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Helly type theorems for the sum of vectors in a normed plane

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The main results here are two Helly type theorems for the sum of (at most) unit vectors in a normed plane. Also, we give a new characterization of centrally symmetric convex sets in the plane.

1. Main results

This paper is about the sum of vectors in a normed plane. We fix a norm $\|\,.\,\|$ in $\mathbb{R}^2$ whose unit ball is B; so B is a 0-symmetric convex body. There are some interesting

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results about sums of unit vectors in normed planes. For instance, it is proved by Swanepoel in [5] (and reproved later in [1]) that for every subset $V = \{v_1, \ldots, v_n\} \subset B$ of unit vectors, with n an odd number, we may choose numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ from $\{1, -1\}$ such that $\|\sum_{v_i \in V} \epsilon_i v_i\| \le 1$. This time we are interested in unit vectors whose sum has length at least 1.

We write $u \cdot v$ for the usual scalar product of $u, v \in \mathbb{R}^2$ and $[n]$ for the set $\{1, 2, \ldots, n\}$. Here comes our first result.

Theorem 1. *Assume $n \ge 3$ is an odd integer and $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^2$ is a set of unit vectors. If $u \cdot v_i \ge 0$ for every $i \in [n]$ with a suitable non-zero vector $u \in \mathbb{R}^2$, then*

$$\|v_1 + v_2 + \ldots + v_n\| \ge 1.$$

Here and in what follows we can assume that V is a multiset, that is, $v_i = v_j$ can happen even if $i \ne j$. Perhaps one should think of V as a sequence of n vectors from $\mathbb{R}^2$.

In accordance to the celebrated Helly's theorem (see [3]), results of the type "if every m members of a family of objects have property P then the entire family has the property P" are called Helly-type theorems. Our main results are two unusual Helly type theorems whose proof uses Theorem 1. For information about Helly type results the reader may consult [4].

Theorem 2. *Assume $n \ge 3$ is an odd integer and $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^2$ is a set of unit vectors. If the sum of any three of them has norm at least 1, then*

$$\|v_1 + v_2 + \ldots + v_n\| \ge 1.$$

Theorem 3. *Assume $n \ge 3$ is an odd integer and $V = \{v_1, v_2, \ldots, v_n\} \subset B$. If the sum of any three elements of V has norm larger than 1, then*

$$\|v_1 + v_2 + \ldots + v_n\| > 1.$$

To our surprise Theorem 3 fails in the following form: If $V \subset B$, $|V|$ is odd, and the sum of any three of its elements has norm at least 1, then $\|v_1 + v_2 + \ldots + v_n\| \ge 1$. The example is with the max norm and the vectors are $v_1 = (1, 1)$, $v_2 = (-1, 1)$, and $v_3 = v_4 = v_5 = (0, -1/2)$. This is also an example showing that Theorem 2 does not hold if we require $V \subset B$ instead of $\|v_i\| = 1$ for all i.

Note that in these theorems n has to be odd. Indeed, let w_1 and w_2 be two antipodal unit vectors. Set $n = 2k$, $v_1 = \ldots = v_k = w_1$ and $v_{k+1} = \ldots = v_n = w_2$. The conditions of Theorems 1 and 2 are satisfied (except that n is even now) but $\|v_1 + v_2 + \ldots + v_n\| = 0$. A minor modification of this example shows that n has to be odd in Theorem 3 as well. Namely, let the segment $[z_1, z_2]$ be a Euclidean diameter of B, and choose w_1, w_2 very close to z_1, z_2 so that $w_1 + w_2$ has norm $< 1/k$ and is orthogonal to z_1. This is clearly possible. Then with $n = 2k$, $v_1 = \ldots = v_k = w_1$ and $v_{k+1} = \ldots = v_n = w_2$ the conditions of Theorem 3 are satisfied but $\sum_1^n v_i \in B$.

For simpler writing let $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$, and given $S \in \binom{[n]}{k}$ define

$$\sigma(S, V) = \sum_{i \in S} v_i,$$

and we call it a k-sum of V. Note that $\sigma(\emptyset, V) = 0$ by definition. Theorem 3 has the following immediate

Corollary 1. *Assume $n \geq 5$ is an integer, $V = \{v_1, v_2, \ldots, v_n\} \subset B$, $k \in [n]$ is odd and $k > 3$. If every 3-sum of V is outside B, then so is every k-sum of V.*

Theorems 1 and 2 have similar corollaries and the interested reader will have no difficulty stating or proving them.

