

# COLOURFUL AND FRACTIONAL $(p, q)$ -THEOREMS

I. BÁRÁNY, F. FODOR, L. MONTEJANO, D. OLIVEROS, AND A. PÓR

ABSTRACT. Let  $p \geq q \geq d+1$  be positive integers and let  $\mathcal{F}$  be a finite family of convex sets in  $\mathbb{R}^d$ . Assume that the elements of  $\mathcal{F}$  are coloured with  $p$  colours. A  $p$ -element subset of  $\mathcal{F}$  is heterochromatic if it contains exactly one element of each colour. The family  $\mathcal{F}$  has the heterochromatic  $(p, q)$ -property if in every heterochromatic  $p$ -element subset there are at least  $q$  elements that have a point in common. We show that, under the heterochromatic  $(p, q)$ -condition, some colour class can be pierced by a finite set whose size we estimate from above in terms of  $d, p$ , and  $q$ . This is a colourful version of the famous  $(p, q)$ -theorem. (We prove a colourful variant of the fractional Helly theorem along the way.) A fractional version of the same problem is when the  $(p, q)$ -condition holds for all but an  $\alpha$  fraction of the  $p$ -tuples in  $\mathcal{F}$ . We show that, in the case that  $d = 1$ , all but a  $\beta$  fraction of the elements of  $\mathcal{F}$  can be pierced by  $p - q + 1$  points. Here  $\beta$  depends on  $\alpha$  and  $p, q$ , and  $\beta \rightarrow 0$  as  $\alpha$  goes to zero.

## 1. INTRODUCTION

Helly's theorem states that if  $\mathcal{F}$  is a finite family of convex sets in  $\mathbb{R}^d$  such that every at most  $(d + 1)$ -element subfamily of  $\mathcal{F}$  has nonempty intersection, then the whole family  $\mathcal{F}$  has nonempty intersection. The condition can be relaxed leading to the so-called  $(p, q)$ -condition of Hadwiger and Debrunner [7] and the conclusion varies accordingly: Assuming  $p \geq q \geq d + 1$ , the family  $\mathcal{F}$  has the  $(p, q)$ -property if among every  $p$  elements of  $\mathcal{F}$  there are  $q$  with nonempty intersection. For example, in Helly's theorem the family of convex sets satisfies the  $(d + 1, d + 1)$ -condition in  $\mathbb{R}^d$ .

A set of points with the property that every element of  $\mathcal{F}$  contains at least one of the points is said to *pierce*  $\mathcal{F}$ . The minimum number of points that can pierce  $\mathcal{F}$  is called the *piercing number* of  $\mathcal{F}$ , and is denoted by  $\tau(\mathcal{F})$ .

Hadwiger and Debrunner [7] asked in 1957 if the  $(p, q)$ -condition implies that  $\tau(\mathcal{F})$  is bounded as a function of  $d, p$ , and  $q$ . They proved this in [7] under the condition that  $(d - 1)p < d(q - 1)$  in stronger form saying that  $\tau(\mathcal{F}) \leq p - q + 1$ . Note that the  $(d - 1)p < d(q - 1)$  condition is always satisfied when  $d = 1$ . The general case had remained open for 35 years and was finally solved by Alon and Kleitman [1] by an ingenious and very powerful method.

**Theorem 1.** (Alon and Kleitman [1]) *Let  $p, q, d$  be positive integers with  $p \geq q \geq d + 1$ . Then there exists a number  $m(p, q, d)$  such that  $\tau(\mathcal{F}) \leq m(p, q, d)$  for every finite family  $\mathcal{F}$  of convex sets in  $\mathbb{R}^d$  satisfying the  $(p, q)$ -condition.*

We remark here that the necessity of the condition that  $p \geq q \geq d + 1$  is shown by the example when  $\mathcal{F}$  is a family of hyperplanes in general position. Note also

that the  $(p, q)$ -property implies the  $(p, q - 1)$ -property. So the most important case of the  $(p, q)$ -problem occurs when  $q = d + 1$ .

In this paper we consider a colourful version of the  $(p, q)$ -problem. Let  $\mathcal{F}_1, \dots, \mathcal{F}_p$  be finite families of convex sets in  $\mathbb{R}^d$ . Their union is denoted by  $\mathcal{F}$ . One can think of  $\mathcal{F}_i$  as containing the elements of  $\mathcal{F}$  coloured by colour  $i$ . A *heterochromatic  $p$ -tuple* of  $\mathcal{F}$  is just a collection of  $p$  sets  $C_1, \dots, C_p$  where  $C_i \in \mathcal{F}_i$  for every  $i \in [p] = \{1, \dots, p\}$ . Lovász [11] found a colourful version of Helly's theorem in 1974, its proof appeared first in Bárány [2] in 1982. The coloured version says the following.

**Theorem 2** (Lovász [11] and Bárány [2]). *Let  $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$  be finite families of convex sets (colour classes) in  $\mathbb{R}^d$  with  $\mathcal{F} = \cup_{j=1}^{d+1} \mathcal{F}_j$ . If every heterochromatic  $(d + 1)$ -tuple of  $\mathcal{F}$  has a point in common, then there exists a family  $\mathcal{F}_i$  whose elements have a point in common.*

The assumption of the colourful Helly theorem can be weakened in a way similar to that of the  $(p, q)$ -problem. The family  $\mathcal{F}$  satisfies the *heterochromatic  $(p, q)$ -condition*, to be denoted by  $(p, q)_H$ , if every heterochromatic  $p$ -tuple of  $\mathcal{F}$  contains an intersecting  $q$ -tuple.

We will use the Alon-Kleitman method to show the following.

**Theorem 3.** *Let  $p, q, d$  be positive integers with  $p \geq q \geq d + 1$ . Then there exists a number  $M(p, q, d)$  such that the following holds. Given finite families  $\mathcal{F}_1, \dots, \mathcal{F}_p$  of convex sets in  $\mathbb{R}^d$  satisfying the  $(p, q)_H$ -property, there are  $q - d$  indices  $i \in [p]$  for which  $\tau(\mathcal{F}_i) \leq M(p, q, d)$ .*

The necessity of the condition  $p \geq q \geq d + 1$  is shown by the example when all the  $\mathcal{F}_i$  consist of hyperplanes in general position. One cannot hope for more than  $q - d$  classes with bounded piercing number: this is shown by  $q - d$  colour classes consisting of many copies of  $\mathbb{R}^d$  and each of the remaining classes consisting of many hyperplanes in general position.

The  $(p, q)$ -property ( $(p, q)_H$ -property) can be weakened by requiring that all but an  $\alpha$  fraction of the  $p$ -tuples (or heterochromatic  $p$ -tuples) of  $\mathcal{F}$  satisfy the  $(p, q)$ -property ( $(p, q)_H$ -property). What can one hope for under this *fractional  $(p, q)$ -condition*? Perhaps  $\mathcal{F}$  contains a subfamily  $\mathcal{G}$  of size  $\gamma|\mathcal{F}|$  with  $\tau(\mathcal{G})$  bounded where  $\gamma$  depends only on  $\alpha, d, p, q$ . It would be desirable to have  $\gamma \rightarrow 1$  when  $\alpha \rightarrow 0$ . We will make a first step in this direction, focusing on the main case  $q = d + 1$ :

**Theorem 4.** *Let  $\alpha > 0$  and let  $p, d$  be positive integers with  $p \geq d + 1$ . Then there exists a real number  $\gamma(\alpha, p, d) > 0$  such that the following holds. Given finite families  $\mathcal{F}_1, \dots, \mathcal{F}_p$  of convex sets in  $\mathbb{R}^d$  satisfying the  $(p, d + 1)_H$ -condition for all but an  $\alpha$  fraction of heterochromatic  $p$ -tuples of  $\mathcal{F}$ , some family  $\mathcal{F}_i$  contains an intersecting subfamily of size  $\gamma|\mathcal{F}_i|$ .*

In the second half of the paper we will consider the same questions in dimension one, that is, when the convex sets in  $\mathcal{F}$  are intervals in  $\mathbb{R}$ . In this case we prove precise results on the piercing number.

