Acta Math. Hungar., **144** (**1**) (2014), 119–131
DOI: 10.1007/s10474-014-0418-0
First published online June 26, 2014

VOLUMES OF CONVEX LATTICE POLYTOPES AND A QUESTION OF V. I. ARNOLD

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(Received October 28, 2013; revised April 9, 2014; accepted May 5, 2014)

Abstract. We show by a direct construction that there are at least $\exp\{cV^{(d-1)/(d+1)}\}$ convex lattice polytopes in $\mathbb{R}^d$ of volume V that are different in the sense that none of them can be carried to an other one by a lattice preserving affine transformation. This is achieved by considering the family $\mathcal{P}^d(r)$ (to be defined in the text) of convex lattice polytopes whose volumes are between 0 and $r^d/d!$. Namely we prove that for $P \in \mathcal{P}^d(r)$, $d! \operatorname{vol} P$ takes all possible integer values between cr^{d-1} and r^d where $c > 0$ is a constant depending only on d.

1. Introduction and main result

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^d$, $d \geqq 2$ and for $r \in \mathbb{N}$ define

$$A(r) = \operatorname{conv}\{re_1, \dots, re_d\} \quad \text{and} \quad S(r) = \operatorname{conv}\left\{ A(r) \cup \{0\} \right\}.$$

We will write $A^d(r)$ and $S^d(r)$ in case of need. We further define a family $\mathcal{P}(r) = \mathcal{P}^d(r)$ as the collection of all convex lattice polytopes P with $A(r) \subset P \subset S(r)$. As is well known, $v(P) = d! \operatorname{vol} P$ is an integer for $P \in \mathcal{P}(r)$, and $0 \leqq v(P) \leqq r^d$ since $A(r)$ and $S(r)$ belong to the family $\mathcal{P}(r)$. It is also clear that if $v(P) \neq 0$, then it is at least r^{d-1} since $P \in \mathcal{P}(r)$ with $v(P) > 0$ contains d-dimensional simplex whose base is $A(r)$.

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Key words and phrases: lattice, polytope, integer convex hull, statistics of convex lattice polytopes.

Mathematics Subject Classification: primary 52B20, secondary 11H06.

The question addressed in this paper concerns the set of values of $\{v(P): P \in \mathcal{P}^d(r)\}$. It has emerged in connection with a problem of V. I. Arnold [1] (see also [2]) as we will explain soon. Our main result is

THEOREM 1.1. *Given $d \geqq 2$ there is a number $c = c_d > 0$ such that for all integers $v \in [cr^{d-1}, r^d]$ there is a polytope $P \in \mathcal{P}^d(r)$ with $v(P) = v$.*

We will prove this result in a more precise form in Section 3. The case $d = 2$ is simple: it is easy to see that $\{v(P): P \in \mathcal{P}^2(r)\} = \{0, r, r+1, \ldots, r^2\}$. We will show in Section 6 that there are other gaps besides $(0, r^{d-1})$ in the set $\{v(P): P \in \mathcal{P}^d(r)\}$ when $d \geqq 3$.

2. Arnold's question

Two convex lattice polytopes are *equivalent* if one can be carried to the other by a lattice preserving affine transformation. This is an equivalence relation and equivalent polytopes have the same volume. Let $N_d(V)$ denote the number of equivalence classes of convex lattice polytopes in $\mathbb{R}^d$ of volume V. (Of course, $d!V$ is a positive integer.) Arnold [1] showed that

$$V^{1/3} \ll \log N_2(V) \ll V^{1/3} \log V.$$

After earlier results by Konyagin and Sevastyanov [6], the upper bound was improved and extended to higher dimensions to

$$\log N_d(V) \ll V^{(d-1)/(d+1)}$$

by Bárány and Pach [4] (for $d = 2$) and by Bárány and Vershik [5] (for $d \geqq 2$). The lower bound $\log N_d(V) \gg V^{(d-1)/(d+1)}$ for all $d \geqq 2$ has recently been proved in [2]. More information about Arnold's question can be found in Arnold [1], Bárány [2], Zong [8] and Liu, Zong [7].

We obtain the same lower bound as a direct and fairly simple application of Theorem 1.1:

COROLLARY 2.1. $V^{(d-1)/(d+1)} \ll \log N_d(V)$.

Some remarks are in place here about notation and terminology. A convex polytope $P \subset \mathbb{R}^d$ is a lattice polytope if its vertex set, $\operatorname{vert} P$ is a subset of $\mathbb{Z}^d$, the integer lattice. The number of vertices of a polytope P is denoted by $f_0(P)$. Throughout the paper we use, together with the usual "little oh" and "big Oh" notation, the convenient $\ll$ symbol, which means, for functions $f, g: \mathbb{R}_+ \to \mathbb{R}$, that $f(V) \ll g(V)$ if there are constants $V_0 > 0$ and $c > 0$ such that $f(V) \leqq cg(V)$ for all $V > V_0$. These constants, to be denoted by $c, c_1, \ldots, b, b_1, \ldots$, may only depend on dimension. The Euclidean norm of the vector $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ is $\|x\| = \sqrt{x_1^2 + \dots + x_d^2}$. B^d denotes the Euclidean unit ball of $\mathbb{R}^d$, and $\operatorname{vol} B_d = \omega_d$. We write $[n] = \{1, 2, \dots, n\}$.