We close this section with a neat *proof* of Theorem 1 for the case of the Euclidean norm. The method (unpublished) is due to Boris Ginzburg who used it for the Euclidean case of Theorem 1 from [1].

We may assume w.l.o.g. that $u = (0, 1)$. The proof is in fact an algorithm that produces a sequence $V = V_0, V_1, \ldots, V_n$ of sets of n unit vectors, satisfying $u \cdot v \geq 0$ for all $v \in V_i$, $i \in [n]$ so that the norm of $s_i = \sum_{v \in V_i} v$ decreases as i increases and $\|s_n\| \geq 1$. Call an element $v \in V_i$ *fixed* if it equals $(1, 0)$ or $(-1, 0)$, and let F_i be the set of fixed elements in V_i, and let $M_i = V_i \setminus F_i$ be the set of *moving* elements in V_i.

At the start $V = V_0 = M_0$ and $F_0 = \emptyset$. Assume V_i has been constructed, and set $f_i = \sum_{v \in F_i} v$ and $m_i = \sum_{v \in M_i} v$. One can rotate the vector m_i so that $\|f_i + m_i\|$ decreases during the rotation (because of the cosine theorem). We rotate m_i in this direction, together with all vectors in M_i as long as one of its elements, say v^*, reaches $(1, 0)$ or $(-1, 0)$. Let M_i^* be this rotated copy of M_i. Define $M_{i+1} = M_i^* \setminus \{v^*\}$ and $F_{i+1} = F_i \cup \{v^*\}$. We indeed have $\|s_i\| \geq \|s_{i+1}\|$. By construction $V_n = F_n$, $M_n = \emptyset$ and $\|f_n\|$ is an odd integer, so $\|s_n\| = \|f_n\| \geq 1$. □

2. Proof of Theorem 1

Proof of Theorem 1. We assume again that $u = (0, 1)$. Let $n = 2k-1$ and let $v_1, \ldots, v_{2k-1}$ be our unit vectors in clockwise order on the boundary of B in the upper halfplane. Let w_1 and w_2 be two unit vectors on the horizontal line through 0 with w_1 to the left of the origin 0. The tangent line L to B at v_k bounds the half-plane H, the one not containing the origin. Set $s = v_1 + \ldots + v_{2k-1}$.

Let ℓ be the line through 0 and v_k. For $v \in \mathbb{R}^2$ let v' be the signed length of its projection in direction L onto ℓ, that is, v' is positive if v' has the same direction as v_k and negative otherwise. Since the projection of the sum of vectors is equal to the sum of their projections, it suffices to prove that

$$v'_1 + v'_2 + \ldots + v'_{2k-1} \geq 1$$

as this implies $s \in H$ and so $\|s\| \geq 1$. We have that $v'_k = \|v_k\| = 1$ and

$$v'_1 + \ldots + v'_{k-1} \geq (k-1)w'_1$$
$$v'_{k+1} + \ldots + v'_{2k-1} \geq (k-1)w'_2.$$

As $w'_1 + w'_2 = 0$, the proof is now complete. □

Remark 1. Using this proof the case of equality can be characterized but the conditions are clumsy. The case when the boundary of B contains no line segment is simple: equality holds iff $(n-1)/2$ of the v_i are equal to some unit vector v and another $(n-1)/2$ are equal to $-v$. This follows easily from the proof above.

We mention further that replacing the condition $u \cdot v_i \geq 0$ by $u \cdot v_i > 0$ for every $i \in [n]$ in Theorem 1 does not imply $\|v_1 + v_2 + ... + v_n\| > 1$. For instance when $\| \, . \, \|$ is the max norm and $v_1 = \ldots = v_k = (-1, \varepsilon)$ and $v_{k+1} = \ldots = v_{2k-1} = (1, \varepsilon)$ and $\varepsilon > 0$ is small enough, $\|s\| = 1$ although $u \cdot v_i > 0$ for all i.

Remark 2. Theorem 1 has no analogue in dimension 3 and higher. For the example showing this let B be the Euclidean unit ball in $\mathbb{R}^3$, let L be a plane at distance ε from the origin with unit normal u, and let P_n be a regular n-gon inscribed in the circle $L \cap B$, with vertices $v_1, \ldots, v_n$. It is clear that $u \cdot v_i > 0$ for all $i \in [n]$ but $\sum_1^n v_i = \varepsilon n u$ whose norm is as small as you wish. The parity of n does not matter.