**Theorem 5.** *Let  $p \geq q \geq 2$  be integers and  $\mathcal{F}$  a finite family of intervals in  $\mathbb{R}$  coloured with  $p$  colours. If  $\mathcal{F}$  has the  $(p, q)_H$ -property, then there exists a colour class  $\mathcal{F}_i \subset \mathcal{F}$  with the property that  $\tau(\mathcal{F}_i) \leq \left\lfloor \frac{p-1}{q-1} \right\rfloor$ . The bound is best possible in the sense that there is a family  $\mathcal{F}$  satisfying the conditions for which  $\tau(\mathcal{F}_i) \geq \left\lfloor \frac{p-1}{q-1} \right\rfloor$  for all  $i \in [p]$ .*

Further, for coloured intervals in  $\mathbb{R}$  the fractional  $(p, q)_H$ -property implies the desired conclusion discussed above. Namely, we prove the following result which is a colourful and fractional version of the classical  $(p, q)$ -theorem of Hadwiger and Debrunner for finite families of intervals in the real line.

**Theorem 6.** *Let  $p \geq q \geq 2$  be integers, set  $\alpha_0 = \frac{1}{2}(p - q + 3)^{-1/(p-q+2)}$  and let  $\alpha \in [0, \alpha_0)$ . Then there is a number  $\beta = \beta(p, q, \alpha) \in [0, 1)$  and an integer  $n_0 = n_0(p, q, \alpha)$  such that the following holds. Let  $\mathcal{F}$  be a finite and coloured family of intervals in  $\mathbb{R}$  with colour classes  $\mathcal{F}_1, \dots, \mathcal{F}_p$  where each  $|\mathcal{F}_i| \geq n_0$ . If  $\mathcal{F}$  satisfies the  $(p, q)_H$ -property with the exception of at most  $\alpha \prod_{j=1}^p |\mathcal{F}_j|$  heterochromatic  $p$ -tuples, then there exists a colour class  $\mathcal{F}_i \subset \mathcal{F}$  such that the elements of  $\mathcal{F}_i$  can be pierced by at most  $p - q + 1$  points with the exception of at most  $\beta |\mathcal{F}_i|$  intervals. Furthermore,  $\beta = O(\alpha^{1/(p-q+2)})$ .*

We will give an example showing that the dependence  $\beta = O(\alpha^{1/(p-q+2)})$  is best possible. In Section 7 we state an extension of Theorem 6 where, under the same conditions, some colour class  $\mathcal{F}_i$  is pierced by  $k$  points except for a small fraction of the intervals in  $\mathcal{F}_i$ . Here  $k$  is any integer from  $\left\{ \left\lfloor \frac{p-1}{q-1} \right\rfloor, \dots, p - q + 1 \right\}$ . The proof is given in Section 8.

Here comes the uncoloured (and fractional) version of Theorem 6. It follows from Theorem 6 quite easily.

**Theorem 7.** *Let  $p \geq q \geq 2$  be positive integers, and let  $\mathcal{F}$  be a finite family of  $n$  intervals in  $\mathbb{R}$ , and  $\alpha \in [0, 1)$ . Then there exists a number  $\beta = \beta(p, q, \alpha) \in [0, 1)$  with the property that if the family  $\mathcal{F}$  has the  $(p, q)$ -property with the exception of at most  $\alpha \binom{n}{p}$   $p$ -tuples, then the elements of  $\mathcal{F}$  can be pierced by  $p - q + 1$  points with the possible exception of at most  $\beta n$  elements. Furthermore  $\beta = O(\alpha^{1/p})$ .*

As a consequence of Theorems 6 and 7, when  $q = 2$ , we obtain the following result that shows how the monochromatic world, for intervals on the line, has influence on the behaviour of the heterochromatic world.

**Corollary 1.** *For every integer  $p \geq 2$  and every  $\alpha > 0$ , there is  $\beta = \beta(p, \alpha) > 0$  such that the following holds. Suppose that  $\mathcal{F}$  is a finite family of intervals in  $\mathbb{R}$  coloured with  $p$  colours. If for every colour  $i$ , the fraction of (monochromatic)  $p$ -tuples in  $\mathcal{F}_i$  that are pairwise disjoint is bigger than  $\alpha$ , then the fraction of heterochromatic  $p$ -tuples of  $\mathcal{F}$  that are pairwise disjoint is larger than  $\beta$ .*

For an overview of this field and for further information we refer to the textbook by Matoušek [12] and the survey papers by Danzer, Grünbaum, and Klee [3], and Eckhoff [4, 5].

## 2. PREPARATIONS

In the above theorems the family  $\mathcal{F}$  consists of general convex sets. However, we can assume that every  $C \in \mathcal{F}$  is a polytope by the following standard argument. Let  $\mathcal{G}$  be a subfamily of  $\mathcal{F}$  with  $\bigcap \mathcal{G} \neq \emptyset$ , and let  $z(\mathcal{G})$  be an arbitrary fixed point in  $\bigcap \mathcal{G}$ . The set  $Z$  consisting of the points  $z(\mathcal{G})$  for all  $\mathcal{G} \subset \mathcal{F}$  with  $\bigcap \mathcal{G} \neq \emptyset$  is finite. Consider now a set  $K \in \mathcal{F}$  and define  $P(K)$  as the convex hull of all points  $z(\mathcal{G}) \in Z$  with  $K \in \mathcal{G}$ . Then  $P(K)$  is a polytope, and the family  $\mathcal{F}^* = \{P(K) : K \in \mathcal{F}\}$  has exactly the same intersection properties and same piercing number as  $\mathcal{F}$  but consists of polytopes only.

As we have seen, the  $(p, q)$ -property implies the  $(p, q - 1)$ -property. So the base case concerns the  $(p, d + 1)$ -property. We will mainly work with this case when  $d > 1$ .

We will need a colourful version of the fractional Helly theorem. The original fractional Helly is due to Katchalski and Liu [10] and says the following.

**Theorem 8.** (Katchalski and Liu [10]) *Assume  $\alpha \in (0, 1]$  and  $\mathcal{F}$  is a family of  $n$  convex sets in  $\mathbb{R}^d$ . If at least  $\alpha \binom{n}{d+1}$  of the  $(d + 1)$ -tuples of  $\mathcal{F}$  are intersecting, then  $\mathcal{F}$  contains an intersecting subfamily of size  $\frac{\alpha}{d+1}n$ .*

The proof of Theorem 1 is based on the Alon-Kleitman lemma that will be stated next. We need the following definition. Given a finite family  $\mathcal{G}$  of convex sets in  $\mathbb{R}^d$ , let  $Z \subset \mathbb{R}^d$  be a finite set that contains one point from every nonempty intersection of elements of  $\mathcal{G}$  (as described above). Now the *fractional packing number*,  $\nu^*(\mathcal{G})$ , of  $\mathcal{G}$  is defined as

$$\nu^*(\mathcal{G}) = \max \sum_{K \in \mathcal{G}} x(K),$$

where the  $x(K)$  are real variables subject to

$$\sum_{z \in K \in \mathcal{G}} x(K) \leq 1 \quad (\forall z \in Z), \quad \text{and } x(K) \geq 0 \quad (\forall K \in \mathcal{G}).$$

In other words, the real variables  $x(K)$  assign weights between 0 and 1 to members of  $\mathcal{G}$  in such a way that the sum of weights does not exceed 1 at any point of  $\mathbb{R}^d$ . Since the sum of  $x(K)$  is the same at any point of the intersection of a subset of  $\mathcal{G}$ , the fractional packing number  $\nu^*$  does not depend on the choice of  $Z$ .