The paper is organized as follows. The main result is proved in the next section. The integer convex hull and some of its properties are given in Section 4. The proof of Corollary 2.1 is the content of Section 5. Then we describe further gaps in $\{v(P) : P \in \mathcal{P}^d(r)\}$ when $d \geqq 3$. We finish with concluding remarks.

3. Proof of Theorem 1.1

As we have mentioned, $\{v(P) : P \in \mathcal{P}^2(r)\} = \{0, r, r+1, \dots, r^2\}$. We assume from now on that $d \geqq 3$. Here comes the more precise version of Theorem 1.1.

THEOREM 3.1. *Assume $d \geqq 3$ and $r > r_0 = 2^d d!$. Then for every non-negative integer $m \leqq (r - 2^d d!)^d$ there is a $P \in \mathcal{P}^d(r)$ with $v(P) = r^d - m$.*

This result implies Theorem 1.1 for $r > r_0$ with $c = 2^d d!$ for instance. For $r \leqq r_0$ the theorem holds by choosing the constant c large enough.

It will be more convenient to work with

$$m(P) = d! \operatorname{vol}\left(S^d(r) \setminus P\right)$$

which we call the *missed volume* of $P \in \mathcal{P}^d(r)$. (It would be more appropriate to call it missed volume times $d!$ though.) With this notation Theorem 3.1 says that the set $\{m(P) : P \in \mathcal{P}^d(r)\}$, that is, the set of missed volumes, contains all integers between 0 and $(r - 2^d d!)^d$ provided $r > r_0$.

The proof is based on the following

LEMMA 3.1. *Assume $d \geqq 2$ and let $g(x) = (2x)^d$ and $m \in \mathbb{N}$. Then there are integers $x_0 \geqq x_1 \geqq \ldots \geqq x_{d-1} \geqq 0$ and an integer $m_d \in \{0, 1, \dots, 2^d d!\}$ such that*

$$m = g(x_0) + g'(x_1) + g''(x_2) + \ldots + g^{(d-1)}(x_{d-1}) + m_d.$$

PROOF. We give an algorithm that outputs the numbers $x_0, \dots, x_{d-1}$ and $m_0 = m, m_1, \dots, m_d$.

Start with $m_0 = m$ and let x_0 be the unique non-negative integer with $g(x_0) \leqq m_0 < g(x_0 + 1)$. If m_{i-1} and x_{i-1} have been defined then set $m_i = m_{i-1} - g^{(i-1)}(x_{i-1})$ and let x_i be the unique non-negative integer with $g^{(i)}(x_i) \leqq m_i < g^{(i)}(x_i + 1)$. We stop with m_{d-1} and x_{d-1} and define $m_d = m_{d-1} - g^{(d-1)}(x_{d-1})$.

We claim that $x_i \leqq x_{i-1}$. Note first that by construction and by the intermediate value theorem

$$m_i = m_{i-1} - g^{(i-1)}(x_{i-1}) < g^{(i-1)}(x_{i-1}+1) - g^{(i-1)}(x_{i-1})$$

$$= g^{(i)}(\xi) \leqq g^{(i)}(x_{i-1}+1),$$

where $\xi \in [x_{i-1}, x_{i-1}+1]$, and we also used that $g^{(i)}(x)$ is increasing for $x \geqq 0$. So if, contrary to the claim, we had $x_i > x_{i-1}$, then $x_{i-1}+1 \leqq x_i$. As $g^{(i)}(x)$ is increasing we have

$$m_i < g^{(i)}(x_{i-1}+1) \leqq g^{(i)}(x_i) \leqq m_i,$$

a contradiction.

The same method gives that $m_d \leqq 2^d d!$:

$$m_d = m_{d-1} - g^{(d-1)}(x_{d-1}) < g^{(d-1)}(x_{d-1}+1) - g^{(d-1)}(x_{d-1})$$

$$= g^{(d)}(\xi) = 2^d d!,$$

for all x. The proof is finished by adding the defining equalities $m_i = m_{i-1} - g^{(i-1)}(x_{i-1})$ for $i = 1, 2, \dots, d$. $\square$

Remark. The same method works for every polynomial g of degree d such that $g^{(i)}(x) > 0$ for all $i = 0, 1, \dots, d$ and $x > 0$.

We return now to the **proof** of Theorem 3.1. So given $r > r_0$ and $m \in \{0, 1, \dots, (r - 2^d d!)^d\}$ we are going to construct $P \in \mathcal{P}(r)$ with $m(P) = m$. This is easy if $m \leqq r$: the simplex Δ with vertices $me_1, e_2, \dots, e_d$ has $v(\Delta) = m$ so the missed volume of the closure of $S(r) \setminus \Delta$, which is a lattice polytope, is m. So assume $m > r$.