Remark 3. The following is a direct consequence of Theorem 1: let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of unit vectors in a normed plane. Then it is always possible to choose numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ from $\{1, -1\}$ such that for every subset $W \subset V$ of odd size, we have that $\|\sum_{v_i \in W} \epsilon_i v_i\| \geq 1$.

3. Proof of Theorem 2

We need some preparations before the proof. We start with a small piece from Euclidean plane geometry. Let a, b, c be distinct unit vectors in the Euclidean plane and define $\triangle = \text{conv}\{a, b, c\}$. It is well known that $h = a + b + c$ is outside $\triangle$ (indeed, outside the unit circle) if the triangle is obtuse, and is inside $\triangle$ if the triangle is acute. (We ignore right angle triangles here.) This is equivalent to saying that $h \in \triangle$ iff $0 \in \triangle$ since $\triangle$ is acute or obtuse depending on whether $0 \in \triangle$ or not.

Is this statement true for any norm in $\mathbb{R}^2$? As we see from the following lemma the answer is yes.

Lemma 4. *Assume $a, b, c \in \partial B$ and set $\triangle = \text{conv}\{a, b, c\}$. Then $0 \in \triangle$ if and only if $h = a + b + c \in \triangle$.*

Proof. If $0 \notin \triangle$, then by separation there is a vector u such that $u \cdot a, u \cdot b, u \cdot c > 0$. Theorem 1 with $V = \{a, b, c\}$ applies and shows that $h \notin \text{int}\, B$. As $\text{int}\, \triangle \subset \text{int}\, B$, $h \in \triangle$

implies $h \in \partial\triangle$, say $h \in [a,c]$. Then $a+b+c = ta + (1-t)c$ for some $t \in [0,1]$ and so

$$\frac{1-t}{2}a + \frac{1}{2}b + \frac{t}{2}c = 0,$$

a convex combination of a, b, c, showing that $0 \in \triangle$. So indeed, $h \notin \triangle$. We remark here for later use that $h \in \partial\triangle$ implies that 0 is on the boundary of the medial triangle of $\triangle$ (because the coefficient of b is $1/2$ above).

Assume next that $0 \in \triangle$. Since 0 is the center of the unit ball, it must be contained in the medial triangle of $\triangle$, that is, $0 = \alpha(\frac{b+c}{2}) + \beta(\frac{a+c}{2}) + \gamma(\frac{a+b}{2})$, with $\alpha, \beta, \gamma \in [0,1]$ and $\alpha + \beta + \gamma = 1$. We have that

$$0 + \frac{\alpha}{2} \cdot a + \frac{\beta}{2} \cdot b + \frac{\gamma}{2} \cdot c = (\alpha + \beta + \gamma)\left(\frac{a+b+c}{2}\right) = \frac{a+b+c}{2},$$

then $a+b+c = \alpha a + \beta b + \gamma c$, that is, $h = a+b+c \in \triangle$. $\square$

Proof of Theorem 2. We assume first the extra condition that V contains no antipodal pair of points. For distinct $i,j,k \in [n]$, the vector $h = v_i + v_j + v_k$ is not in $\operatorname{int} B$. Then, as $\triangle = \operatorname{conv}\{v_i, v_j, v_k\} \subset B$, $h \notin \operatorname{int}\triangle$. So either $h \notin \triangle$ for all i, j, k or $h \in \partial\triangle$ for some $i,j,k \in [n]$.

Assume first that $h \notin \triangle$ for all $i,j,k \in [n]$ which is the simpler case. Then $0 \notin \triangle$ follows from Lemma 4. Carathéodory's theorem (see [2]) shows that $0 \notin \operatorname{conv} V$, too. By separation, there is a vector $u \neq 0$ with $u \cdot v > 0$ for every $i \in [n]$. Theorem 1 applies and gives $\|v_1 + \ldots + v_n\| \geq 1$.

So we are left with the case when $h \in \partial\triangle$ for some $i,j,k \in [n]$. For simpler writing set $a = v_i$, $b = v_j$, $c = v_k$ and suppose $h \in [a,c]$ as in the proof of Lemma 4. As $h \notin \operatorname{int} B$ and $\triangle \subset B$, we have $h \in \partial B$. Thus $a,h,c \in \partial B$. As $h = a$ (resp. $h = c$) would imply $b + c = 0$ (and $a + b = 0$), the whole segment $[a,c]$ is contained in ∂B. Now let ℓ be the line through a and c and set $L = \ell \cap B$. L is a segment on the boundary of B and so is $-L$. If every v_i is contained in $L \cup -L$, then $\sum_1^n v_i$ cannot be between in the strip delimited by ℓ and $-\ell$ as n is odd. Suppose now that some $v_i \notin L \cup -L$ and let L' be the chord of B parallel with ℓ and containing v_i, see Fig. 1.