Here comes the Alon-Kleitman lemma [1].

**Lemma 1.** *Let  $\mathcal{G}$  be a finite family of convex sets in  $\mathbb{R}^d$ . Then  $\tau(\mathcal{G})$  is bounded by a function of  $d$  and  $\nu^*(\mathcal{G})$ .*

When  $\mathcal{G}$  is a finite family of convex sets in  $\mathbb{R}^d$ , a *blown-up copy* of  $\mathcal{G}$ ,  $\mathcal{G}^b$ , is simply the same as  $\mathcal{G}$  with some sets repeated (possibly deleted). The size of  $\mathcal{G}^b$ ,  $|\mathcal{G}^b|$  is the number of sets in it counted with multiplicities. The following lemma, also from [1], gives a simple and direct way to check whether  $\nu^*(\mathcal{G}) \leq \gamma$  for some  $\gamma > 0$ .

**Lemma 2.** *Let  $\mathcal{G}$  be a finite family of convex sets in  $\mathbb{R}^d$  and  $\gamma > 0$ . Then  $\nu^*(\mathcal{G}) \leq \gamma$  iff every blown-up copy of  $\mathcal{G}$ , say  $\mathcal{G}^b$ , contains an intersecting subfamily of size at least  $\gamma^{-1}|\mathcal{G}^b|$ .*

It will often be convenient to use the language of hypergraphs. A finite family  $\mathcal{F}$  of convex sets in  $\mathbb{R}^d$ , which is partitioned into  $p$  colour classes  $\mathcal{F}_1, \dots, \mathcal{F}_p$ , gives rise to a  $p$ -partite hypergraph  $\mathcal{H}$  with partition classes  $\mathcal{F}_1, \dots, \mathcal{F}_p$ . The vertices of  $\mathcal{H}$  are the convex sets  $C \in \mathcal{F}$ , its edges are of the form  $e = (C_1, \dots, C_p)$ , where  $C_1, \dots, C_p$  is a heterochromatic  $p$ -tuple of  $\mathcal{F}$  satisfying certain conditions. For instance  $e \in \mathcal{H}$  if the heterochromatic  $p$ -tuple  $C_1, \dots, C_p$  contains an intersecting  $q$ -tuple. We mention further that a blown-up copy  $\mathcal{F}^b$  of the family  $\mathcal{F}$  gives rise to a blown-up copy  $\mathcal{H}^b$  of the corresponding hypergraph  $\mathcal{H}$ : the partition classes are simply  $\mathcal{F}_i^b$  and  $e = (C_1, \dots, C_p)$  is an edge in  $\mathcal{H}^b$  iff it is an edge in  $\mathcal{H}$ .

### 3. PROOF OF THEOREM 3

The proof uses the colourful version of the fractional Helly theorem.

**Lemma 3.** *Let  $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$  be finite families of convex sets (colour classes) in  $\mathbb{R}^d$ , write  $\mathcal{F}$  for their union and assume that  $\alpha \in (0, 1)$ . If an  $\alpha$  fraction of heterochromatic  $(d+1)$ -tuples of  $\mathcal{F}$  are intersecting, then some  $\mathcal{F}_i$  contains an intersecting subfamily of size  $\frac{\alpha}{d+1}|\mathcal{F}_i|$ .*

**Proof.** The following is the standard method. Let  $\mathcal{H}$  be the  $(d+1)$ -partite hypergraph with class  $i$  identified with  $\mathcal{F}_i$  and edges  $e \in \mathcal{H}$  corresponding to intersecting heterochromatic  $(d+1)$ -tuples of  $\mathcal{F}$ . Thus  $e$  is simply  $(C_1, \dots, C_{d+1})$  with  $C_i \in \mathcal{F}_i$  and  $\bigcap_1^{d+1} C_i \neq \emptyset$ . Set  $C(e) = \bigcap_1^{d+1} C_i$ . Define a *partial edge* as  $f = (C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{d+1})$  if the intersection,  $C(f)$ , of these  $d$  convex sets is nonempty. Assume as we may that all  $C \in \mathcal{F}$  are polytopes. Then all  $C(e)$  and  $C(f)$  are polytopes as well, and we can choose a vector  $a \in \mathbb{R}^d$  so that the minimum of the scalar product  $ax$  over all  $x$  in  $C(e)$  and the minimum over all  $x$  in  $C(f)$  is reached at unique points  $x(e)$  and  $x(f)$ .

To the best of our knowledge, the following claim was proved first by Wegner in [13]. For the sake of completeness, we present a short and simple proof here.

**Claim 1.** *For every  $e \in \mathcal{H}$  there is a partial edge  $f \subset e$  with  $x(e) = x(f)$ .*

**Proof.** Let  $H = \{x \in \mathbb{R}^d : ax < ax(e)\}$ , this is an open halfspace and the definition of  $x(e)$  implies that

$$H \cap C(e) = H \cap C_1 \cap \dots \cap C_{d+1} = \emptyset.$$

So these  $d+2$  convex sets have empty intersection. By Helly's theorem some  $d+1$  of them have empty intersection. This  $(d+1)$ -tuple cannot be  $C_1, \dots, C_{d+1}$  so it is  $H, C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{d+1}$  for some  $i$ . This means that  $\bigcap_{j \neq i} C_j$  is disjoint from  $H$ . But it contains  $x(e)$  so  $x(f) = x(e)$  with  $f = (C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{d+1})$ .  $\square$

Now let  $N_i = |\mathcal{F}_i|$  for all  $i$  and let  $N = N_1 \dots N_{d+1}$ . Write  $\mathcal{H}_i$  for the  $d$ -partite hypergraph whose edges are the partial edges  $f$  missing class  $i$ . Clearly,  $|\mathcal{H}_i| \leq N/N_i$ . For  $f \in \mathcal{H}_i$  let  $\mathcal{F}_i(f) = \{C \in \mathcal{F}_i : x(f) \in C\}$ . Note that  $\mathcal{F}_i(f)$  is an intersecting subfamily of  $\mathcal{F}_i$ . We define  $\alpha_i$  by

$$\alpha_i N_i = \max_{f \in \mathcal{H}_i} |\mathcal{F}_i(f)|.$$

We finish the proof by double-counting the pairs  $(e, f)$  with  $e \in \mathcal{H}$ ,  $f \subset e$ ,  $f \in \mathcal{H}_i$  for some  $i$ , and  $x(e) = x(f)$ . Claim 1 says that the number of such pairs is at least  $\alpha N_1 \dots N_{d+1} = \alpha N$ . Hence

$$\begin{aligned} \alpha N &\leq \text{number of such pairs } (e, f) \\ &= \sum_{i=1}^{d+1} \sum_{f \in \mathcal{H}_i} \text{number of } e \in \mathcal{H} \text{ with } (e, f) \text{ being such a pair} \\ &\leq \sum_{i=1}^{d+1} \sum_{f \in \mathcal{H}_i} |\{C \in \mathcal{F}_i : x(f) \in C\}| \leq \sum_{i=1}^{d+1} \sum_{f \in \mathcal{H}_i} \alpha_i N_i \\ &\leq \sum_{i=1}^{d+1} \alpha_i N_i \frac{N}{N_i} = \sum_{i=1}^{d+1} \alpha_i N. \end{aligned}$$

This implies that  $\alpha \leq \sum_1^{d+1} \alpha_i$  and so  $\alpha_i \geq \frac{\alpha}{d+1}$  for some  $i$ .  $\square$

**Proof of Theorem 3.** We are going to use the Alon-Kleitman lemma (Lemma 1). We set  $\gamma = (d+1)\binom{p}{d+1}$  and want to show first that  $\nu^*(\mathcal{F}_i) \leq \gamma$  for some  $i \in [p]$ . So we have to prove, by using Lemma 2, that in every blown-up copy  $\mathcal{F}^b$  of  $\mathcal{F}$  some  $\mathcal{F}_i^b$  contains an intersecting subfamily of size  $\gamma^{-1}|\mathcal{F}_i^b|$ .