Apply Lemma 3.1 with m which is at most $(r - 2^d d!)^d$, to get numbers $x_0, \dots, x_{d-1}$ and m_d. Note that $2x_0 \leqq r - 2^d d!$ and also $x_0 \geqq 2$ as $m > r \geqq r_0 = 2^d d!$. Next we are going to define simplices $\Delta_0, \dots, \Delta_d$ that are lattice polytopes contained in $S^d(r)$, are pairwise internally disjoint, and $v(\Delta_i) = g^{(i)}(x_i)$ for $i = 0, 1, \dots, d-1$ and $v(\Delta_d) = m_d$.

We set $e_i^* = 2x_0 e_i$ and define $\Delta_0 = \operatorname{conv}\{e_1^*, \dots, e_d^*\}$, a non-degenerate simplex because $x_0 \geqq 2$. Clearly, $v(\Delta_0) = 2^d x_0^d = g(x_0)$ and $\Delta_0 \subset S(r)$.

Next, for $i \in [d]$, we let Δ_i be the convex hull of vectors

$$e_i^*, e_i^* + 2^d d(d-1)\dots(d-i+1)e_i \text{ and}$$

$$e_i^* + (e_j - e_i) \ (j < i) \text{ and } e_i^* + x_i(e_j - e_i) \ (j > i),$$

this is a simplex whose vertices are lattice points in $S(r)$, as one can check easily. Its edges starting from vertex e_i^* are the vectors $e_j - e_i$ for $j < i$, $2^d d(d-1)\dots(d-i+1)e_i$, and $x_i(e_j - e_i)$ for $j > i$. As the vectors $e_1 - e_i$, $\dots, e_{i-1} - e_i, e_i, e_{i+1} - e_i, \dots, e_d - e_i$ form a basis of the lattice $\mathbb{Z}^d$,

$$v(\Delta_i) = 2^d d(d-1)\dots(d-i+1)x_i^{(d-i)} = g^{(i)}(x_i).$$

Note that Δ_i may be degenerate (exactly when $x_i = 0$) but only for $i > 0$.

These simplices are internally disjoint. Indeed, Δ_0 lies on one side of the hyperplane aff $\left\{ e_1^*, \dots, e_d^* \right\}$ and all other simplices are on the other side. Further, as $2x_0 \leqq r - 2^d d!$ and $x_i \leqq x_0$, every Δ_i with $i > 0$ is contained in the simplex whose vertices are e_i^*, $e_i^* + 2^d d! e_i$ and $\frac{1}{2}\left(e_i^* + e_j^*\right)$ for $j \neq i$, and these larger simplices are internally disjoint.

We check next that the closure of $S(r) \setminus \bigcup_0^d \Delta_i$ is a lattice polytope P in $\mathcal{P}(r)$. Write X_0 for the set of vertices of Δ_0 except 0, and X_i for the set of vertices of Δ_i except e_i^*. Let Y be the set of vertices of $A(r)$. P is obtained from $S(r)$ by deleting $\Delta_0, \Delta_1, \dots, \Delta_d$ in this order. Deleting Δ_0 results in $P_0 = \operatorname{conv}(Y \cup X_0) \in \mathcal{P}(r)$. Deleting Δ_1 from P_0 gives $P_1 = \operatorname{conv}(Y \cup X_0 \cup X_1) \in \mathcal{P}(r)$. Similarly, deleting Δ_i from P_{i-1} results in a convex lattice polytope $P_i \in \mathcal{P}(r)$ whose vertices are $\left(Y \cup \bigcup_0^i X_i\right)$. This works even if Δ_i is degenerate; then $P_i = P_{i-1}$.

Finally we have $P = P_d \in \mathcal{P}(r)$ and the missed volume of P is

$$m(P) = \sum_0^d v(\Delta_i) = \sum_0^{d-1} g^{(i)}(x_i - 1) + m_d = m.$$

This finishes the construction and gives a polytope $P \in \mathcal{P}(r)$ with $m(P) = m$ if $r \geqq r_0$. $\square$

Remark. The same construction with minor modification works for all $m \leqq \left(r - 2^d d\right)^d$ and $r \geqq r_0 = 2^d d$.

4. The integer convex hull

Suppose $K \subset \mathbb{R}^d$ is a bounded convex set. Its *integer convex hull, $I(K)$*, is defined as

$$I(K) = \operatorname{conv}(K \cap \mathbb{Z}^d),$$

which is a convex lattice polytope if nonempty.

To avoid some trivial complications we assume that r is large enough, $r \geqq r_d$ say. One important ingredient of our construction is

$$Q_r = I(rB^d) = \operatorname{conv}\left(\mathbb{Z}^d \cap rB^d\right).$$

Trivially $\operatorname{vol} Q_r \leqq \omega_d r^d$. It is proved in Bárány and Larman in [3] that $\operatorname{vol}(rB_d \setminus Q_r) \ll r^{d\frac{d-1}{d+1}}$. The last exponent will appear so often that we write $D = d\frac{d-1}{d+1}$. The number of vertices of Q_r is estimated in [3] as

$$r^D \ll f_0(Q_r) \ll r^D.$$

The vertices of Q_r are very close to the boundary of rB^d. More precisely, we have the following estimate which is also used in [3].