Here L' is at least as long as L, while $a+b$ and $c+b$ are parallel with ℓ, they point in opposite directions, and both are shorter than L (because 0 is in the relative interior of the medial segment $[(a+b)/2, (c+b)/2]$). Consequently either $v = v_i + a + b$ or $v = v_i + c + b$ lies in $\operatorname{int} B$, and so one of them has norm less than one. A contradiction.

The general case goes by induction on n. The starting case $n = 3$ is trivial. In the induction step $n - 2 \to n$ (when $n \geq 5$) $V = \{v_1, \ldots, v_n\}$ either satisfies the extra condition and we are done, or V contains an antipodal pair, v_{n-1}, v_n say. By induction, $\|v_1 + \ldots + v_{n-2}\| \geq 1$, and the equality $\sum_1^n v_i = \sum_1^{n-2} v_i$ finishes the proof. $\square$

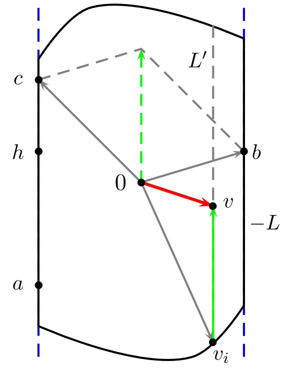

Fig. 1. If $h \in \partial\triangle$ then v is contained in int B.

Remark 4. Theorem 2 has no direct analogue in $\mathbb{R}^3$. For instance if V is the set of vertices of a regular tetrahedron centered at the origin and inscribed in the Euclidean unit ball, then every triple sum has (Euclidean) norm 1 yet the sum of the vectors is zero. A second example is when u is a unit vector in $\mathbb{R}^3$, v_1, v_2, v_3 are the vertices of a regular triangle in the plane orthogonal to u and center at u and $v_{i+3} = v_i - 2u$ $(i = 1, 2, 3)$, $V = \{v_1, \ldots, v_6\}$ and $B = \operatorname{conv}\{\pm v_1, \ldots, \pm v_6\}$. The sum of any three vectors from V has norm at least one but $\sum_1^6 v_i = 0$. The same example works for Theorem 3, this time every 4-sum has norm larger than one but $\sum_1^6 v_i = 0$ again.

4. Preparations for the proof of Theorem 3

We need a lemma about 6 vectors in the plane.

Lemma 5. *Assume $z_1, \ldots, z_6 \in B$ and $\sum_1^6 z_i = 0$. Then there are distinct i, j, k with $z_i + z_j + z_k \in B$.*

Proof. Assume for the time being that there are two linearly independent vectors among the z_i. We will deal with the remaining case soon. Define $D = \operatorname{conv}\{\pm z_1, \ldots, \pm z_6\}$, D is a 0-symmetric convex polygon with at most 12 vertices. Clearly $Z = \{z_1, \ldots, z_6\} \subset D$ and $D \subset B$. This implies that it suffices to prove Lemma 5 when $B = D$.

Let vert D denote the set of vertices of D. We distinguish two cases:

Case 1. When $|Z \cap \operatorname{vert} D| = 2$. Then D is a parallelogram with vertices a, b, $-a$, $-b$ where $a, b \in Z \cap \operatorname{vert} D$. As the assumptions and statement of the lemma are invariant under a non-degenerate linear transformation we may assume that $a = (1, 1)$ and $b = (-1, 1)$. This is in fact the case of the max norm. We need the following

Claim 1. *If the sum of real numbers $z_1, \ldots, z_6$ is zero and all of them lie in $I = [-1, 1]$, then there are at least 12 distinct triplets among them whose sum lies in I as well.*

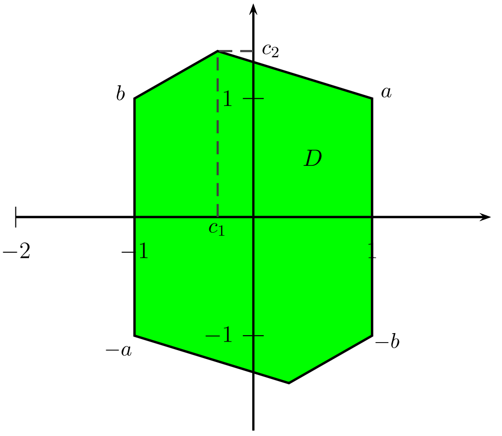

Fig. 2. There is a point $c = (c_1, c_2) \in Z \cap \operatorname{vert} D$ with $c_1 \in (-1, 1)$ and $c_2 > 1$.