We are going to use the complete  $p$ -partite hypergraph  $\mathcal{H}$  associated with the family  $\mathcal{F}$ , and its blown-up copy  $\mathcal{H}^b$ . When  $e = (C_1, \dots, C_p)$  is an edge of  $\mathcal{H}^b$  (or what is the same, of  $\mathcal{H}$ ) and  $J$  is a subset of  $[p]$ , we write  $e(J)$  for the *partial edge*  $(C_j : j \in J)$ . For  $I \in \binom{[p]}{d+1}$  define the  $(d+1)$ -partite hypergraph  $\mathcal{H}^b(I)$  whose classes are  $\mathcal{F}_i^b, i \in I$ , and  $f = (C_i : i \in I)$  is an edge of  $\mathcal{H}^b(I)$  if  $\bigcap_{i \in I} C_i \neq \emptyset$ .

**Claim 2.** *Some  $\mathcal{H}_i^b$  has at least  $\delta|\mathcal{H}_i^b|$  edges where*

$$\delta = \binom{p}{d+1}^{-1}.$$

This follows from double-counting the pairs  $(e, f)$  with  $e \in \mathcal{H}^b$  and  $f = e(I) \in \mathcal{H}^b(I)$ . Set  $|\mathcal{F}_i^b| = N_i$  (repeated sets counted with their multiplicity) and define  $N = N_1 \dots N_p$ . The  $(p, d+1)_H$ -condition implies that for every  $e \in \mathcal{H}^b$  there is an  $I \in \binom{[p]}{d+1}$  such that  $e(I) \in \mathcal{H}^b(I)$ . This gives the first inequality below.

$$\begin{aligned} N &\leq \text{number of such pairs } (e, f) \\ &= \sum_{\text{all } I} \sum_{f \in \mathcal{H}^b(I)} |\{e \in \mathcal{H}^b : f = e(I)\}| \\ &\leq \sum_{\text{all } I} \sum_{f \in \mathcal{H}^b(I)} \prod_{j \notin I} N_j \\ &= N \sum_{\text{all } I} \frac{1}{\prod_{i \in I} N_i} |\mathcal{H}^b(I)|. \end{aligned}$$

This implies that some  $\mathcal{H}^b(I)$  indeed has at least  $\delta|\mathcal{H}^b(I)|$  edges.  $\square$

This finishes the proof quite quickly. The edge density in some  $\mathcal{H}^b(I)$  is at least  $\delta$ . By the coloured fractional Helly theorem (Lemma 3), some  $\mathcal{F}_i^b$  with  $i \in I$  has an intersecting subfamily of size  $\delta/(d+1)|\mathcal{F}_i^b|$ . Consequently, by Lemma 2,  $\nu^*(\mathcal{F}_i) \leq (\delta/(d+1))^{-1} = \gamma$ .

This was the proof for the base case  $q = d+1$ . For the general case of Theorem 3 we need to find  $q-d$  families  $\mathcal{F}_i$  with bounded piercing number. This is quite easy: We find the first one, say  $\mathcal{F}_1$ , with the previous proof. Then the family  $\mathcal{F} \setminus \mathcal{F}_1$  is  $p-1$  coloured, and satisfies the  $(p-1, q-1)$  condition. The previous proof gives another family, say  $\mathcal{F}_2$  with bounded  $\tau$ . We repeat this process  $q-d$  times and get  $q-d$  families with bounded piercing number.  $\square$

#### 4. PROOF OF THEOREM 4

The proof is simple and short. Let  $\mathcal{H}$  be the  $p$ -partite hypergraph whose classes are  $\mathcal{F}_1, \dots, \mathcal{F}_p$  and where  $e = (C_1, \dots, C_p)$  is an edge if the  $p$ -tuple  $C_1, \dots, C_p$  contains an intersecting  $(d+1)$ -tuple. Set  $N_i = |\mathcal{F}_i|$  and  $N = N_1 \dots N_p$  as before. Also, for  $I \in \binom{[p]}{d+1}$  let  $\mathcal{H}(I)$  be the  $(d+1)$ -partite hypergraph with classes  $\mathcal{F}_i, i \in I$  and where  $f = (C_i : i \in I)$  is an edge if  $\bigcap_{i \in I} C_i \neq \emptyset$ . Apply the previous double

counting to the hypergraph  $\mathcal{H}$  (instead of  $\mathcal{H}^b$ ). The  $(p, d + 1)_H$ -condition with  $\alpha$  fraction exceptions guarantees that  $\mathcal{H}$  has  $(1 - \alpha)N$  edges. The rest of the double counting is the same and we conclude that some  $\mathcal{H}(I)$  has at least  $(1 - \alpha)\delta \prod_{i \in I} N_i$  edges with the same  $\delta$  as before. The colourful fractional Helly theorem implies that some  $\mathcal{F}_i$  (with  $i \in I$ ) has an intersecting subfamily of size  $(1 - \alpha)\delta/(d + 1)|\mathcal{F}_i|$ .  $\square$

## 5. COLOURED FAMILIES OF INTERVALS IN $\mathbb{R}$

Let  $p$  be a positive integer, and let  $\mathcal{F}$  be a finite family of intervals in  $\mathbb{R}$ , coloured with  $p$  colours. The intervals with colour  $i$  form the subfamily  $\mathcal{F}_i$ . We may assume (after applying the standard method from Section 2) that all intervals in  $\mathcal{F}$  are closed. Clearly, there is a  $\delta > 0$  such that any two disjoint intervals in  $\mathcal{F}$  are at least at distance  $\delta$  from each other. Now replace each interval  $I \in \mathcal{F}$  by an open interval  $I^*$  containing  $I$  and contained in a  $\delta/3$  neighbourhood of  $I$ . This gives rise to a new family  $\mathcal{F}^*$ . It is evident that this can be done in such a way that no two intervals in  $\mathcal{F}^*$  have a common endpoint. It is also clear that  $\mathcal{F}^*$  has the same intersection pattern and the same values for  $\tau(\mathcal{F}^*)$  and  $\tau(\mathcal{F}_i^*)$  as  $\mathcal{F}$ . From now on we assume that  $\mathcal{F}$  consists of bounded open intervals no two of which have a common endpoint.

The following lemma, in a slightly different setting, was proved by Gyárfás and Lehel in [6]. For the sake of completeness, we present the short and simple proof.