Lemma 4.1. *If x is a vertex of Q_r, then* $r - \|x\| \ll r^{-(d-1)/(d+1)}$.

Proof. The set $rB^d \cap \big(2x - rB^d\big)$ is convex and centrally symmetric with center $x \in \mathbb{Z}^d$. It does not contain any lattice point $z \neq x$: if it does, then both z and $2x - z$ are lattice points in rB^d and $x = \frac{1}{2}\big(z + (2x - z)\big)$ is not a vertex of Q_r. By Minkowksi's classical theorem, $\operatorname{vol}\big(rB^d \cap \big(2x - rB^d\big)\big) < 2^d$. The estimate in the lemma follows from this by a simple computation. □

We let $\mathbb{R}^d_+$ denote the set of $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ with $x_i \geqq 0$ for every $i \in [d]$. In the proof of Corollary 2.1 we will consider $Q^r = Q_r \cap \mathbb{R}^d_+$. It is clear that $Q^r = I\big(rB^d \cap \mathbb{R}^d_+\big)$ and further, that

$$\operatorname{vol}\big(\big(rB^d \cap \mathbb{R}^d_+\big) \setminus Q^r\big) \ll r^D.$$

Let X be the set of those vertices $x = (x_1, \dots, x_d)$ of Q^r for which $x_i > 0$ for all $i \in [d]$. We claim that

$$r^D \ll |X| \ll r^D.$$

Only the lower bound needs some explanation. The number of vertices of Q_r with $x_i = 0$ for some $i \in [d]$ is less than d times the number of vertices of Q^{d-1}_r which is of order $r^{(d-1)(d-2)/d} = o\big(r^D\big)$. Then $|X| \geqq 2^{-d}\big(f_0(Q_r) - o\big(r^D\big)\big)$, so indeed $r^D \ll |X|$.

Lemma 4.1 says that $\|x\| \geqq r - b_1 r^{-(d-1)/(d+1)}$ where $b_1 > 0$ depends only on d. Define $r_0 = r - b_1 r^{-(d-1)/(d+1)}$, so X lies in the annulus $rB^d \setminus r_0 B^d$. Consequently all lattice points in $r_0 B^d \cap \mathbb{R}^d_+$ are contained in $Q^r \setminus X$ and so $I\big(r_0 B^d \cap \mathbb{R}^d_+\big) \subset I(Q^r \setminus X)$. The result from [3] cited above applies to $I\big(r_0 B^d \cap \mathbb{R}^d_+\big)$ and gives that

$$\operatorname{vol}\big(\big(r_0 B^d \cap \mathbb{R}^d_+\big) \setminus I\big(r_0 B^d \cap \mathbb{R}^d_+\big)\big) \ll r_0^D \ll r^D$$

as $r_0 < r$. This implies that with a suitable constant $b > 0$

$$2^{-d}\omega_d r^d - br^D \leqq \operatorname{vol} I(Q^r \setminus X) \leqq 2^{-d}\omega_d r^d.$$

After these preparations we are ready for the proof of Corollary 2.1.

5. Proof of Corollary 2.1

Given a (large enough) number V with $d!V \in \mathbb{N}$ we are going to construct many non-equivalent convex lattice polytopes whose volume equals V. As a first step, we define r via the equation

$$V = 2^{-d}\omega_d r^d - br^D.$$

Consider Q^r from the previous section. For $Z \subset X$ we define $Q(Z) = I(Q^r \setminus Z)$, $Q(Z)$ is a convex lattice polytope, it contains $Q(X)$. So $V \leqq \operatorname{vol} Q(X) \leqq \operatorname{vol} Q(Z) \leqq 2^{-d}\omega_d r^d$. Set $m(Z) = \operatorname{vol} Q(Z) - V$. Then

$$m(Z) = \operatorname{vol} Q(Z) - V \leqq 2^{-d}\omega_d r^d - V = br^D.$$

Since $r^D \ll |X|$, the number of polytopes $Q(Z)$ is $2^{|X|} \geqq \exp\left\{b_2 r^D\right\} = \exp\left\{b_3 V^{(d-1)/(d+1)}\right\}$ with suitable positive constants b_2, b_3. This is what we need in Corollary 2.1. But the volumes of the $Q(Z)$ are larger than V. So we are going to cut off volume $\operatorname{vol} Q(Z) - V = m(Z)/d!$ from $Q(Z)$ so that what is left is a lattice polytope of volume exactly V. We will do so using Theorem 1.1 or rather Theorem 3.1 the following way.

Set $\rho = \lfloor r/10 \rfloor \in \mathbb{N}$ and assume r_d is so large that $\rho > r_0 = 2^d d!$. Consider $A(\rho)$, $S(\rho)$ and $\mathcal{P}(\rho)$ from Section 1. Theorem 3.1 says that given if $m = d!m(Z) \leqq \left(\rho - 2^d d!\right)^d$, there is a polytope $P = P(Z) \in \mathcal{P}$ with $m(P) = m$. Then

$$P^*(Z) = \left[Q(Z) \setminus S(r)\right] \cup P(Z) = Q(Z) \setminus \left[S(r) \setminus P(Z)\right]$$

is a convex lattice polytope, and its volume is exactly V.

