

Holding Circles and Fixing Frames

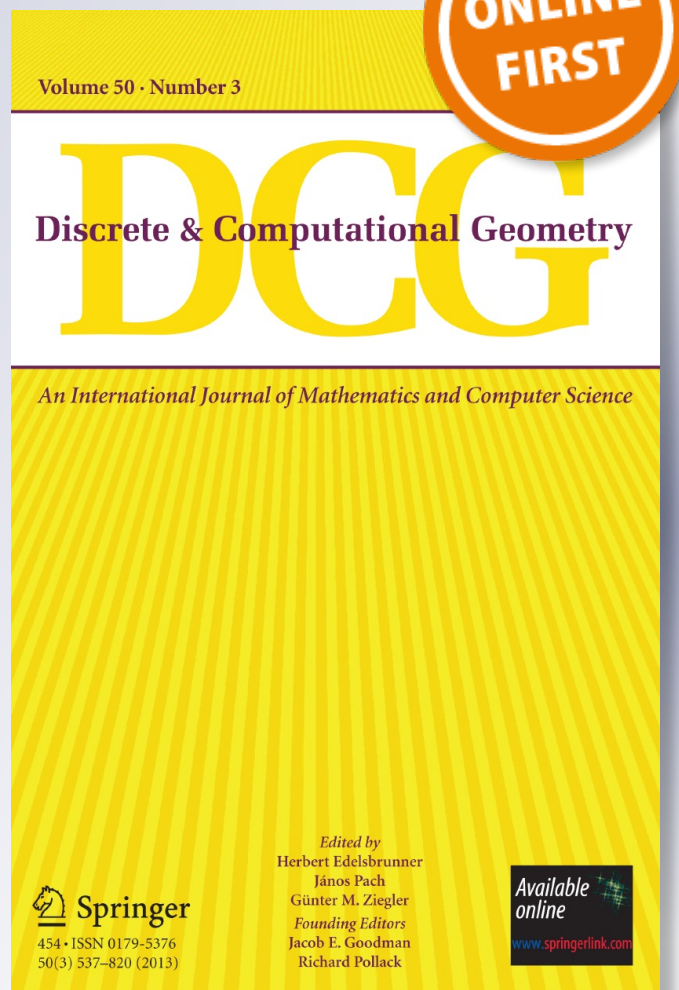
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Holding Circles and Fixing Frames

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Abstract A circle C holds a convex body $K \subset \mathbb{R}^3$ if K can't be moved far away from its position without intersecting C . One of our results says that there is a convex body $K \subset \mathbb{R}^3$ such that the set of radii of all circles holding K has infinitely many components. Another result says that the circle is unique in the sense that every frame different from the circle holds a convex body K (actually a tetrahedron) so that every nontrivial rigid motion of K intersects the frame.

Keywords Frames · Convex bodies · Fixing and holding

Mathematics Subject Classification Primary 52A15 · Secondary 15B40

1 Circles Holding Convex Bodies

Can a circle hold a convex body? Yes, it can. Not every convex body, but many, most of them: it is proved in [8] that those convex bodies which cannot be held by a circle form a nowhere dense family.

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What does this holding mean, mathematically?

Let $K \subset \mathbb{R}^3$ be a convex body with interior $\text{int } K \neq \emptyset$. Consider the space \mathcal{C} of all circles in \mathbb{R}^3 disjoint from $\text{int } K$, equipped with the Pompeiu–Hausdorff metric, and $\mathcal{C}(r)$ the set of all circles in \mathcal{C} of fixed radius $r > 0$.

We say that a circle *holds* K if it belongs to a bounded component of $\mathcal{C}(r)$ for some r .

We say that a holding circle C *fixes* K if the only possible movement of C in \mathcal{C} is a rotation which keeps C fixed.

The midpoints of the edges of a hamiltonian cycle in the 1-skeleton of a regular tetrahedron are the vertices of a square. The circle circumscribed to that square fixes the tetrahedron, and many circles slightly larger than but close to it hold it. Assuming the tetrahedron T has edge-length 1, it can be held by circles of diameter in the interval $[1/\sqrt{2}, 0.896\dots]$, a result of Itoh et al. [4]. There are three circles of diameter $1/\sqrt{2}$, each of which fixes T . And T can pass through circles of diameter at least $0.896\dots$. Remark that the (circular) section of the *circumscribed cylinder of T* (this is the infinite circular cylinder of smallest section surrounding $\text{int } T$) has diameter 1.