The proof is postponed to Section 6. We note first that Claim 1 justifies our assumption about the existence of two linearly independent vectors among the z_i. Indeed, if all the z_i are on a line through the origin, then they can be thought of as real numbers. Claim 1 says then that there are three among them with the required property (actually, 12 such triplets).

We show next how the claim finishes Case 1. Both the first and the second components of the z_i satisfy the conditions of Claim 1. So there are 12 triplets whose first component, and 12 further triplets whose second component, sum to a number in I. As there are 20 triplets altogether, there is a triplet whose first and second components sum to a number in I, that is, there are distinct i, j, k with $z_i + z_j + z_k \in D$.

Case 2. When $|Z \cap \operatorname{vert} D| \geq 3$. If there are $a, b, c \in Z \cap \operatorname{vert} D$ such that $a + b + c \in D$, then we are done. Otherwise Lemma 4 (together with Carathéodory's theorem) says that $0 \notin \operatorname{conv}(Z \cap \operatorname{vert} D)$. So we may assume that every point of $Z \cap \operatorname{vert} D$ is in the open upper halfplane. Let a be the first and b be the last vertex as we walk around ∂D in the upper halfplane in anticlockwise direction. By a non-degenerate linear transformation we can achieve $a = (1, 1)$ and $b = (-1, 1)$. Clearly, $[a, -b]$ and $[b, -a]$ lie on ∂D. Note that there is $c = (c_1, c_2) \in Z \cap \operatorname{vert} D$, different from a, b implying that $c_1 \in (-1, 1)$ and $c_2 > 1$ (see Fig. 2).

For simpler writing let u_1, u_2, u_3 be the z_i distinct from a, b, c. We are going to show that $a + b + u_i \in D$ for some i. Otherwise $u_i \notin D - a - b$ for all i. In other words, $u_1, u_2, u_3 \in D \setminus (D - a - b)$. It is easy to see that the second component of every vector in $D \setminus (D - a - b)$ is larger than -1. Thus the second component of $u_1 + u_2 + u_3$ is larger than -3. The second component of $a + b + c$ is $2 + c_2 > 3$. This contradicts the assumption $z_1 + \ldots + z_6 = 0$. $\square$

To close this section we prove Theorem 3 in the case when V does not contain two linearly independent vectors. In this case V can be thought of as real numbers $x_1, \ldots, x_n$

with $x_1 \geq \ldots \geq x_n$. By symmetry and scaling we may assume that $x_1 = 1 \geq |x_n|$ and $B = [-1, 1]$. There is nothing to prove if $x_n \geq 0$. Also, $x_1 = -x_n$ is impossible since then $x_1 + x_2 + x_n = x_2 \in B$, contrary to the conditions. Thus $x_1 + x_n > 0$ and $x_{n-1} > 0$ as otherwise $x_1 + x_{n-1} + x_n \in B$. Consequently $x_1 + \ldots + x_n \geq x_1 + x_{n-1} + x_n > 1$.

5. Proof Theorem 3

The result is trivially true for $n = 3$. Next comes the case $n = 5$: Set $z_i = v_i$, $i = 1, 2, 3, 4, 5$ and $z_6 = -(v_1 + \ldots + v_5)$. If $\|z_6\| \leq 1$ were the case, then Lemma 5 implies that a 3-sum, $z_i + z_j + z_k$ say, lies in B. This contradicts the condition if z_6 is not present among z_i, z_j, z_k. But if it is, then the complementary 3-sum goes without z_6, and its norm equals $\|z_i + z_j + z_k\| \leq 1$, a contradiction again.