Now we have $\exp\left\{bV^{(d-1)/(d+1)}\right\}$ convex lattice polytopes $P^*(Z)$, each of volume V. We claim that any one of them is equivalent to at most $d!$ other $P^*(W)$. This will clearly finish the proof of $\log N_d(V) \gg V^{(d-1)/(d+1)}$.

To prove this note first that $P^*(Z)$ has a long edge on the segment $[0, re_i]$ for all $i \in [d]$. This edge is of the form $E_i = E_i(Z) = \left[\alpha_i e_i, \lfloor r \rfloor e_i\right]$ and $\alpha_i \leqq \rho \leqq r/10$. It is quite easy to check (we omit the details) that $P^*(Z)$ has exactly these d edges whose length is larger than $0.9r$. So if $P^*(Z)$ and $P^*(W)$ are equivalent, then the lattice preserving affine transformation T that carries $P^*(Z)$ to $P^*(W)$, has to map each $E_i(Z)$ to a uniquely determined $E_j(W)$. As the lines containing $E_i(Z)$ all pass through the origin, $T(0) = 0$ and so T is a linear map. Thus T permutes the axes and is lattice preserving. Then it has to permute the vectors $e_1, \dots, e_d$. There are $d!$ such transformations. Consequently, $P^*(Z)$ is equivalent with at most $d!$ other polytopes of the form $P^*(W)$. □

REMARK. The above construction can be modified so that all $P^*(Z)$ are non-equivalent. Namely, set $t = \lfloor r \rfloor$ and replace $Z \subset X$ by

$$Z^0 = Z \cup \bigcup_1^d \left\{ te_i, (t-1)e_i, \dots, (t-i+1)e_i \right\}.$$

Set $Q\big(Z^0\big) = I\big(Q^r \setminus Z^0\big)$ and $P^*\big(Z^0\big) = \big[Q\big(Z^0\big) \setminus S(r)\big] \cup P(Z^0)$ where $P\big(Z^0\big) \in \mathcal{P}(\rho)$ is again chosen so that $\operatorname{vol} P^*\big(Z^0\big) = V$. The long edges of $P^*\big(Z^0\big)$ are almost the same $E_i(Z)$ except that this time each carries a 'marker', namely the last i lattice points are missing from $E_i(Z)$. So if $P^*\big(Z^0\big)$ and $P^*\big(W^0\big)$ are equivalent, then the corresponding lattice preserving affine transformation has to be the identity.

6. Gaps in $v(\mathcal{P})$

Let $A_k = \big\{ (x_1, x_2, \dots, x_d) : x_1 + x_2 + \dots + x_d = k,\ x_i \in \{0, 1, 2, \dots, k\},\ i \in [d] \big\}$, where $k = 0, 1, \dots, r$. For any $P \in \mathcal{P}(r)$, if $P \cap A_{r-2} \neq \emptyset$, then clearly $v(P) \geqq 2r^{d-1}$. Now suppose that $P \cap A_{r-2} = \emptyset$ but $P \cap A_{r-1} \neq \emptyset$. Assume that $P = \operatorname{conv}\big\{ A(r) \cup B \big\}$, where $B \subset A_{r-1}$, $B = \{b_1, b_2, \dots, b_t\}$, $b_i = \big(b_1^i, b_2^i, \dots, b_d^i\big)$, $b_j^i \in \{0, 1, 2, \dots, r-1\}$ for all $i \in [t]$ and $j \in [d]$. For any $b_i, b_k \in B$, let $q_{ik} = \max\big\{ \big|b_j^i - b_j^k\big| : j \in [d] \big\}$ and $q = \max\{q_{ik} : b_i,\ b_k \in B\}$.

THEOREM 6.1. *For any* $d \geqq 3$ *and* $P \in \mathcal{P}(r)$, $v(P) \notin \big[r^{d-1} + 1, r^{d-1} + r^{d-2} - 1 \big]$.

PROOF. If $|B| = 1$, then P is actually a d-dimensional simplex with base $A(r)$, which means that $v(P) = r^{d-1}$.