**Lemma 4.** (Gyárfás and Lehel [6]) *Assume that  $\mathcal{F}$  is a finite family of intervals in  $\mathbb{R}$ , coloured with  $p$  colours such that each colour class contains at least  $p$  pairwise disjoint intervals. Then there exists a pairwise disjoint heterochromatic  $p$ -tuple in  $\mathcal{F}$ .*

The **proof** goes by induction on  $p$ . The case  $p = 1$  is obvious. For the induction step  $p - 1 \rightarrow p$ , ( $p \geq 2$ ) let  $a$  be the leftmost right endpoint of all intervals in  $\mathcal{F}$ . We assume, without loss of generality, that  $a$  is the right endpoint of some interval  $I_1$  from the first colour class  $\mathcal{F}_1$ . Delete all intervals from  $\mathcal{F} \setminus \mathcal{F}_1$  that contain  $a$ . The resulting family  $\mathcal{F}'$  of intervals is coloured with  $p - 1$  colours, and each colour class  $\mathcal{F}'_j$  contains at least  $p - 1$  disjoint intervals as only intervals containing the point  $a$  have been deleted from  $\mathcal{F}_i$ . The induction hypothesis guarantees the existence of disjoint intervals  $I_j \in \mathcal{F}'_j \subset \mathcal{F}_j$ ,  $j \in \{2, \dots, p\}$ . All of these  $p - 1$  intervals are to the right of  $a$ , and so  $I_1, I_2, \dots, I_p$  is a heterochromatic  $p$ -tuple consisting of disjoint intervals.  $\square$

We need the following lemma.

**Lemma 5.** *Let  $p \geq q \geq 2$  be integers and  $\mathcal{F}$  a finite family of intervals in  $\mathbb{R}$  coloured with  $p$  colours. If  $\mathcal{F}$  has the  $(p, q)_H$ -property, then there is a colour class  $\mathcal{F}_i$  such that  $\tau(\mathcal{F}_i) \leq p - q + 1$ .*

Note that for  $p = 2$ , Lemma 5 is the colourful Helly theorem (Theorem 2) in one dimension.

The **proof** is indirect, elementary and constructive. We describe the argument in detail because the construction will be used later to improve the upper bound on  $\tau(\mathcal{F}_i)$ .

Assume, on the contrary, that  $\tau(\mathcal{F}_i) \geq p - q + 2$  for each  $i = 1, \dots, p$ . We will find a heterochromatic  $p$ -tuple in  $\mathcal{F}$  in which no  $q$  elements intersect, and thus reach a contradiction.

The indirect assumption implies that each colour class  $\mathcal{F}_i$  must contain at least  $p - q + 2$  pairwise disjoint intervals. Lemma 4 yields the existence of a pairwise disjoint heterochromatic  $(p - q + 2)$ -tuple of intervals  $\{I_1, \dots, I_{p-q+2}\}$  with  $I_j \in \mathcal{F}_j$  for  $j = 1, \dots, p - q + 2$ .

Select one arbitrary interval  $I_k \in \mathcal{F}_k$  from each one of the remaining colour classes  $k = p - q + 3, \dots, p$ . Clearly, the set of intervals  $\{I_1, \dots, I_p\}$  is a heterochromatic  $p$ -tuple with the property that any  $q$ -element subset of it must contain two disjoint intervals from the set  $\{I_1, \dots, I_{p-q+2}\}$  and thus cannot be intersecting.  $\square$

Note that in the case  $q = 2$ , the upper bound in Lemma 5 is best possible. This fact is shown by the following example.

**Example 1.** Let  $p \geq q = 2$  be positive integers. For every  $i \in [p]$  the family  $\mathcal{F}_i$  consists of the same  $p - 1$  pairwise disjoint intervals  $I_1, \dots, I_{p-1}$ . So  $\mathcal{F}$  consists of  $p$  copies of each  $I_j$ . The pigeonhole principle shows that  $\mathcal{F}$  has the  $(p, 2)_H$ -property. At the same time,  $\tau(\mathcal{F}_i) = p - 1$  for each colour class.

## 6. PROOF OF THEOREM 5

Lemma 5 implies that  $\tau(\mathcal{F}_i) \leq p - q - 1$  for at least one colour class. It is easy to see (we omit the details) that

$$\left\lfloor \frac{p-1}{q-1} \right\rfloor = \max \left\{ m \in \mathbb{N} \mid q \leq \left\lceil \frac{p}{m} \right\rceil \right\}. \quad (1)$$

Set

$$m := \min \{ \tau(\mathcal{F}_i) : i = 1, \dots, p \}.$$

This implies that there are at least  $m$  pairwise disjoint intervals in each colour class  $\mathcal{F}_i \subset \mathcal{F}$ . According to Lemma 5,  $1 \leq m \leq p - q + 1$ . Let

$$p = km + r, \text{ where } k, r \in \mathbb{N} \text{ and } 0 \leq r < m.$$

For each  $0 \leq l \leq k - 1$ , Lemma 4 yields the existence of  $m$  pairwise disjoint intervals  $\{I_{lm+1}, \dots, I_{(l+1)m}\}$  of mutually different colours with  $I_{lm+j} \in \mathcal{F}_{lm+j}$  for  $j = 1, \dots, m$ .

If  $r > 0$ , then, again by Lemma 4, there exist  $r$  pairwise disjoint intervals  $\{I_{km+1}, \dots, I_p\}$  of mutually different colours, one from each of the remaining  $r$  colour classes  $\mathcal{F}_{km+1}, \dots, \mathcal{F}_p$ . The set  $\{I_1, \dots, I_p\}$  just constructed is a pairwise disjoint heterochromatic  $p$ -tuple of intervals, which consists of  $\lceil p/m \rceil$  groups and each group contains  $m$  disjoint intervals (all of them of distinct colours) except the last group which contains  $r$  disjoint intervals.

If  $q > \lceil p/m \rceil$ , then the pigeonhole principle guarantees that any  $q$ -element subset of  $\{I_1, \dots, I_p\}$  contains two intervals from the same group and so they are disjoint. This contradicts the hypothesis of the theorem, implying that  $q \leq \lceil p/m \rceil$ . Formula (1) then shows that indeed  $m \leq \left\lfloor \frac{p-1}{q-1} \right\rfloor$ .  $\square$

The following example shows that upper bound in Theorem 5 is best possible.



**Example 2.** Let  $p \geq q \geq 2$  be positive integers and let  $m = \lfloor \frac{p-1}{q-1} \rfloor$ . Let the family  $\mathcal{F}$  consist of  $m$  pairwise disjoint intervals  $I_1, I_2, \dots, I_m$ , each taken with multiplicity  $p$ , and let the colour classes be  $\mathcal{F}_i := \{I_1, \dots, I_m\}$ , for all  $i = 1, \dots, p$ .

It is clear that  $\mathcal{F}$  satisfies the  $(p, q)_H$ -property because any heterochromatic  $p$ -tuple of intervals must contain at least  $q$  copies of one of the intervals  $I_1, \dots, I_m$ , again by the pigeonhole principle. Further,  $\tau(\mathcal{F}_i) = \lfloor \frac{p-1}{q-1} \rfloor$  for all  $i = 1, \dots, p$ .

**Remark 1.** There is no similar theorem in the uncoloured case: the  $(p, q)$ -condition implies  $\tau(\mathcal{F}) \leq p - q + 1$  (by the Hadwiger-Debrunner results [7]) and this bound is best possible, as shown by  $p - q + 1$  disjoint intervals, one of them taken with arbitrary (large) multiplicity, and the others with multiplicity one. This means that, not surprisingly, the  $(p, q)_H$ -condition on  $p$  repeated copies of  $\mathcal{F}$  is stronger than the  $(p, q)$ -condition on  $\mathcal{F}$ .

