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Circles holding typical convex bodies

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Abstract: We prove here that, for most convex bodies, the space of all holding circles has infinitely many components.

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Dedicated to the memory of Professor Francesco S. De Blasi

Can a circle hold a convex body? Yes, it can. Not every convex body, but many. Most of them: it was proved in [4] that those convex bodies which cannot be held by a circle form a nowhere dense family.

What does this holding mean mathematically?

Let $K \subset \mathbb{R}^3$ be a convex body with interior int $K \neq \emptyset$. Consider the space $\mathcal{C}_K$ of all circles in $\mathbb{R}^3$ disjoint from int K, equipped with the Pompeiu-Hausdorff metric, and $\mathcal{C}_K(r)$ the set of all circles in $\mathcal{C}_K$ of fixed radius $r > 0$.

We say that a circle *holds* K if it belongs to a bounded component of $\mathcal{C}_K(r)$ in $\mathcal{C}_K$, for some r. Let $\mathcal{H}_K \subset \mathcal{C}_K$ denote the space of all holding circles for the convex body K. For $\mathcal{H}' \subset \mathcal{H}_K$, let $r(\mathcal{H}')$ denote the *range of* $\mathcal{H}'$, that is the set of radii of all circles in $\mathcal{H}'$.

The midpoints of the edges of a hamiltonian cycle in the 1-skeleton of a regular tetrahedron T are the vertices of a square. The circle circumscribed to that square and many circles slightly larger than, but close to, it hold the tetrahedron. There are three such squares, and each corresponding circle belongs to a different component of $\mathcal{H}_T$.

H. Maehara [3] proved that, for the regular icosahedron I, $r(\mathcal{H}_I)$ has 2 components, and exhibited a polytope P for which $r(\mathcal{H}_P)$ has 3 components. In [1] we showed that there exist convex bodies K, for which $r(\mathcal{H}_K)$ has infinitely many components.

Here, we study the generic case. Let $\mathcal{K}$ be the Baire space of all convex bodies in $\mathbb{R}^3$. A convex body is *typical* as soon as it enjoys a property **P** such that the

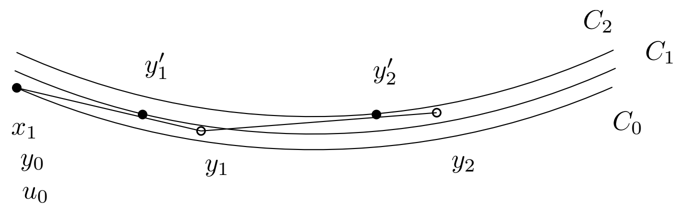

Figure 1:

set of all convex bodies not enjoying **P** is of the first Baire category in $\mathcal{K}$. Then we also say that *most* convex bodies enjoy the *generic* property **P**. For generic results in Convex Geometry, see [2], [6].

Theorem 1. *Let n be a natural number. For all convex bodies K, except for a nowhere dense family, $\mathcal{H}_K$ has at least n components, the ranges of which are components of $r(\mathcal{H}_K)$.*

Proof. A regular polygon $P_0 = x_1 x_2 ... x_m$ is inscribed in the unit circle C_0, which lies in the plane $x\mathbf{0}y$ and has centre $\mathbf{0}$. Consider also the circle $C_i \subset x\mathbf{0}y$ of radius $1 - i\varepsilon$ and centre $\mathbf{0}$ $(i = 1, ..., n)$. Put $y_0 = x_1$. Choose $y_i \in x\mathbf{0}y$ such that $y_{i-1}y_i$ be tangent to C_i and $\|y_i - y'_i\| = \nu$, where $\{y'_i\} = C_i \cap y_{i-1}y_i$ $(i = 1, ..., n)$, see Figure 1.

If ν and ε are small enough, $\|y_i\| < 1 - (i-1)\varepsilon$ $(i = 1, ..., n)$ and $\angle x_1 \mathbf{0} y_n \leq \pi/m$.

Choose a large $t \in \mathbb{R}$, and put $v = (0, 0, t)$. Consider planes Π_i given by $z = \delta i$ for some $\delta > 0$.

Denote by p the orthogonal projection onto $x\mathbf{0}y$. The cone with apex v and basis C_0 cuts Π_i along a circle C'_i, and δ will be chosen so that $pC'_i = C_i$ $(i = 1, ..., n)$. As δ depends linearly on ε, it can be made arbitrarily small.

There are points u_i and $u'_i \in u_{i-1}u_i$ such that $u_0 = x_1$, $pu_i = y_i$ and $u'_i \in C'_i$ $(i = 1, ..., n)$.

Let ω be the rotation of angle $2\pi/m$ around $\mathbf{0}z$ (for which $\omega(x_1) = x_2$), and put

$$\Omega(u) = \{u, \omega(u), \omega^2(u), ..., \omega^{m-1}(u)\}.$$

Then $\mathrm{conv}\Omega(x_1) = P_0$, $\mathrm{conv}\Omega(y'_i)$ is another regular polygon in the plane Π_i, and $\mathrm{conv}\Omega(y_i)$ is a third regular polygon.