If $|B| = 2$, then suppose that $B = \{b_1, b_2\}$. In this case $q = q_{12} = \max\big\{ \big|b_j^1 - b_j^2\big| : j \in [d] \big\}$ and $q \geqq 1$ as $b_1 \neq b_2$. Thus we may assume without loss of generality, that $b_d^1 - b_d^2 = q \geqq 1$. Let $P_1 = \operatorname{conv}\big\{ A(r) \cup \{b_1\} \big\}$, $P_2 = \operatorname{conv}\{re_1, re_2, \dots, re_{d-1}, b_1, b_2\}$. Since $b_d^1 \neq b_d^2$, P_2 is a d-dimensional simplex. It is easy to check that P_1 and P_2 are internally disjoint which implies $v(P) = v(P_1) + v(P_2)$. Clearly, $v(P_1) = r^{d-1}$. Furthermore, we have

$$v(P_2) = \left| \det \begin{pmatrix} 1 & \dots & 1 & 1 & 1 \\ re_1 & \dots & re_{d-1} & b_1 & b_2 \end{pmatrix} \right| = \big|b_d^1 - b_d^2\big| r^{d-2},$$

which means that $v(P) = r^{d-1} + qr^{d-2} \geqq r^{d-1} + r^{d-2}$. Clearly $v(P) \geqq r^{d-1} + r^{d-2}$ still holds for any $P \in \mathcal{P}(r)$ satisfying $|B| \geqq 3$. The proof is complete. $\square$

Now we consider the special case when $d=3$. Define a graph $G=\big(V(G),E(G)\big)$ such that $V(G)=A_{r-1}$, $E(G)=\big\{b_ib_k:\ \big|b_1^i-b_1^k\big|+\big|b_2^i-b_2^k\big|+\big|b_3^i-b_3^k\big|=2\big\}$. Clearly G is a triangular grid graph with boundary

$$\operatorname{conv}\big\{(r-1)e_1,(r-1)e_2\big\}\cup\operatorname{conv}\big\{(r-1)e_1,(r-1)e_3\big\}$$

$$\cup\operatorname{conv}\big\{(r-1)e_2,(r-1)e_3\big\}.$$

Furthermore, it is not difficult to see that for $b_i,b_k\in A_{r-1}$, $b_ib_k\in E(G)$ if and only if $q_{ik}=1$.

For any $b_i,b_k\in V(G)$, let $l_{b_ib_k}$ denote the line determined by b_i, b_k. In the hyperplane determined by A_{r-1}, let H_{ik}^+ denote the open halfplane bounded by $l_{b_ib_k}$ such that $\big|H_{ik}^+\cap\big\{(r-1)e_1,(r-1)e_2,(r-1)e_3\big\}\big|=2$ and H_{ik}^- the open halfplane bounded by $l_{b_ib_k}$ satisfying $\big|H_{ik}^-\cap\big\{(r-1)e_1,(r-1)e_2,(r-1)e_3\big\}\big|=1$. Furthermore, let $d_G(b_i,b_k)$ denote the graphic distance between b_i and b_k, i.e., the length of the shortest paths between b_i and b_k. Clearly, $d_G(b_i,b_k)=q_{ik}$. For any $b_i\in A_{r-1}$, let $D_{b_i}(s)=\big\{b_k:\ b_k\in A_{r-1},\ d_G(b_i,b_k)\leqq s\big\}$.

THEOREM 6.2. *If* $d=3$, *then for any* $P\in\mathcal{P}(r)$, $v(P)\notin\big[r^2+r+2,\ r^2+2r-1\big]$.

PROOF. If $|B|=1$, clearly $v(P)=r^2$. If $|B|=2$, then by the proof of Theorem 6.1 we know that $v(P)\in\big\{r^2+kr:\ k=1,2,\dots,r-1\big\}$. Now suppose that $|B|=t(t\geqq 3)$ and $B=\{b_1,b_2,\dots,b_t\}$, where $b_i=\big(b_1^i,b_2^i,b_3^i\big)$, $b_j^i\in\{0,1,2,\dots,r-1\}$ for all $i\in[t]$ and $j\in[3]$.

If $q\geqq 2$, assume without loss of generality that $q_{12}=q\geqq 2$, then we have $v(P)\geqq v\big(\operatorname{conv}\big\{A(r)\cup\{b_1,b_2\}\big\}\big)=r^2+qr\geqq r^2+2r$.

Now suppose that $q=1$. Then any two points in B are adjacent in G, which forces $t=3$ and $\operatorname{conv}B$ is a 2-dimensional simplex homothetic to $A(r)$. Furthermore, $\operatorname{conv}B=\theta_B+\frac{\lambda_B}{r}A(r)$, where $\lambda_B=\pm 1$. Suppose without loss of generality that $l_{b_1b_2}\parallel l_{re_1,re_2}$ and the vectors $\overrightarrow{b_1b_2}$ and $\overrightarrow{re_1,re_2}$ have the same direction. If $b_3\in H_{12}^-$, then $\lambda_B=1$ and

$$v(P)=v\big(\operatorname{conv}\big\{A(r)\cup\{b_1,b_2\}\big\}\big)+v\big(\operatorname{conv}\{re_3,b_1,b_2,b_3\}\big)=r^2+r+1.$$

If $b_3\in H_{12}^+$, then $\lambda_B=-1$ and

$$v(P)=v\big(\operatorname{conv}\big\{A(r)\cup\{b_1,b_2\}\big\}\big)+v\big(\operatorname{conv}\{re_1,re_2,b_1,b_3\}\big)$$

$$+v\big(\operatorname{conv}\{re_2,b_1,b_2,b_3\}\big)=\big(r^2+r\big)+r+1=r^2+2r+1.$$