Assume now that the theorem fails and let $V = \{v_1, \ldots, v_n\}$ be a counterexample with the smallest possible n and let B be the unit ball of the corresponding norm. Here $n \geq 7$ clearly and V contains two linearly independent vectors. Define $v_0 = \sum_1^n v_i$. Then $D = \operatorname{conv}\{\pm v_0, \pm v_1, \ldots, \pm v_n\}$ is a 0-symmetric convex body (actually a convex polygon) that is the unit ball of a norm $\|\,.\,\|$. As $D \subset B$, V is a counterexample with this norm. This means that $\|v_i\| \leq 1$ for all $i = 0, 1, \ldots, n$ and all 3-sums have norm > 1. From now on we keep this norm fixed and consider V a counterexample with this norm.

We choose $\lambda < 1$ but very close to 1 so that $\lambda v_1, \ldots, \lambda v_n$ is still a counterexample, this time with $\|\sum_1^n \lambda v_i\| < 1$. By continuity there is an $\varepsilon > 0$ so that if $\|u_i - \lambda v_i\| < \varepsilon$ for all $i \in [n]$, then $U = \{u_1, \ldots, u_n\}$ is still a counterexample meaning that $\|u_i\| < 1$ for all $i \in [n]$, $\|\sigma(S, U)\| > 1$ for all $S \in \binom{[n]}{3}$, and $\|\sum_1^n u_i\| < 1$. Here of course $\sigma(S, U)$ stands for $\sum_{i \in S} u_i$.

Claim 2. *One can choose U so that for all $S \in \binom{[n]}{3}$ and all $T \in \binom{[n]}{3} \cup \binom{[n]}{5}$, $\|\sigma(S, U)\| = \|\sigma(T, U)\|$ implies $S = T$.*

The technical proof is postponed to Section 6. Now we return to the proof by fixing U as in the claim.

The numbers $\|\sigma(S, U)\|$ with $S \in \binom{[n]}{3} \cup \binom{[n]}{5}$ are all larger than one. Let $\mu > 1$ be the smallest among them. We claim that $\mu = \|\sigma(S, U)\|$ for some unique $S \in \binom{[n]}{3}$. Indeed, if the minimal S is a 5-tuple, $S = \{1, 2, 3, 4, 5\}$ say, then the five vectors $\mu^{-1}u_1, \ldots, \mu^{-1}u_5$ are all in D, all of their 3-sums are outside D but their sum is in D, contradicting case $n = 5$ of the theorem.

Consequently $\mu = \|\sigma(S, U)\|$ for a unique $S \in \binom{[n]}{3}$. We assume w.l.o.g. that $S = \{1, 2, 3\}$. Choose $\nu < \mu^{-1} < 1$ so that $\nu\|\sigma(T, U)\| > 1$ for all $T \in \binom{[n]}{3} \cup \binom{[n]}{5}$ except for $T = S$ and $\nu\|\sigma(S, U)\| < 1$. Set $w_0 = \nu(u_1 + u_2 + u_3)$, $w_i = \nu u_i$ for $i > 3$, and define $W = \{w_0, w_4, \ldots, w_n\}$.

We show finally that W is another counterexample with the norm $\|\,.\,\|$. This would contradict the minimality of n as $|W| = n - 2 < n$ and so finish the proof.

It is clear that $W \subset D$ and $w_0 + w_4 + \ldots + w_n \in D$. All 3-sums of W that do not contain w_0 are outside D since such a 3-sum equals $\nu(u_i + u_j + u_k)$ with $4 \le i < j < k$ which is outside D by the definition of ν. A 3-sum of the form $w_0 + w_i + w_j$ for $4 \le i < j$ is equal to $\nu(w_1 + w_2 + w_3 + w_i + w_j)$ which is again outside D because of the definition of ν. □

6. Proofs of the claims

Proof of Claim 1. Write $x_1, x_2, \ldots, x_s$ resp $-y_1, \ldots, -y_t$ for the positive and non-positive elements of our set Z of real numbers, here $s + t = 6$ and we assume w.l.o.g. that $s \le t$. We assume further that $x_1 \ge x_2 \ge \ldots \ge x_s$ and $y_1 \ge \ldots \ge y_t$. The case $s = 0$ is trivial, and so is case $s = 1$: then all 3-sums of Z lie in $I = [-1, 1]$.

If $s = 2$, then $x_1 - y_i - y_j \in I$ for all distinct i, j. Indeed, this is clear if $x_1 \ge y_i + y_j$ since then $0 \le x_1 - y_i - y_j \le x_1 \le 1$. Assume next that $x_1 < y_i + y_j$ and $y_i \ge y_j$ say, then $-1 \le -y_i \le -y_i + (x_1 - y_j) < 0$ provided $x_1 \ge y_j$. But case $x_1 < y_j$ is impossible: then we'd have $x_1, x_2 < y_i, y_j$ and $x_1 + x_2 < y_i + y_j$, so the sum of our six numbers cannot be zero. Thus there are $\binom{4}{2} = 6$ distinct 3-sums in I and no two of them are complementary. The 6 complementary 3-sums lie in I, too.