The set $H(K)$ of all diameters of circles holding K is called the *holding range* of K . Put

$$d_h(K) = \inf H(K), \quad d^h(K) = \sup H(K).$$

Further, let $d_p(K)$ be the lower bound of all diameters of circles through which K can pass, and let $d_c(K)$ denote the diameter of the (normal) section of the circumscribed cylinder.

It seems quite reasonable to think that in general there exist circles of diameter $d_h(K)$ fixing K , and

$$d^h = d_p < d_c,$$

as in the case of the regular tetrahedron.

However, in fact the situation is often different. Generally true are only the trivial inequalities directly derived from the definitions.

The number $d_p(K)$ is, for various K , rather independent of $d_h(K)$ and $d^h(K)$. It can be smaller than $d_h(K)$, larger than $d^h(K)$, or in between.

While, indeed, for most convex bodies K , $d_p(K) < d_c(K)$ (this was proved by the second author [9]), there are still lots of them for which equality holds.

Fruchard [3] proved that, while $d_h(K)$ can be less than the width $w(K)$ of K , we always have $d_h(K)/w(K) > 2/3$. He also showed [3] that $d^h(K)/d_c(K)$ can be arbitrarily small (which comes close to our Theorem 3).

The set $H(K)$ of all diameters of circles holding K , the holding range of K , may be disconnected. Maehara [7] showed that the holding range of the regular icosahedron has two components. He also constructed an octahedron whose holding range has three components, and asked whether the number of components can be arbitrarily large [7].

Theorem 1 *There are convex bodies K such that $H(K)$ has arbitrarily many components.*

The proof will be clear after reading the proof of Theorem 2.

Let ab denote the line through $a, b \in \mathbb{R}^3$, let ab denote the line-segment from a to b , and let $]ab[$ denote ab deprived of its endpoints. The convex hull, interior, and boundary of $S \subset \mathbb{R}^3$ are denoted by $\text{conv}S$, $\text{int}S$, and ∂S respectively. If $a_1, \dots, a_k \in \mathbb{R}^3$, we also write $a_1a_2 \dots a_k$ for the convex hull of the set $\{a_1, \dots, a_k\}$.

Theorem 2 *There are convex bodies K for which $H(K)$ has infinitely many components.*

Proof Let $\Delta_0, \Delta_1, \Delta_2, \dots$ be consecutive half-lines in the plane xOy , starting at O , such that their angles $\angle \Delta_i \Delta_j$ satisfy

- (1) $\angle \Delta_0 \Delta_1 = \pi/24$,
- (2) $\angle \Delta_i \Delta_{i+1} = \pi/2^{i+2}$ ($i = 1, 2, \dots$).

Let a_1 be a point on the trisector of the angle $\Delta_1 \Delta_2$ closer to Δ_1 , and consider the orthogonal projections b_1, b_2 of a_1 onto Δ_1, Δ_2 , respectively. Let a_0 be the intersection of the line a_1b_1 with Δ_0 , and a_2 the intersection of a_1b_2 with the trisector of the angle $\Delta_2 \Delta_3$ closer to Δ_2 (see Fig. 1).

The obvious iteration leads to the sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$, both convergent to the same point a on the half-line Δ , limit of $\{\Delta_n\}_{n=1}^\infty$. $\angle \Delta_0 \Delta = (\pi/4) + (\pi/24)$. It follows from the construction that $a_{i+1} < b_i < a_i$.

Finally, let c be the point of coordinates $(0, 0, \varepsilon)$ for a small $\varepsilon > 0$.