Consider the polytope $Q = \mathrm{conv} \cup_{i=0}^n \Omega(u_i)$. The circle C'_i holds Q, for each i. Its component in $\mathcal{H}_Q$ contains circles of radii filling a small interval $[1 - i\varepsilon, \alpha_i[$. Notice that

$$\alpha_i < \|y_i\| < 1 - (i-1)\varepsilon.$$

The first inequality follows from results in [5]. For our purpose here its non-strict version, which is obvious, suffices.

Thus, the total range of these n components $\mathcal{H}_1, ..., \mathcal{H}_n$ of the space $\mathcal{H}_Q$ has itself n components.

Note that Q has two facets that are regular m-gons (on the top and bottom level), and all other facets are triangles of the form $u_i u_{i+1} \omega(u_i)$ or $u_{i+1}\omega(u_i)\omega(u_{i+1})$ and their images under rotations ω^j. Moreover, none of the facets is parallel with the line $\mathbf{0}z$.

We are going to show that for all convex bodies B, except for a nowhere dense family, $r(H_B)$ has at least n components.

Let $\mathcal{O}$ be an open set in $\mathcal{K}$, and let $P \in \mathcal{O}$ be a polytope with vertex set V which we suppose to be in general position, for instance the coordinates of the vertices are algebraically independent. (Weaker general position assumptions would also work for our purposes.) For each triple $a, b, c \in V$ let $R(a, b, c)$ be the radius of the smallest disk $D(a, b, c)$ in the (unique) plane $S(a, b, c)$ containing a, b, c. The number $R(a, b, c)$ is uniquely defined by the triple a, b, c.

Choose a triple $a^*, b^*, c^* \in V$ with $R(a^*, b^*, c^*) = \max\{R(a, b, c) : a, b, c \in V\}$. There are two cases here:

(1) $a^*b^*c^*$ is an acute triangle and $D(a^*, b^*, c^*)$ is its circumscribed disk. The general position of V guarantees that this triple is unique.

(2) $a^*b^*c^*$ is obtuse and then $R(a^*, b^*, c^*)$ is half of the longest edge of $a^*b^*c^*$, say a^*b^*. The general position of V ensures again that a^* and b^* are unique.

Now let p be the orthogonal projection on the plane $S(a^*, b^*, c^*)$, and define the cylinder U as $p^{-1}D(a^*, b^*, c^*)$. It is easy to check that U contains int$P$ and $P \cap \mathrm{bd}U$ equals $\{a^*, b^*, c^*\}$ in case (1) and $\{a^*, b^*\}$ in case (2). Let C be the circle bounding the disk $D(a^*, b^*, c^*)$.

Now consider a polytope $L \in \mathcal{O}$ approximating P with the following properties:

(*) in case (1), $L \cap C = \{a^*, b^*, c^*\}$,

(**) in case (2), $L \cap C$ consists of a^* plus two vertices a^+ and a^-, symmetrical with respect to the line through a^*, b^* and close to b^*

(***) all vertices of L except for a^+ and a^- in case (2) lie in P.

At this point, every circle from $\mathcal{C}_P$ close to C can slip out of P. Because P passes through circles smaller than C (see [5]), this property is preserved for L close enough to P.

Now remember the circle C_0 considered above. Identify, after scaling, C_0 with C and take three arcs A_l, A_2, A_3 in C containing respectively a^*, b^*, c^* in case (1), and a^*, a^+, a^- in case (2).

Remark that the two definitions of p now agree with each other.

We consider the polytopes

$$Q' = Q \cap p^{-1}\mathrm{conv}(A_1 \cup A_2 \cup A_3)$$

and

$$W = \operatorname{conv}(L \cup Q').$$

The arcs A_i can be chosen arbitrarily short. Then ε can be chosen so small and m so large that Q' is arbitrarily thin and still has lots of triangular faces close to each of the arcs A_i. Thus we can achieve that $W \in \mathcal{O}$. Clearly, Q' has the same holding circles as Q. From (***) it follows that no facet of L meeting C is parallel to $\mathbf{0}z$. Let β be the minimal angle between such a facet and $\mathbf{0}z$. Thus, for r large enough, all triangular facets of Q' make with $\mathbf{0}z$ an angle less than β, and are therefore facets of W.

Hence, $\mathcal{H}_1, ..., \mathcal{H}_n$ are components of $\mathcal{H}_W$ with pairwise disjoint ranges.

As each circle in $\cup_{i=1}^n \mathcal{H}_i$ is close to C, no $r(\mathcal{H}_i)$ interferes with the range of any other component of $\mathcal{H}_W$. Thus, $r(\mathcal{H}_1)$,..., $r(\mathcal{H}_n)$ are components of $r(\mathcal{H}_W)$.

An easy lower semicontinuity argument shows that, for each convex body K in a whole neighbourhood of W, $\mathcal{H}_K$ will admit at least n components with pairwise disjoint ranges and, moreover, $r(\mathcal{H}_K)$ has at least n components. □

Theorem 2. *For most convex bodies K, $r(\mathcal{H}_K)$ has infinitely many components. Consequently, $\mathcal{H}_K$ has infinitely many components, too.*

This follows easily from Theorem 1.

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