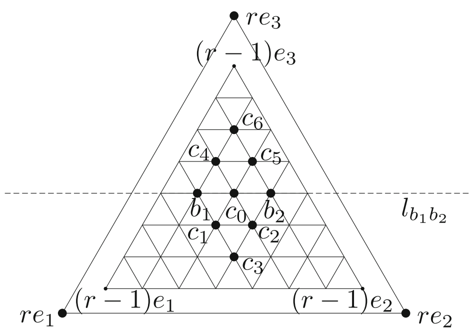
Combining the above discussions we see that there is no $v(P)$ of $P \in \mathcal{P}(r)$ lying in the interval $\left[r^2+r+2, r^2+2r-1\right]$, and the proof is complete. □

THEOREM 6.3. *If* $d=3$ *and* $r \geqq 6$, *then for any* $P \in \mathcal{P}(r)$, $v(P) \notin \left[r^2+2r+5, r^2+3r-1\right]$.

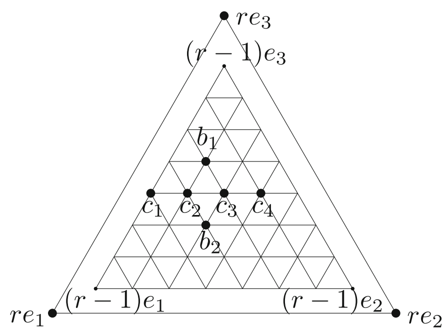
PROOF. Now still suppose that $|B|=t$ and $B=\{b_1, b_2, \ldots, b_t\}$, where $b_i=\left(b_1^i, b_2^i, b_3^i\right)$, $b_j^i \in \{0,1,2,\ldots,r-1\}$ for all $i \in [t]$ and $j \in [3]$.

If $q \geqq 3$, say, $q_{12}=q \geqq 3$, then $v(P) \geqq v\big(\operatorname{conv}\{A(r) \cup \{b_1, b_2\}\}\big) = r^2+qr \geqq r^2+3r$.

If $q \leqq 2$, then combining the proof of Theorem 6.2, we only need to consider the cases when B satisfies $t \geqq 3$ and $q=2$.



(a) *Case 1*



(b) *Case 2*

Fig. 1: $t \geqq 3$, $q=2$.

Case 1: $\exists\ b_i, b_k \in B$ such that $q_{ik}=2$, $l_{b_ib_k}$ is parallel to one side of $A(r-1)$.

Assume without loss of generality that $b_1, b_2 \in B$ such that $q_{12}=2$, $l_{b_1b_2} \parallel l_{(r-1)e_1,(r-1)e_2} \parallel l_{re_1,re_2}$ and the vectors $\overrightarrow{b_1b_2}$ and $\overrightarrow{re_1, re_2}$ have the same direction. Since $q=2$, $B \subset D_{b_1}(2) \cap D_{b_2}(2) = \{b_1, b_2, c_0, c_1, c_2, c_3, c_4, c_5, c_6\}$, as shown in Fig. 1(a), where $c_0 \in l_{b_1b_2}$, $\{c_1, c_2, c_3\} \subset H_{12}^+$ and $\{c_4, c_5, c_6\} \subset H_{12}^-$. Clearly,

$$v\big(\operatorname{conv}\{A(r) \cup \{b_1, b_2, c_1\}\}\big) = v\big(\operatorname{conv}\{A(r) \cup \{b_1, b_2\}\}\big)$$

$$+\, v\big(\operatorname{conv}\{b_1, c_1, re_1, re_2\}\big) + v\big(\operatorname{conv}\{b_1, c_1, b_2, re_2\}\big)$$

$$= \left(r^2+2r\right)+r+2 = r^2+3r+2.$$

Similarly, $v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_2\}\}\,\big)=r^2+3r+2$. As a result, if $B\cap H_{12}^+\neq\emptyset$, then $v(P)\geqq r^2+3r+2$. If $B\cap H_{12}^+=\emptyset$, then all the possible values of $v(P)$ are:

$$v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_0\}\}\,\big)=r^2+2r;$$

$$v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_4\}\}\,\big)=r^2+2r+2;$$

$$v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_5\}\}\,\big)=r^2+2r+2;$$

$$v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_4,c_5\}\}\,\big)=r^2+2r+3;$$

$$v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_6\}\}\,\big)=r^2+2r+4.$$

Case 2: $b_1,b_2\in B$ such that $l_{b_1b_2}\perp l_{(r-1)e_1,(r-1)e_2}$. Then we have $B\subset\{b_1,b_2,c_1,c_2,c_3,c_4\}$ and B contains at most two adjacent points among $\{c_1,c_2,c_3,c_4\}$, as shown in Fig. 1(b). Clearly,

$$v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_2\}\}\,\big)$$

$$=v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2\}\}\,\big)+v\big(\text{conv}\{b_1,b_2,c_2,re_1\}\big)=r^2+2r+1.$$