We define our convex body

$$K = \text{conv}\{a_0, a_1, a_2, \dots, a, c, -c\}.$$

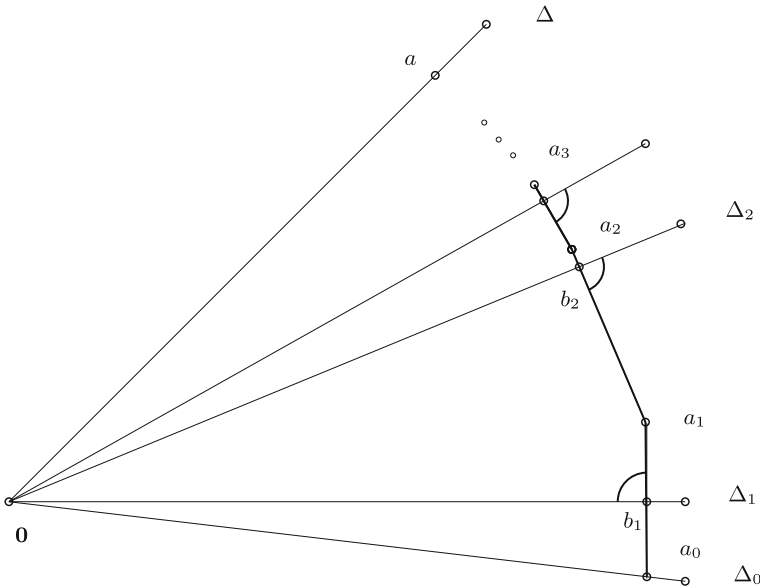


Fig. 1 The construction for Theorem 2

Each circle on which $\mathbf{0}$ and b_i are diametrically opposite, and whose plane makes an angle of $\pi/4$ with $x\mathbf{0}y$, holds K ($i = 1, 2, \dots$). Close to it there are circles of diameters in $[\|b_i\|, \|a_i\|]$ holding K . Circles with diameter in $[\|a_{i+1}\|, \|b_i\|]$ do not hold K .

The circles with centres close to $\mathbf{0}$ which hold K have diameter close to 2ε , smaller than $\|a_i\|$. Thus, $[\|b_i\|, \|a_i\|]$ are connected components of $H(K)$ for all $i \geq 1$.

This proves the theorem. □

Theorem 3 *There are convex bodies K with $d^h(K)/d_p(K)$ as small as wished.*

Proof Choose $a, b \neq \mathbf{0}$ in the plane $x\mathbf{0}y$ such that $\pi/2 < \angle \mathbf{0}ab < \pi$, and take $c = (0, 0, \varepsilon)$, for small $\varepsilon > 0$ and set $c' = -c$. Then

$$K = \text{conv}\{a, b, c, c'\}$$

is an appropriate convex body. Indeed, all circles holding K have centres close to $\mathbf{0}$, and $d^h(K) = 2\varepsilon$. On the other hand, $d_p(K)$ equals the minimal value v of the diameter of the circle circumscribed to axx' , where x, x' are running on bc , respectively bc' , and v is larger than the height, h , of the triangle $\mathbf{0}ab$ at a . Thus, $d^h(K)/d_p(K) < \varepsilon/h$ can be made arbitrarily small. □

Theorem 4 (a) *There are convex bodies K with $d_c(K)/d_h(K)$ as small as wished.*
 (b) *There are convex bodies admitting holding circles, but admitting no fixing circle.*

Proof Let $\varepsilon > 0$ be small, and C_i be the circles

$$y^2 + z^2 = 4\varepsilon^2$$

in the plane $x = i$. Consider the segment σ connecting the points $(3, 0, \varepsilon)$ and $(3, 0, -\varepsilon)$, and the segment τ connecting $(0, \varepsilon, 0)$ and $(0, -\varepsilon, 0)$. The convex body

$$K = \text{conv}(\tau \cup \sigma \cup C_1 \cup C_2)$$

obviously has $d_h(K) = 3$ and $d_p(K) = d_c(K) = 4\varepsilon$.

This K has holding circles, but no fixing circles. □

We saw in the preceding example that a convex body K with holding circles may have no fixing circle, but the holding circles of minimal diameter allow only rotations of K .

A holding circle C fixes K up to rotations, if the only possible movements of K are rotations fixing the sphere circumscribed to C . The natural question arises, whether the existence of holding circles yields the existence of fixing circles up to rotations.

