# Notes About the Carathéodory Number

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**Abstract** In this paper we give sufficient conditions for a compactum in $\mathbb{R}^n$ to have Carathéodory number less than $n + 1$, generalizing an old result of Fenchel. Then we prove the corresponding versions of the colorful Carathéodory theorem and give a Tverberg-type theorem for families of convex compacta.

## 1 Introduction

The Carathéodory theorem [7] (see also [10]) asserts that every point $x$ in the convex hull of a set $X \subset \mathbb{R}^n$ is in the convex hull of one of its subsets of cardinality at most $n + 1$. In this note we give sufficient conditions for the *Carathéodory number* to be less than $n + 1$ and prove some related results. In order to simplify the reasoning, we always consider compact subsets of $\mathbb{R}^n$.

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There are results about lowering the Carathéodory constant: A theorem of Fenchel [11] (see also [12]) asserts that a compactum $X \subset \mathbb{R}^n$ either has the Carathéodory number $\leq n$ or can be separated by a hyperplane into two nonempty parts. By *separated* we mean "divided by a hyperplane disjoint from $X$ into two nonempty parts." In order to state more results, we need formal definitions.

**Definition 1.1** For a compactum $X \subset \mathbb{R}^n$, we denote by $\text{conv}_k X$ the set of points $p \in \mathbb{R}^n$ that can be expressed as a convex combination of at most $k$ points in $X$. We denote by $\text{conv}\, X$ (without subscript) the standard convex hull of $X$.

**Definition 1.2** The *Carathéodory number* of $X$ is the smallest $k$ such that $\text{conv}\, X = \text{conv}_k X$.

*Remark 1.3* So, when $X \subset \mathbb{R}^n$, Carathéodory's theorem [7] is equivalent to the equality $\text{conv}\, X = \text{conv}_{n+1} X$. We will give an alternative definition for $\text{conv}_k X$ in Sect. 4 as the $k$-fold join of $X$.

**Definition 1.4** A compactum $X \subset \mathbb{R}^n$ is $k$-convex if every linear image of $X$ to $\mathbb{R}^k$ is convex.

We give some examples of $k$-convex sets. What is needed in Fenchel's theorem is 1-convexity, and every connected set is 1-convex. The $k$-skeleton of a convex polytope is $k$-convex (though for such $k$-convex sets, most results of this paper are trivial). In [6] (see also [5, Chap. II, Sect. 14]) it is shown that the image of the sphere under the Veronese map $v_2 : S^{n-1} \to \mathbb{R}^{n(n+1)/2}$ (with all degree 2 monomials as coordinates) is 2-convex.

In [12, Corollary 1] the following remarkable result is proved:

**Theorem 1.5** (Hanner–Rådström, 1951) *If $X$ is a union of at most $n$ compacta $X_1, \ldots, X_n$ in $\mathbb{R}^n$ and each $X_i$ is 1-convex, then $\text{conv}_n X = \text{conv}\, X$.*

It is also known [5, 15] that a *convex curve* in $\mathbb{R}^n$ (that is, a curve with no $n + 1$ points in a single affine hyperplane) has Carathéodory number at most $\lfloor \frac{n+2}{2} \rfloor$. It would be interesting to obtain some nontrivial bounds for the Carathéodory number of the orbit $Gx$ of a point $x$ in a representation $V$ of a compact Lie group $G$ in terms of $\dim V$ and $\dim G$ (or the rank of $G$). The latter question is mentioned in [18, Question 3] and would be useful in results like those in [17].

In Sects. 2 and 3 of this paper we show that the Carathéodory number is at most $k + 1$ for $(n - k)$-convex sets. In Sect. 4 we prove the corresponding analogue of the colorful Carathéodory theorem, and in Sect. 5 we develop another topological approach to colorful Carathéodory-type results, which may be of independent interest. In Sect. 6 we give a related Tverberg-type result.

We note that the content of Sects. 2–4 may be trivially generalized to *positive hulls* in place of convex hulls, as was done, for example, in [2]. But we do not state the corresponding results explicitly to simplify the exposition.

## 2 The Carathéodory Number and $k$-Convexity

We are going to give a natural generalization of the reasoning in [12]:

**Theorem 2.1** *Suppose that $X_1, \ldots, X_{n-k}$ are compacta in $\mathbb{R}^n$ and $p$ does not belong to $\mathrm{conv}_{k+1} X_i$ for any $i$. Then there exists an affine $k$-plane $L \ni p$ that has empty intersection with any $X_i$.*

*Remark 2.2* If we replace $\mathrm{conv}_{k+1} X_i$ by the honest convex hull $\mathrm{conv}\, X_i$, then the result is simply deduced by induction from the Hahn–Banach theorem.

