$, for each sufficiently large j. Let t be the limit point of any convergent subsequence of $\{q_{n_j}\}$. Since $\|t\| = R$, we have $t \notin Q$, in other words,

$$\langle a_i, t\rangle > 1 \quad \text{for some } i \quad (1 \leqslant i \leqslant k).$$

On the other hand

$$\langle a_{n_j i}, q_{n_j}\rangle \leqslant 1 \qquad (j = 1, 2, \ldots)$$

obviously holds, and from here, taking the limit, we obtain

$$\langle a_i, t\rangle \leqslant 1.$$

This contradiction proves the theorem for systems of halfspaces.

The rest of the proof is entirely routine. Let $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary family of convex sets satisfying the requirements of the Theorem. By using Helly's theorem for the interior sets $\operatorname{int} C_i$, we obtain $\operatorname{Vol}(\cap\mathscr{C}) > 0$. Further, we may suppose that $\cap\mathscr{C}$ is bounded, otherwise $\operatorname{Vol}(\cap\mathscr{C}) = \infty$ and our theorem holds. For every C_i consider the system $\mathscr{H}_i$ of all closed halfspaces containing C_i. It is obvious that the intersection of any $2d$ halfspaces belonging to $\mathscr{H} = \cup_{i=1}^m \mathscr{H}_i$ has volume at least $V(d)$, and $\cap\mathscr{H} = \cap\mathscr{C}$.

Let $\varepsilon > 0$ and let U_ε be defined as the set of those points of $\mathbb{R}^d$ whose distance from $\cap\mathscr{C}$ is at most ε. The boundary of U_ε is compact and is covered by the complements of the closed halfspaces belonging to $\mathscr{H}$. It follows from the Borel covering theorem that there exists a finite covering of the boundary. This, in turn, implies that there is a finite subfamily $\mathscr{H}'$ of $\mathscr{H}$ such that $\cap\mathscr{H}' \subseteq U_\varepsilon$. If we apply our theorem to $\mathscr{H}'$, we get $\operatorname{Vol} U_\varepsilon \geqslant \operatorname{Vol}(\cap\mathscr{H}') \geqslant 1$. Taking into account that

$$\lim_{\varepsilon\to 0}[\operatorname{Vol} U_\varepsilon - \operatorname{Vol}(\cap\mathscr{C})] = 0,$$

this implies $\operatorname{Vol}(\cap\mathscr{C}) \geqslant 1$. This completes the proof.

REMARKS. One can easily see that the condition of finiteness of $\mathscr{C}$ is inessential in the Theorem, and can be replaced for instance by the condition that $\mathscr{C}$ consists of convex closed sets at least one of which is bounded (compact). Concerning the exact value of $V(d)$ we know only $V(d) \leqslant d^{2d^2}$, but we conjecture $V(d) \leqslant d^{cd}$ with a suitable constant c. Similar results are valid for the diameter and the surface area of $\cap \mathscr{C}$.

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THE TEACHING OF MATHEMATICS

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AN ALTERNATE METHOD FOR FINDING THE PARTIAL FRACTION DECOMPOSITION OF A RATIONAL FUNCTION

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The so-called partial fraction decomposition method is fundamental for the integration of rational functions. Furthermore, as Chrystal points out, this decomposition is "a fruitful source of complicated algebraical identities." In this note we describe an efficient method for determining the partial fraction decomposition of a rational function. Although this method can be found in several classical algebra texts such as Chrystal [**1**] and van der Waerden [**3**], it seems to have disappeared from current calculus texts.

The usual methods found in modern calculus textbooks essentially require the solution of a linear algebraic system. As such, they involve a good deal of algebra and take roughly n^3 steps to determine the constants in the decomposition, where n is the degree of the denominator of the rational function. As will be shown, the alternative method we recommend requires roughly only n^2 steps to find the necessary constants. Even more important to the student, this method, being recursive, is algebraically simpler and is to a large extent self-checking.

We begin with an example in order to contrast our method with those found in current calculus texts. Consider the problem of determining the decomposition of $r(x) = p(x)/q(x)$, where $p(x) = 4x^4 + 5x^3 - 25x^2 - 5x + 12$ and $q(x) = (x+2)^2(x-1)^3$. As usual, we assume the existence of constants A_i $(i = 1,2)$ and B_j $(j = 1,2,3)$ such that

$$r(x) = \sum_i \frac{A_i}{(x+2)^i} + \sum_j \frac{B_j}{(x-1)^j}.$$