Theorem 5 *There are convex bodies admitting holding circles but admitting no fixing circle up to rotations.*

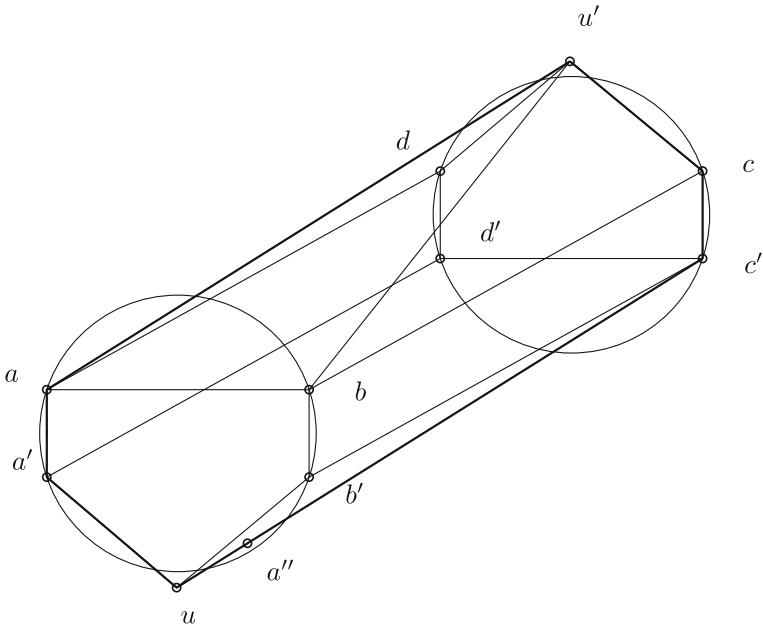


Fig. 2 The construction for Theorem 5

Proof Let $abcd a' b' c' d'$ be a box (a parallelepiped with all angles right), with centre $\mathbf{0}$, opposite faces $abcd$ and $a' b' c' d'$, long edges $bc, da, b' c', d' a'$, and very short edges aa', bb', cc', dd' (see Fig. 2). Let C be the circle circumscribed to $abb'a'$. Choose u in the plane Π of C , outside of C but close to C , such that $\|u - a\| = \|u - b\| > \|u - a'\|$. Set $u' = -u$ and consider

$$K' = \text{conv}\{a, b, c, d, a', b', c', d', u, u'\}$$

and the ball B of centre $\mathbf{0}$ and radius $\|u\|$.

Let \mathcal{H} be the family of all closed half-spaces containing K' and bounded by the planes of all faces of K' except $abb'ua'$ and $c'd'du'c$. Then

$$K = B \cap \bigcap_{H \in \mathcal{H}} H$$

is a suitable convex body. This can be seen as follows.

Let Z be the bounded cylinder $\partial \text{conv}(C \cup -C)$. Let a'' be the intersection of Z with uc' .

Obviously, all planes parallel to Π and separating a'' from $-a''$ meet Z along circles, of diameter $d_h(K) = \text{diam}C$, which hold but do not fix K .

The spherical parts of ∂K prohibit the existence of other components of $H(K)$. Thus, no circle fixes K . □

2 Frames Fixing Convex Bodies

A frame is a planar closed convex curve. The notions of holding or fixing can be easily generalized from the circle to any frame.

We noticed that a circle cannot fix any convex body K in the sense that K doesn't move at all; so, we allowed rotation in the definition for the circle. Is only the circle in this situation? No, Maehara, Tokushige and the first author [1] showed that no triangle can hold any convex body, and a fortiori it cannot fix it. A more detailed analysis follows.

A fixed plane $P \subset \mathbb{R}^3$ and a convex compact set $F \subset P$, whose boundary ∂F is our frame, are given. We always assume that $\text{int} F$ is nonempty. A *rigid motion* is a continuous map $M : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$ with $M(\cdot, 0) = I$ (the identity) such that $M_t = M(\cdot, t)$ is a congruence of \mathbb{R}^3 for all $t \in [0, 1]$. So a rigid motion M_t is a homotopy (with congruences) between the identity I and M_1 . Given a convex body $K \subset \mathbb{R}^3$, the rigid motion M_t moves K through F if $(M_t K) \cap P \subset F$ for all $t \in [0, 1]$. This definition models the case when F is a hole in the wall P and one wants to move K through the hole, and also, when one wants to see that K is stuck in the hole.