Finally $s = 3$. By symmetry we assume that $x_1 \ge y_1$. If $x_1, x_2 \ge y_1$, then $x_k - y_i - y_j \in I$ for $k = 1, 2$ and for all distinct i, j. This follows in the same way as above. This is already 6 distinct 3-sums in I (with no two complementary), giving 12 distinct 3-sums that lie in I.

So suppose $y_1 > x_2$. Again $x_1 - y_i - y_j \in I$ for all distinct i, j and both $-y_1 + x_1 + x_2$ and $-y_1 + x_1 + x_3$ lie in I as both are non-negative and each smaller than x_1. This is five distinct (and non-complementary) 3-sums. We only need to find one more.

The missing one is $-y_1 + x_2 - y_2$ if $x_1 \ge y_1 > x_2 \ge y_2$, and $x_1 - y_2 + x_2$ if $x_1 \ge y_1 \ge y_2 \ge x_2$. □

Proof of Claim 2. Our unit ball D is a 0-symmetric convex polygon with edge set E. For an edge $e = [x, y]$ define ℓ_e as the (unique) linear function $\mathbb{R}^2 \to \mathbb{R}$ such that $\ell_e(x) = \ell_e(y) = 1$. It follows that for all $z \in \mathbb{R}^2$, $\|z\| = \min\{\ell_e(z) : e \in E\}$.

Recall the definition of $\binom{[n]}{k}$ and $\sigma(S, V)$ from Section 1. We are going to choose the vectors $u_1, u_2, \ldots, u_n$ in this order where u_i is in the ε-neighborhood $N_\varepsilon(\lambda v_i)$ of λv_i ($i \in [n]$) so that the following holds. The sets $U_k = \{u_1, \ldots, u_k\}$ for $k \in [n]$ satisfy

(1) $\ell_e(\sigma(S, U_k)) \ne \ell_f(\sigma(T, U_k))$ for all distinct $e, f \in E$ and all $s, t \in \{0, 1, \ldots, 5\}$ and all $S \in \binom{[k]}{s}$ and $T \in \binom{[k]}{t}$ with $S \ne T$,
(2) $\ell_e(\sigma(S, U_k)) \ne \ell_e(\sigma(T, U_k))$ for all $e \in E$, for all $s, t \in \{0, 1, \ldots, 5\}$ and all $S \in \binom{[k]}{s}$ and $T \in \binom{[k]}{t}$ with $S \ne T$.

These conditions guarantee that in $U = U_n$ all 3-sums have different norms and no 3-sum and 5-sum have the same norm. This is the requirement in Claim 2.

The proof goes by induction. The first vector u_1 is chosen from $N_\varepsilon(\lambda v_1)$ so that $\ell_e(u_1) \neq 0$ for all $e \in E$. So the forbidden region for u_1 is the union of finitely many lines, and consequently there is a suitable u_1. Assume U_k has been constructed satisfying conditions (1) and (2) and $k \geq 1$.

We start with condition (1). For a fixed pair $e, f \in E$ ($e \neq f$), and for a fixed $S \in \binom{[k+1]}{s}$ and fixed $T \in \binom{[k+1]}{t}$, (1) says something for $u_{k+1} \in N_\varepsilon(\lambda v_{k+1})$ only if $k+1 \in S \cup T$, otherwise it is satisfied by the induction hypothesis. If $k+1$ only appears in S (resp. in T), then (1) says that $\ell_e(u_{k+1}) \neq \alpha$ (and $\ell_f(u_{k+1}) \neq \alpha$) for a particular value of α depending only on e, f, S, T. So the forbidden region is a line $L = L(e, f, S, T)$. When $k+1 \in S \cap T$ then the condition is $\ell_e(u_{k+1}) - \ell_f(u_{k+1}) \neq \alpha$. So the forbidden region is a line again as $\ell_e - \ell_f$ is a non-identically zero linear function.