**Remark 2.** Under the hypotheses of Theorem 5, there exists a colour class, say  $\mathcal{F}_1 \subset \mathcal{F}$ , with  $\tau(\mathcal{F}_1) \leq \lfloor \frac{p-1}{q-1} \rfloor$ . Then the subfamily  $\mathcal{F} \setminus \mathcal{F}_1$  satisfies the  $(p-1, q-1)_H$  property and Theorem 5 guarantees the existence of a colour class, say  $\mathcal{F}_2 \subset \mathcal{F} \setminus \mathcal{F}_1$ , with  $\tau(\mathcal{F}_2) \leq \lfloor \frac{p-2}{q-2} \rfloor$ . Repeating this argument  $q-2$  times, we obtain  $q-2$  colour classes, say  $\mathcal{F}_k$ ,  $k = 1, \dots, q-2$ , with  $\tau(\mathcal{F}_k) \leq \lfloor \frac{p-k}{q-k} \rfloor$ .

Let  $p \geq 3$ . Assume that the family  $\mathcal{F}$  is coloured with  $p$  colours and has the  $(p, p-1)_H$ -property. Applying the above argument to  $\mathcal{F}$ , we obtain that  $p-3$  of the colour classes of  $\mathcal{F}$  have piercing number one and one colour class has piercing number at most two.

## 7. AN EXTENSION OF THEOREM 6 AND A CONSTRUCTION

Theorem 5 says that, under the  $(p, q)_H$ -condition, some colour class of the family  $\mathcal{F}$  of intervals can be pierced by  $\lfloor \frac{p-1}{q-1} \rfloor$  points. Thus, it is not surprising that Theorem 6 can be generalized so that all intervals of some colour class are pierced by  $k$  points, where  $k \in \{ \lfloor \frac{p-1}{q-1} \rfloor, \dots, p-q+1 \}$ :

**Theorem 9.** *Let  $p \geq q \geq 2$  be integers,  $k$  another integer with  $\lfloor \frac{p-1}{q-1} \rfloor \leq k \leq p-q+1$ ,  $h = q-1 + \lfloor (q-p-1)/k \rfloor$ , and  $\alpha \in [0, \alpha_0)$  where  $\alpha_0 = \frac{1}{2}(k+2)^{-1/(p-h)}$ . Then there is a number  $\beta = \beta(p, q, k, \alpha) \in [0, 1)$  and an integer  $n_0 = n_0(p, q, k, \alpha)$  such that the following holds. Let  $\mathcal{F}$  be a finite and coloured family of intervals in  $\mathbb{R}$  with colour classes  $\mathcal{F}_1, \dots, \mathcal{F}_p$  where each  $|\mathcal{F}_i| \geq n_0$ . If  $\mathcal{F}$  satisfies the  $(p, q)_H$ -property with the exception of at most  $\alpha \prod_{j=1}^p |\mathcal{F}_j|$  heterochromatic  $p$ -tuples, then there exists a colour class  $\mathcal{F}_i \subset \mathcal{F}$  such that the elements of  $\mathcal{F}_i$  can be pierced by at most  $k$  points with the exception of at most  $\beta |\mathcal{F}_i|$  intervals. Furthermore,  $\beta = O(\alpha^{1/(p-h)})$ .*

Note that this is exactly Theorem 6 when  $k = p - q + 1$  and  $h = q - 2$ . We mention further that, as one can easily see, the  $h$  defined above is the largest integer  $l$  satisfying  $\lfloor \frac{p-l}{q-l} \rfloor \leq k$ .

In the next section we shall prove Theorems 9 and 6 simultaneously. The proof will use the following construction. Assume that  $\mathcal{G}$  is a finite family of bounded open intervals in  $\mathbb{R}$  with no two intervals having the same endpoint. Suppose that

$a$  is the right endpoint of some interval from  $\mathcal{G}$ . We construct a subfamily  $\mathcal{G}(a)$  of  $\mathcal{G}$  as follows. Denote by  $T(a)$  the collection of all intervals  $I \in \mathcal{G}$  lying to the left of  $a$  and by  $\mathcal{G}(a)$  the collection of all intervals to the right of  $a$ .

Now let  $\mathcal{G} = \{I_1, \dots, I_n\}$ , each  $I_i$  is open and no two intervals have a common endpoint. Define  $t := \lceil \gamma n \rceil$  where  $\gamma > 0$  is a parameter.

The right endpoints of the  $I_j$ s form an increasing sequence of  $n$  distinct numbers. Let  $a_1$  be its  $t$ th element, in other words,  $a_1$  is the  $t$ th smallest right endpoint of the intervals in  $\mathcal{G}$ . Then  $T_1 = T(a_1)$  consists of exactly  $t$  intervals and every interval in  $\mathcal{G}^1 = \mathcal{G}(a_1)$  is to the right of  $a_1$ .

Assume that the families  $\mathcal{G}^j \subset \mathcal{G}^{j-1} \subset \dots \subset \mathcal{G}$  have already been constructed. Assuming that  $|\mathcal{G}^j| \geq t$ , let  $a_{j+1}$  the  $t$ th smallest right endpoint of the intervals in  $\mathcal{G}^j$ . Then  $T_{j+1} = T(a_{j+1})$  consists of exactly  $t$  intervals, and we set  $\mathcal{G}^{j+1} = \mathcal{G}^j(a_{j+1})$ . We can continue this construction as long as  $|\mathcal{G}^j| \geq t$ .

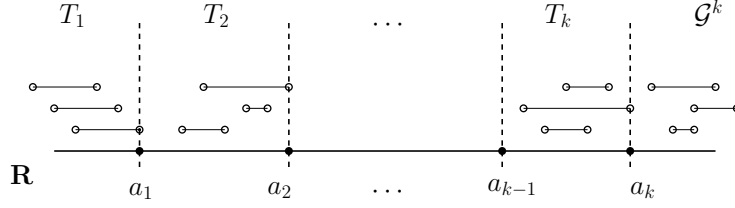


FIGURE 1

**Fact.** *The points  $a_1, \dots, a_k$  pierce all but  $kt + |\mathcal{G}^k|$  intervals from  $\mathcal{G}$ .*

## 8. PROOF OF THEOREMS 9 AND 6

We assume again that all intervals in  $\mathcal{F}$  are open and no two of them have a common endpoint. Let  $n_i = |\mathcal{F}_i|$ ,  $t_i = \lceil \gamma n_i \rceil$  where  $\gamma = (2\alpha)^{1/(p-h)}$ , and define  $\beta = (k+2)\gamma$ . Note that  $\beta < 1$  follows because  $\alpha < \alpha_0$ .

For each colour class  $\mathcal{F}_i \subset \mathcal{F}$ ,  $i \in [p]$  we apply the above construction giving points  $a_1^i, \dots, a_j^i$  and sets  $T_1^i, \dots, T_j^i$ , and call the class *short* if the construction cannot be continued up to  $j = k$ . We note that we are done if some  $\mathcal{F}_i$  is short; the Fact from Section 7 shows that points  $a_1^i, \dots, a_j^i$  pierce all but at most  $jt_i + |\mathcal{F}_i^j| < (j+1)t_i < (k+1)\lceil \gamma n_i \rceil < \beta n_i$  intervals from  $\mathcal{F}_i$ . Here the last inequality follows from the choice of  $\beta$  and  $n_i \geq n_0$  and  $\alpha < \alpha_0$ .