Thus, the frame ∂F holds K if no rigid motion M_t that moves K through F satisfies $M_1 K \cap P = \emptyset$. The frame *fixes* K if every rigid motion moving K through F satisfies $M_t P = P$ for all $t \in [0, 1]$. If F fixes K then, of course, it holds K as well. The following result is known.

Theorem 6 [1]. *Every frame that is not a triangle fixes some tetrahedron.*

In the definition of fixing we did not require that M_t equal the identity because, for instance, even if a circle fixes some convex body K , there is a rigid motion M_t (the one that rotates the disk around its centre) with $(M_t K) \cap P \subset F$ for all $t \in [0, 1]$ which is not the identity. We are going to show that the circle is, essentially, the only exception. We need one more definition: We say that a convex body $K \subset \mathbb{R}^3$ is *fixed without motion* by the frame ∂F if every rigid motion M_t of K through F satisfies $M_t = I$ for all t .

Theorem 7 *If F is neither a triangle nor a circular disk, then some tetrahedron $T \subset \mathbb{R}^3$ is fixed without motion by F .*

Some preparations are needed before the proof. Let $f : F \times F \rightarrow \mathbb{R}$ be the Euclidean distance function, that is, $f(x, y) = \|x - y\|$, restricted to F .

The proof uses the following two ingredients of the proof of Theorem 6.

Lemma 1 *Assume that f has a local maximum at (a, b) , that $ab \subset \text{int} F$, and that $c, d \in \partial F$ are on opposite sides of \overline{ab} . Then there is a tetrahedron T that is fixed by ∂F and satisfies $T \cap P = abcd$.*

Although such a T need not be unique, the proof of the lemma constructs it explicitly. We will denote it by $T = T(a, b; c, d)$.

Lemma 2 *Assume ab is a diameter of F , $ab \subset \partial F$, and the function $f_a(x) = f(a, x)$ is non-decreasing as x moves from a to b on $\partial F \setminus ab$. Then F fixes a tetrahedron T without motion.*

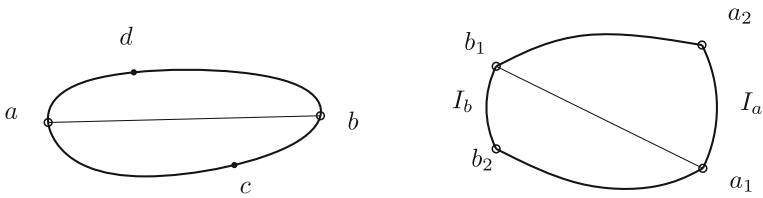


Fig. 3 Case 1 and 2 of Theorem 7

Proof of Theorem 7 Assume first that F has a diameter ab such that $]ab[\subset \text{int} F$. Let I_a be an arc on ∂F containing a and maximal with respect to the property that for every $x \in I_a$ there is $y \in F$ such that xy is a diameter of F . Of course, I_a may consist of a single point (namely a) and may coincide with ∂F , in which case F is of constant width. The arc I_b is defined analogously.

Case 1 When $I_a = \{a\}$. Choose $c, d \in \partial F \setminus ab$ from opposite sides of ab . As (a, b) is a local maximum of f , Lemma 1 applies and the tetrahedron $T(a, b; c, d)$ is fixed by ∂F . Now let M_t be a continuous rigid motion with $M_0 = I$ and $(M_t T) \cap P \subset F$ for all $t \in [0, 1]$. Then $M_t P = P$ and $M_t a = a$ for all t , as M_t maps a diameter of F to another diameter. Then M_t is a rotation with centre a , and it follows easily that $M_t = I$ for all t (see Fig. 3 left).

Case 2 When neither I_a nor I_b is a single point or the whole ∂F . Then I_a , resp. I_b , is an arc from a_1 to a_2 , resp. b_1, b_2 , on ∂F . We choose the notation so that a_1, a_2, b_1, b_2 are in this order on ∂F . It is easy to see that $a_1 b_1$ and $a_2 b_2$ are diameters of F . Thus, Lemma 1 applies again and gives a tetrahedron $T(a_1, b_1; a_2, b_2)$ which is fixed by F . Any rigid motion M_t (as in the definition of fixing) would move the endpoints of $a_1 b_1$ or $a_2 b_2$ outside I_a or I_b . Thus M_t must be identity (see Fig. 3 right).