Checking condition (2) is similar. For a fixed $e \in E$, and for fixed $S \in \binom{[n]}{s}$ and $T \in \binom{[n]}{t}$, condition (2) says something for u_{k+1} only if again $k+1 \in S \cup T$, otherwise it is satisfied by the induction hypothesis. If $k+1 \in S \cap T$, then condition (2) says that $\ell_e(\sigma(S \setminus \{k+1\}, U_{k+1})) \neq \ell_e(\sigma(T \setminus \{k+1\}, U_{k+1}))$. This follows from the induction hypothesis. Finally, if $k+1$ is in $S \setminus T$, condition (2) says that $\ell_e(u_{k+1}) \neq \alpha$ with a particular value of α depending only on e, f, S, T. So the forbidden region is a line, again. The same applies when $k+1 \in T \setminus S$.

As there are finitely many such forbidden lines for u_{k+1}, the Lebesgue measure of the forbidden region is zero. Thus almost all choices of u_{k+1} avoid the forbidden region. $\square$

7. Characterization of central symmetry

Theorem 2 is about a norm whose unit ball is a 0-symmetric convex body B. In the particular case $n = 3$ it says that if a, b, c are unit vectors and their convex hull is separated from 0, then their sum has norm at least 1. The next theorem is a kind of converse.

Theorem 6. *Let $K \in \mathbb{R}^2$ be a convex body with $0 \in \operatorname{int} K$. Then K is centrally symmetric with center at 0 under either one of the following conditions.*

- (i) *For any three distinct vectors $a, b, c \in \partial K$ contained in a closed halfplane whose bounding line goes through 0, the vector $a + b + c \notin \operatorname{int} K$.*
- (ii) *For any three distinct vectors $a, b, c \in \partial K$ with $0 \in \operatorname{int conv}\{a, b, c\}$, the vector $a + b + c \in \operatorname{int} K$.*

Proof of (i). Suppose on the contrary that K is not centrally symmetric. Then we can choose a chord ac (of K) containing 0 with $a + c \neq 0$. Further let b be a vector on ∂K, very close to a, and let bw be the chord which is parallel to ac. It is very easy to see that $h = a + b + c \in \operatorname{relint}(bw)$ if b is close enough to a. This implies that $h \in \operatorname{int} K$, a contradiction (see Fig. 3). $\square$

Proof of (ii). Again, let ac be a chord of K containing 0 such that $a + c \neq 0$ and further, let b be the point on ∂K where the tangent line ℓ at b to K is parallel to ac. We choose

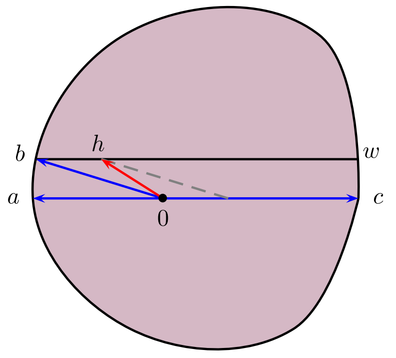

Fig. 3. If K is not centrally symmetric then there are three points $a, b, c \in \partial K$ contained in a closed half-space with $h = a + b + c$ in $\operatorname{int} K$.

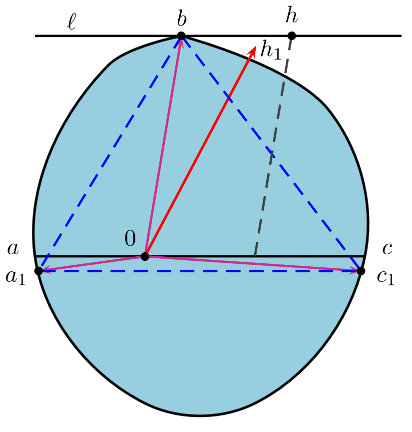

Fig. 4. If K is not centrally symmetric then there are three points $a, b, c \in \partial K$ with $0 \in \operatorname{int}\operatorname{conv} K$ and such that $h = a + b + c$ is not in K.

a and c so that this b is a single point (on either side of the chord ac). This is clearly possible.

This way $h = a+b+c \in \ell$ and consequently h is outside K (see Fig. 4). Now, replace a resp. c, by a_1 and c_1 very close to a and c so that the chord a_1c_1 is parallel to ℓ and so that the line through a and c separates b and a_1, c_1. In this case $0 \in \operatorname{int}(\operatorname{conv}\{a_1, b, c_1\})$. Since the norm of the sum of vectors is a continuous function, we have that $h_1 = a_1 + b + c_1$ is not in $\operatorname{int} K$ provided the line through a_1c_1 is close enough to the chord ac. $\square$

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