So we assume that there are no short colour classes, that is,  $a_k^i$  exists for all  $i$ . Let  $T^i$  denote the set of intervals in  $\mathcal{F}_i$  that are to the right of  $a_k^i$ . It follows that  $|T_j^i| = t_i$  for  $j = 1, \dots, k$  and any two intervals from two different sets among  $T_1^i, \dots, T_k^i, T^i$  are disjoint.

We are going to show that  $|T^i| < t_i$  for some  $i$ . This will finish the proof since then  $\mathcal{F}_i$  is pierced by the points  $a_1^i, \dots, a_k^i$  except for at most  $kt_i + |T^i| < (k+1)t_i = (k+1)\lceil \gamma n_i \rceil < \beta n_i$  intervals where, again, the last inequality follows the same way as above. So assume, on the contrary, that  $|T^i| \geq t_i$  for all  $i$ .

For  $i \in [p-h]$  we define a family of intervals  $\mathcal{G}_i$  by setting

$$\mathcal{G}_i := \{(-\infty, a_1^i), (a_1^i, a_2^i), \dots, (a_k^i, \infty)\},$$

their union,  $\mathcal{G}$ , is a family of intervals coloured with  $p-h$  colours.

**Claim 3.** For each  $i \in [p - h]$  there is an interval  $I_{j(i)} \in \mathcal{G}_i$  such that no  $q - h$  of the  $I_{j(i)}$ s intersect.

**Proof.** If  $k = p - q + 1$ , then  $h = q - 2$ , and Lemma 4 guarantees the existence of a pairwise disjoint heterochromatic  $(k + 1)$ -tuple in  $\mathcal{G}$ . If  $k < p - q + 1$ , then no  $\mathcal{G}_i$  can be pierced by  $k$  points, and so by Theorem 5,  $\mathcal{G}$  does not have the  $(p - h, q - h)_H$ -property. (This is where we use the choice of  $h$ .) Consequently, there are intervals  $I_{j(i)} \in \mathcal{G}_i$  for each  $i \in [p - h]$  such that no  $q - h$  of the  $I_{j(i)}$ s intersect.  $\square$

Define  $S_i$  as the set of intervals from  $\mathcal{F}_i$  that are contained in  $I_{j(i)}$ , so  $S_i$  coincides with some  $T_j^i$  or  $T^i$ . Consequently,  $|S_i| \geq t_i$  for all  $i$ .

We count those heterochromatic  $p$ -tuples that contain one interval from every  $S_i$ ,  $i \in [p - h]$ . Such a  $p$ -tuple cannot contain an intersecting  $q$ -tuple. Their number is at least

$$\prod_{i=1}^{p-h} |S_i| \prod_{j=p-h+1}^p |\mathcal{F}_j| \geq \prod_{i=1}^{p-h} t_i \prod_{j=p-h+1}^p n_j \geq \gamma^{p-h} \prod_{i=1}^p n_i = 2\alpha \prod_{i=1}^p n_i,$$

a contradiction, as  $\mathcal{F}$  contains at most  $\alpha \prod_{i=1}^p n_i$  heterochromatic  $p$ -tuples with no intersecting  $q$ -tuple.  $\square$

**Remark 3.** This proof gives a little more, namely the following. Under the conditions of the theorem there are at least  $h + 1$  colour classes  $\mathcal{F}_i$  that can be pierced by  $k$  points except for  $\beta n_i$  intervals. The argument is easy: assume there are  $l$  short colour classes. We are done if  $l \geq h + 1$ . Suppose then that  $l \leq h$ . There are  $p - l \geq p - h$  non-short colour classes and any  $p - h$  of them can be used in the above proof to give another non-short colour class with the required piercing property. We can repeat the argument getting further and further non-short colour classes until we have a total of  $h + 1$  colour classes, each pierced by a set of size at most  $k$  except for a  $\beta$  fraction of the intervals in the class.

The following example shows that the order of magnitude of  $\beta$  in Theorem 9 is optimal.

**Example 3.** Let  $p \geq q \geq 2$  be positive integers, define  $k$  and  $h$  as above, let  $0 < \beta < 1/(p - h + 1)$  be a real number to be specified later, and set  $\delta = (k + 1)\beta$ . Fix pairwise disjoint intervals  $I_1, \dots, I_{k+1}$  and a big interval  $I$  containing their union. The family  $\mathcal{F}_i$  is the same for all  $i \in [p]$ : it contains each of  $I_1, \dots, I_{k+1}$  with multiplicity  $\beta n$ , and the interval  $I$  with multiplicity  $(1 - \delta)n$ . Hence such an  $\mathcal{F}_i$  is pierced by  $k$  points except for  $\beta n$  intervals.

Suppose that a given heterochromatic  $p$ -tuple  $P$  of  $\mathcal{F}$  is *bad* in the sense that it does not contain an intersecting  $q$ -tuple. Say, the  $p$ -tuple contains exactly  $l$  copies of  $I$  and  $s_j$  copies of  $I_j$ ,  $j \in [k + 1]$ . We check that  $l \leq h$ . This is trivial if  $k = p - q + 1$  since then  $h = q - 2$  and  $l > h$  would imply  $l \geq q - 1$ . Thus  $P$  would contain an intersecting  $p$ -tuple. If  $k < p - q + 1$  and  $l > h$ , then  $s_j \leq q - 1 - l$  for all  $j$ , and the definition of  $h$  would give

$$p = s_1 + \dots + s_{k+1} + l \leq (k + 1)(q - 1 - l) + l = k(q - 1 - l) + q - 1 < p,$$

a contradiction.

We call the sequence  $s_1, \dots, s_{k+1}, l$  the *profile* of  $P$ . The number of possible profiles of bad  $p$ -tuples with  $l$  copies of  $I$  is an integer  $f(p, q, l)$ , independent of  $n$ . Set  $f(p, q) = \sum_0^h f(p, q, l)$ .

The number of bad  $p$ -tuples with a fixed profile  $s_1, \dots, s_{k+1}, l$  is

$$((1 - \delta)n)^l (\beta n)^{s_1} (\beta n)^{s_2} \dots (\beta n)^{s_{k+1}} = (1 - \delta)^l \beta^{p-l} n^p.$$

As  $\beta < 1/(p - h + 1)$  the total number of bad  $p$ -tuples is

$$\begin{aligned} \sum_{l=0}^h f(p, q, l) (1 - \delta)^l \beta^{p-l} n^p &\leq \sum_{l=0}^h f(p, q, h) (1 - \delta)^h \beta^{p-h} n^p \\ &= f(p, q) (1 - (k + 1)\beta)^h \beta^{p-h} n^p = \alpha n^p, \end{aligned}$$

when we define  $\beta$  by requiring  $f(p, q) (1 - (k + 1)\beta)^h \beta^{p-h} = \alpha$ . It is easy to see that for  $\alpha$  small enough there is a unique solution  $\beta$  in the interval  $(0, 1/(p - h + 1))$  and  $\beta = \Omega(\alpha^{1/(p-h)})$ . The order of magnitude  $\beta = O(\alpha^{1/(p-h)})$  in Theorem 9 is indeed best possible.

## 9. PROOF OF THEOREM 7

Set  $|\mathcal{F}| = n$ ,  $t = \lceil \gamma n \rceil$  where  $\gamma = (q - 1)^{(p-1)/p} \alpha^{1/p}$ , and  $k = p - q + 1$ . We apply the construction of Section 7 to  $\mathcal{F}$ . If it stops before reaching  $a_k$ , then we are done the same way as before. So assume the construction produces points  $a_1, \dots, a_k$  and families of intervals  $T_1, \dots, T_k, T$  from  $\mathcal{F}$ . Then  $|T_i| = t$  for all  $i$  and we are done, again, if  $|T| < t$ . So assume, for a contradiction, that  $|T| \geq t$ .