Case 3 When $I_a = I_b = \partial F$, i.e. F is of constant width, then for every $x \in \partial F$ there is $y \in \partial F$ such that xy is a diameter of F . Let B be the unique smallest area Euclidean disk containing F ; assume its centre is $\mathbf{0}$. Set $H = \partial F \cap \partial B$. It is well known that H is nonempty and $\mathbf{0} \in \text{conv} H$. If H contains a pair of antipodal points then F and B coincide. Then F would be a circular disk, which is excluded by the hypothesis of the theorem.

So H contains no antipodal pair of points. Then there are points $a, b, c \in H$ with $\mathbf{0} \in abc$. Let aa^*, bb^*, cc^* be diameters of F .

Assume that the diameter aa^* does not separate the points b and c , meaning that they are on the same side of the diameter aa^* . Let $I(a, b)$ (resp. $I(a, c)$) be the arc on ∂F between a and b not containing c (and between a and c not containing b). Thus, a^* lies in $I(a, b)$ or $I(a, c)$. It lies on $I(a, c)$, say. As any two diameters have a common point, bb^* and aa^* intersect, implying that $b^* \in I(a, c)$. Hence, a and c are separated by bb^* .

We showed that either aa^* separates b and c or bb^* separates a and c . For simpler writing, we assume that bb^* separates a and c . Now Lemma 1 applies since (b, b^*) is a local maximum of f and a, c are on opposite sides of bb^* . Thus, the tetrahedron $T(b, b^*; a, c)$ is fixed by F .

Assume there is a continuous rigid motion M_t , as in the definition. Then $M_t(abc) \subset F \subset B$. The triangles abc and $M_t(abc)$ are congruent, so B is the minimal area disk containing $M_t(abc)$. Then $M_t B = B$ and consequently M_t is a rotation with centre $\mathbf{0}$.

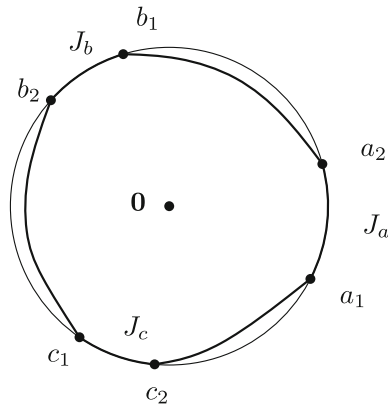


Fig. 4 Case 3 of Theorem 7

Let $J_a, J_b, J_c \subset H$ be maximal arcs containing a, b, c , respectively. If any of these arcs is a single point, then $M_t = I$. So assume none of them is a single point, and write a_1, a_2 resp. b_1, b_2 and c_1, c_2 for the endpoints of J_a, J_b, J_c , so that the points $a_1, a_2, b_1, b_2, c_1, c_2$ come in this order on ∂F , see Fig. 4. Lemma 1 gives the tetrahedron $T(b, b^*; a_2, c_1)$. Now let N_t be the rigid motion with $N_0 = I$ and $(N_t T) \cap P \subset F$. As we have seen before, N_t is a rotation with center $\mathbf{0}$. But if $N_t \neq I$ then either $N_t a_2$ moves outside J_a or $N_t c_1$ moves outside J_c , which is impossible. This finishes Case 3 and, further, all cases when a diameter of F does not lie on its boundary.

Now let $ab \subset \partial F$ be a diameter of F . If $f_a(x)$ is non-decreasing (in the sense of Lemma 2), then this lemma guarantees the existence of a tetrahedron T that is fixed by F without motion. Assume now that $f_a(\cdot)$ has a local maximum at $c \in \partial F \setminus ab$.

Case 4 When $]ac[\subset \partial F$. Then, as it is easy to check, (c, b) is a local maximum of f . Note that $]cb[\subset \text{int} F$, as otherwise F is a triangle. Choose a point d from the arc connecting c, b and not containing a . Lemma 1 applies and gives a tetrahedron $T(b, c; a, d)$ fixed by ∂F . It is clear that this fixing is without motion: the triangle abc has only one congruent copy contained in F , namely abc .