Similarly, $v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_3\}\}\,\big)=r^2+2r+1$. Furthermore,

$$v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_1\}\}\,\big)=v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_1,c_2\}\}\,\big)$$

$$=v\big(\text{conv}\,\{\,A(r)\cup\{b_2,c_1\}\}\,\big)+v\big(\text{conv}\{b_1,b_2,re_2,re_3\}\big)$$

$$+\,v\big(\text{conv}\{b_1,b_2,c_1,re_3\}\big)=\big(r^2+2r\big)+r+3=r^2+3r+3.$$

Similarly,

$$v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_4\}\}\,\big)=v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_3,c_4\}\}\,\big)$$

$$=r^2+3r+3.$$

Finally,

$$v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2,c_2,c_3\}\}\,\big)=v\big(\text{conv}\,\{\,A(r)\cup\{b_1,b_2\}\}\,\big)$$

$$+\,v\big(\text{conv}\{b_1,b_2,c_2,re_1\}\big)+v\big(\text{conv}\{b_1,b_2,c_3,re_2\}\big)=r^2+2r+2.$$

Combining all the discussions above, we have the following facts:
if $|B| = 1$, then $v(P) = r^2$;
if $|B| = 2$, then $v(P) \in \{ r^2 + kr : k = 1, 2, \dots, r-1 \}$;
if $|B| \geqq 3$ and $q = 1$, then $v(P) \in \{ r^2 + r + 1, r^2 + 2r + 1 \}$;
if $|B| \geqq 3$ and $q = 2$, then either $v(P) \in [r^2 + 2r, r^2 + 2r + 4]$ or $v(P) \geqq r^2 + 3r$;
if $|B| \geqq 3$ and $q \geqq 3$, then $v(P) \geqq r^2 + 3r$.

If $r \geqq 6$, then $(r^2 + 3r - 1) - (r^2 + 2r + 5) = r - 6 \geqq 0$, which means that $[r^2 + 2r + 5, r^2 + 3r - 1]$ is nonempty. Clearly there is no $P \in \mathcal{P}(r)$ satisfying $v(P) \in [r^2 + 2r + 5, r^2 + 3r - 1]$. The proof is complete. □

COROLLARY 6.1. *For every $v \in [r^2 + 2r, r^2 + 2r + 4]$, there exists a $P \in \mathcal{P}(r)$ such that $v(P) = v$.*

7. Concluding remarks

We mention further that, as Arnold [1] suggests, the paraboloid

$$D_r = \{ x \in \mathbb{R}_+^d : x_1^2 + \dots + x_{d-1}^2 \leqq x_d \leqq r^2 \}$$

can be used to give the lower bound in Corollary 2.1 as the following proof, or rather sketch of a proof, shows. Given $V > 0$ with $d!v \in \mathbb{N}$ we are going to construct many non-equivalent convex lattice polytopes of volume V.

The integer convex hull $I(D_r)$ of D_r has a vertex corresponding to each lattice point $z = (z_1, \dots, z_{d-1}, 0) \in rB^d \cap \mathbb{R}_+^d$, namely the point $z + \|z\|^2 e_d$. Denoting this set of vertices by X we have $r^{d-1} \ll |X| \ll r^{d-1}$. Also, $\operatorname{vol} I(D_r)$ is of order r^{d+1}, and

$$\operatorname{vol} I(D_r) - \operatorname{vol} I(D_r \setminus X) \ll r^{d-1}$$

as one can check easily.

Define r by $V = \operatorname{vol} I(D_r \setminus X)$ and note that $V \gg r^{d+1} \gg |X|^{(d+1)/(d-1)}$. For $Z \subset X$ consider the polytopes $D(Z) = I(D_r \setminus Z)$. These are $2^{|X|} \geqq \exp \{ c_1 V^{(d-1)/(d+1)} \}$ convex lattice polytopes, each having volume between V and $V + c_2 r^{d-1}$ (where $c_1, c_2 > 0$ are constants depending only on d).

At the vertex $(0, \dots, 0, \lfloor r^2 \rfloor)$ of $D(Z)$ one can place a congruent copy of $S(\rho)$; here one chooses $\rho \in \mathbb{N}$ to be of order $r^{(d-1)/d}$. We denote this copy by $S(\rho)$ as well. Given $Z \subset X$ set $m = m(Z) = d!(\operatorname{vol} D(Z) - V)$, Theorem 3.1 implies the existence of $P(Z) \in \mathcal{P}(\rho)$ with $m(P(Z)) = m(Z)$. We define

$$D^*(Z) = [D(Z) \setminus S(\rho)] \cup P(Z),$$

which is a convex lattice polytope and $\mathrm{vol}\, D^*(Z) = V$. One shows again that the number of $D^*(W)$'s equivalent to a fixed $D^*(Z)$ is at most $d!$. We leave the details to the interested reader.

Acknowledgements. Support from ERC Advanced Research Grant no 267165 (DISCONV) and from Hungarian National Research Grant K 83767 is acknowledged with thanks. The second author gratefully acknowledges financial supports by NNSF of China (11071055); NSF of Hebei Province (A2012205080, A2013205189); Program for New Century Excellent Talents in University, Ministry of Education of China (NCET-10-0129); the project of Outstanding Experts' Overseas Training of Hebei Province.

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