Next we derive a lower bound on the number of  $p$ -tuples in  $\mathcal{F}$  that contain no intersecting  $q$ -tuple. We only consider the following specific types of  $p$ -tuples: all intervals are from  $T_1 \cup \dots \cup T_k \cup T$  with at least one interval and at most  $q - 1$  intervals from every set  $T_1, \dots, T_k$  and  $T$ . We will call such a  $p$ -tuple *bad*. Every  $q$ -tuple from a bad  $p$ -tuple contains intervals from at least two of the sets  $T_1, \dots, T_k, T$  and thus its intersection is empty. Therefore a bad  $p$ -tuple does not have the  $q$ -intersection property.

A bad  $p$ -tuple has, say,  $s_i$  intervals from  $T_i$  for  $i = 1, \dots, k$ , and  $l$  intervals from  $T$ . Then  $p = s_1 + \dots + s_k + l$  and  $s_1, \dots, s_k$  and  $l$  are integers from  $[q - 1]$ . Call the sequence  $s_1, \dots, s_k, l$  the *profile* of the given  $p$ -tuple, and let  $g(p, q, l)$  be the number of profiles of bad  $p$ -tuples with  $|T| = l$ . The number of bad  $p$ -tuples with given profile  $s_1, \dots, s_k, l$  is

$$\begin{aligned} \binom{|T|}{l} \prod_{i=1}^k \binom{t}{s_i} &\geq \left(\frac{|T|}{l}\right)^l \prod_{i=1}^k \left(\frac{t}{s_i}\right)^{s_i} > \left(\frac{|T|}{q-1}\right)^l \prod_{i=1}^k \left(\frac{t}{q-1}\right)^{s_i} \\ &= \left(\frac{|T|}{q-1}\right)^l \left(\frac{t}{q-1}\right)^{p-l}. \end{aligned}$$

Let  $N$  denote the total number of bad  $p$ -tuples. As  $g(p, q, l) \geq 1$ ,

$$N > \sum_{l=1}^{q-1} g(p, q, l) \left(\frac{|T|}{q-1}\right)^l \left(\frac{t}{q-1}\right)^{p-l} \geq \frac{1}{(q-1)^p} \sum_{l=1}^{q-1} |T|^l t^{p-l},$$

which is a non-decreasing function of  $|T|$ . As  $|T| \geq t$ , we have

$$N > (q-1) \frac{1}{(q-1)^p} t^p \geq \frac{1}{(q-1)^{p-1}} \left( (q-1)^{\frac{p-1}{p}} \alpha^{\frac{1}{p}} \right)^p n^p = \alpha n^p > \alpha \binom{n}{p}.$$

This contradicts the assumption of Theorem 7, and so  $|T| < t$  must be true. Further,  $a_1, \dots, a_k$  pierce all but at most  $(k + 1)t$  intervals from  $\mathcal{F}$  and so  $\beta = O(\alpha^{1/p})$ .  $\square$

Under the conditions of Theorem 7 one can give a better bound, namely,  $\beta = O(\alpha^{1/(p-q+2)})$  provided  $n > p^p/\alpha$ . To prove this one should take each set in  $\mathcal{F}$  with multiplicity  $p$  giving colour classes  $\mathcal{F}_1, \dots, \mathcal{F}_p$  and apply Theorem 6 to this new family. We omit the details. We mention that the monochromatic version of Example 3 shows that this  $\beta$  is of order  $\alpha^{1/(p-q+2)}$  when  $\alpha$  is small and  $n > p^p/\alpha$ .

## 10. ACKNOWLEDGEMENTS

The authors wish to acknowledge the generous support of this research by the Hungarian-Mexican Intergovernmental S&T Cooperation Programme TÉT.10-1-2011-0471 and NIH B330/479/11 “Discrete and Convex Geometry”.

This paper was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

The first and the last author were partially supported by ERC Advanced Research Grant no 267165 (DISCONV), and the first author by Hungarian National Research Grants no. 78439 and 83767 as well.

The second author was also partially supported by Hungarian National Research Grant no. 75016 and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

The third and fourth author acknowledge partial support from Conacyt under project 166306.

## REFERENCES

- [1] N. Alon, and D.J. Kleitman, Piercing convex sets and the Hadwiger-Debrunner  $(p, q)$ -problem, *Adv. Math.* **96** (1992), no.1, 103–112.
- [2] I. Bárány, A generalization of Carathéodory’s theorem, *Discrete Math.* **40** (1982), no. 2-3, 141–152.
- [3] L. Danzer, B. Grünbaum, and V. Klee, *Helly’s theorem and its relatives*, in Convexity (V. Klee ed.), Proceedings of Symposia in Pure Mathematics, vol. 7, American Mathematical Society, Providence, RI, 1963, pp. 100–181.
- [4] J. Eckhoff, *Helly, Radon, and Carathéodory type theorems*, Handbook of convex geometry, Vol. A, B, pp. 389–448, North-Holland, Amsterdam, 1993.
- [5] J. Eckhoff, *A survey of the Hadwiger-Debrunner  $(p, q)$ -problem*, Discrete and computational geometry, pp. 347–377, Algorithms Combin., 25, Springer, Berlin, 2003.
- [6] A. Gyárfás, and J. Lehel, A Helly-type problem in trees, *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pp. 571–584. North-Holland, Amsterdam, 1970.
- [7] H. Hadwiger, H. Debrunner, *Über eine Variante zum Hellyschen Satz (in German)*, Arch. Math. (Basel) **8** (1957), 309–313.
- [8] H. Hadwiger, H. Debrunner, and V. Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart & Winston, New York, 1964.
- [9] G. Kalai, Intersection patterns for convex sets, *Israel J. Math.* **48** (1984), 161–174.
- [10] M. Katchalski, and A. Liu, A problem of geometry in  $R^n$ , *Proc. Am. Math. Soc.* **75** (1979), 284–288.
- [11] L. Lovász, Exercise 206 (in Hungarian), *Mat. Lapok*, (1974), 394–396.
- [12] J. Matoušek, Lectures on discrete geometry, Graduate Texts in Mathematics, 212. Springer-Verlag, New York, 2002.
- [13] G. Wegner,  $d$ -collapsing and nerves of families of convex sets, *Arch. Math. (Basel)* **26** (1975), 317–321.

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, PO BOX 127, H-1364 BUDAPEST, HUNGARY,  
AND DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, GOWER STREET, LONDON,  
WC1E 6BT, U.K.

*E-mail address:* `barany@renyi.hu`

DEPARTMENT OF GEOMETRY, BOLYAI INSTITUTE, UNIVERSITY OF SZEGED, ARADI VÉRTANÚK  
TERE 1, H-6720 SZEGED, HUNGARY, AND DEPARTMENT OF MATHEMATICS AND STATISTICS, UNI-  
VERSITY OF CALGARY, 2500 UNIVERSITY DR. N.W., CALGARY, ALBERTA, CANADA, T2N 1N4

*E-mail address:* `fodorf@math.u-szeged.hu`

INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

*E-mail address:* `luis@matem.unam.mx`

INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

*E-mail address:* `dolivero@matem.unam.mx`

DEPARTMENT OF MATHEMATICS, WESTERN KENTUCKY UNIVERSITY, BOWLING GREEN, KY  
42101, USA

*E-mail address:* `Attila.Por@wku.edu`