Case 5 When $]ac[\subset \text{int} F$. Choose d from the arc connecting a, c on ∂F which is not containing b . Lemma 1 applies again: (a, c) is a local maximum of f and b, d are on opposite sides of ac . The tetrahedron $T = T(a, c; b, d)$ is fixed by ∂F . If this fixing goes with a rigid motion M_t then $M_t(ab)$ is a diameter of F . As any two diameters of F intersect, and the diameter ab lies on the boundary of F , $M_t a = a$ or $M_t b = b$ holds for every $t \in [0, 1]$, and M_t is a rotation with centre a or b . It follows again that $M_t = I$. □

A frame ∂F fixes tightly a convex body K , if it fixes it and lies on its boundary. Note that this does not imply that the frame fixes K without motion: let K be the convex hull of the union of the regular tetrahedron $abcd$ with the smallest disk D circumscribed to the square with vertices $(a + b)/2, (a + c)/2, (d + b)/2, (d + c)/2$. The frame ∂D fixes K (it even fixes the regular tetrahedron) but not without motion. The following theorem gives a positive answer to Problem 2 from [1].

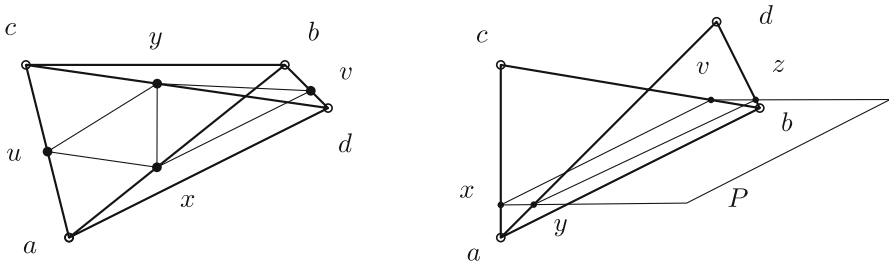


Fig. 5 Case 1 and 2 of Theorem 8

Theorem 8 Every tetrahedron is tightly fixed by some frame.

Proof Let a, b, c, d be the vertices of T , and ab be the longest edge of T (or one of the longest ones). We distinguish two cases.

Case 1 The angle between the faces abc and abd of T is at least $\pi/2$.

Case 2 The angle between the faces abc and abd of T is less than $\pi/2$.

Case 1 is simpler. The shortest distance between the lines \overline{ab} and \overline{cd} is reached on a line-segment xy with $x \in ab$ and $y \in cd$, as one can easily check (this is the point where we need the large angle). Consider the circular cylinder with radius ρ whose axis is the line \overline{xy} . For small $\rho > 0$ this cylinder is disjoint from the broken path through a, c, b, d, a . So there is a smallest $\rho > 0$ when they have a common point, say u . So u is (one of) the point(s) on the broken path which is closest to the segment xy (ties broken arbitrarily). It is clear that u does not coincide with either of the points a, b, c, d . We can assume without loss of generality that $u \in ac$. Let P be the plane spanned by x, y, u . P intersects T in a quadrilateral $Q = T \cap P$ with vertices x, y, u, v where $v \in bd$. We claim that, as a frame, ∂Q fixes T .

Fact. (x, y) is a strict local maximum of the distance function $f : Q \times Q \rightarrow \mathbb{R}$. (This is trivial.)

Assume that M_t is a rigid motion of \mathbb{R}^3 with $P \cap M_t T \subset Q$ for all $t \in [0, t_0]$ where $t_0 > 0$. The Fact implies that $M_t(xy) = xy$. Consequently, M_t is a rotation around the line \overline{xy} . It follows that $M_t(xyu)$ is not contained in xyu unless M_t is the identity, as $M_t u \in ac$, and every point of ac , including $M_t u$, is farther from xy than u . It follows that $M_t Q$ cannot be contained in Q unless M_t is the identity (see Fig. 5 left).

In *Case 2* choose a plane P parallel with ab , perpendicular to abc , and close to ab . (How close will be clear later.) Let P intersect the edges ac, ad, bd, bc respectively in points x, y, z, v . (The existence of y, z follows from the assumption that the angle is smaller than $\pi/2$.) We show that the frame ∂Q , with $Q = xyzv$, fixes T (see Fig. 5 right). Note that Q is a trapezoid contained in the triangle Δ^* with vertices $a^*b^*d^*$ (where a^*, b^*, d^* are the projections of a, b, d onto P), and that the angles of Q at z and y are obtuse.

Suppose again that a rigid motion M_t of \mathbb{R}^3 moves T , with $P \cap M_t T \subset Q$ for all $t \in [0, t_0]$.

Set $Q_t = x_t y_t z_t v_t = M_t^{-1} P \cap T$. Thus, Q_t fits into Q . Let $Q^t = x^t y^t z^t v^t$ be the orthogonal projection of Q_t onto the plane P . By Kovalyov's result in [6], (see also Kós and Törőcsik [5] or Debrunner and Mani-Levitska [2]), Q^t also fits into Q . Then Q^t fits into the triangle Δ^* , since $Q \subset a^*b^*d^*$.

Note that, given $\varepsilon > 0$, we can choose the plane P so close to ab that $|x - a|, |v - a|, |y - b|, |z - b| < \varepsilon$, which implies that $|x - a^*|, |v - a^*|, |y - b^*|, |z - b^*| < \varepsilon$. Further, given any $\delta > 0$, for all small enough $t > 0$ the angles of Q_t at z_t and v_t are obtuse, and $|x - x_t|, |y - y_t|, |z - z_t|, |v - v_t| < \delta$.

Fact. If $\varepsilon, \delta > 0$ are small enough, then Q^t fits into $a^*b^*d^*$ in a unique way, that is, there is a unique congruent copy of Q^t contained in $a^*b^*d^*$, namely Q^t .

The proof is a direct 2-dimensional argument in the plane P . As x_t, y_t, z_t, v_t resp. are on edges ac, bd, bc, ad of T , their projections x^t, y^t, z^t, v^t resp. are on edges $a^*d^*, b^*d^*, b^*c^*, a^*d^*$ of Δ^* .

Take a congruent copy of Q^t contained in $a^*b^*d^*$, Q^1 say, with vertices x^1, y^1, z^1, v^1 . Apply a parallel translation to Q^1 towards a^*b^* by a translation orthogonal to a^*b^* as long as the translated copy of Q^1 lies in Δ^* . Let Q^2 be the final position of Q^1 in this translation, with vertices x^2, y^2, z^2, v^2 . Then either $x^2 \in a^*b^*$ or $v^2 \in a^*b^*$ because a^*b^* is the longest edge of Δ^* and Q^1 is very close to $[a^*, b^*]$ when ε and t are small enough.

By symmetry we can assume $x^2 \in a^*b^*$. Now rotate Q^2 around its vertex x^2 so that v^2 moves towards a^*b^* as long as the rotated copy of Q^2 is contained in Δ^* . So we stop when v^2 reaches a^*b^* or z^2 reaches b^*d^* or when y^2 reaches d^*a^* , whichever happens first. As the angles of Q^2 at vertices z^2 and y^2 are obtuse, the points z^2 resp. y^2 move away from the edges b^*d^* and d^*a^* during the rotation. So we stop when v^2 reaches a^*b^* . Let Q^3 be the final position of Q^2 during this rotation, its vertices are x^3, y^3, z^3, v^3 .

It follows that $Q^3 \subset \Delta^*$ and $x^3, v^3 \in a^*b^*$. Also, $Q^t \subset \Delta^*$ and $x^t, v^t \in a^*b^*$ (for small enough t). But Q^3 is congruent to Q^t , and their vertices are on the same edges of Δ^* . So they coincide. Then there was no rotation, and there was no translation, so $Q^1 = Q^2 = Q^3 = Q^t$. \square

The Fact shows that Q^t fits into Q in a unique way as well. Then $z^t = z$ because z^t resp. z is the unique common point of Q^t and Q with $[b^*d^*]$. Then $z_t = z^t = z$. Similarly $y_t = y^t = y$, implying that M_t is a rotation around the axis \overline{xy} .

Finally, P is perpendicular to abc , so the width of $T \cap M_t^{-1}P$ is larger than that of $xyuv$, unless M_t is the identity.

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