*Remark 2.3* In [16] a somewhat related result was proved: For a compactum $X \subset \mathbb{R}^n$ and a point $p \notin X$, there exists an affine $k$-plane $L$ (for a prescribed $k < n$) such that the intersection $L \cap K$ is not acyclic modulo 2. Here *acyclic* means having the Čech cohomology of a point.

The proof of Theorem 2.1 is given in Sect. 3. Now we deduce the following generalization of Fenchel's theorem [11] (stated in the second paragraph of the introduction):

**Corollary 2.4** *If a compactum $X \subset \mathbb{R}^n$ is $(n-k)$-convex, then $\mathrm{conv}_{k+1} X = \mathrm{conv}\, X$.*

*Proof* Assume the contrary and let $p \in \mathrm{conv}\, X \setminus \mathrm{conv}_{k+1} X$. Applying Theorem 2.1 to the family $\underbrace{X, \ldots, X}_{n-k}$, we find a $k$-dimensional $L \ni p$ disjoint from $X$. Now project $X$ along $L$ with $\pi : \mathbb{R}^n \to \mathbb{R}^{n-k}$. Since $X$ is $(n-k)$-convex, $\pi(L)$ must be separated from $\pi(X)$ by a hyperplane. Hence, $L$ is separated from $X$ by a hyperplane, and therefore $p$ cannot be in $\mathrm{conv}\, X$. $\qquad\square$

*Remark 2.5* In the above Corollary 2.4 and its proof we could consider $n - k$ different $(n-k)$-convex compacta $X_1, \ldots, X_{n-k}$ and by the same reasoning obtain the following conclusion:

$$\bigcup_{i=1}^{n-k} \mathrm{conv}_{k+1} X_i = \bigcup_{i=1}^{n-k} \mathrm{conv}\, X_i.$$

But this result trivially follows from Corollary 2.4 by taking the union.

*Remark 2.6* For the image $v_2(S^{n-1})$ of the Veronese map, the Carathéodory constant is roughly of order $n$, see [5, Chap. II, Sect. 14, Theorem 14.3]. Hence, Corollary 2.4 is not optimal for this set.

## 3 Proof of Theorem 2.1

Let us replace $X_i$ by a smooth nonnegative function $\rho_i$ such that $\rho_i > 0$ on $X_i$ and $\rho_i = 0$ outside some $\varepsilon$-neighborhood of $X_i$. Let $p$ be the origin.

Assume the contrary: For any $k$-dimensional linear subspace $L \subset \mathbb{R}^n$ some intersection $L \cap X_i$ is nonempty. The space of all possible $L$ is the Grassmann manifold $G_n^k$. Denote by $D_i$ the open subset of $G_n^k$ consisting of $L \in G_n^k$ such that $\int_L \rho_i > 0$. Note that 0 cannot lie in the convex hull $\mathrm{conv}(L \cap X_i)$ because in this case by the ordinary Carathéodory theorem, 0 would be in $\mathrm{conv}_{k+1}(L \cap X_i) \subseteq \mathrm{conv}_{k+1} X_i$, contradicting the hypothesis. Hence (if we choose small enough $\varepsilon > 0$), the "momentum" integral

$$m_i(L) = \int_L \rho_i x \, dx$$

never coincides with 0 over $D_i$. Obviously, $m_i(L)$ is a continuous section of the canonical vector bundle $\gamma : E(\gamma) \to G_n^k$, which is nonzero over $D_i$. Now we apply the following:

**Lemma 3.1** *Any $n - k$ sections of $\gamma : E(\gamma) \to G_n^k$ have a common zero because of the nonzero Euler class $e(\gamma)^{n-k}$.*

This lemma is a folklore fact, see, for example, [9, 22]. Applying this lemma to the sections $m_i$, we obtain that the sets $D_i$ do not cover the entire $G_n^k$. Hence, some $L \in G_n^k$ has an empty intersection with every $X_i$.

*Remark 3.2* In the proof of Theorem 1.5 in [12, the proof of Theorem 3] Hanner and Rådström find a maximum of certain volume function over a convex subset of the sphere $S^{n-1}$. This may also be done by an application of the Brouwer fixed-point theorem similar to the above proof, thus exhibiting the topological nature of that result.

## 4 The Colorful Carathéodory Number

Let us introduce some notation and restate the colorful Carathéodory theorem [2].

**Definition 4.1** Denote by $A * B$ the *geometric join* of two sets $A, B \subseteq \mathbb{R}^n$, which is

$$\{ta + (1 - t)b : a \in A, b \in B, \text{ and } t \in [0, 1]\}.$$

This is actually the alternative definition of $\mathrm{conv}_k X$ as $\underbrace{X * \cdots * X}_{k}$.

**Theorem 4.2** (Bárány, 1982) *If $X_1, \ldots, X_{n+1} \subset \mathbb{R}^n$ are compacta and $0 \in \mathrm{conv}\, X_i$ for every $i$, then $0 \in X_1 * X_2 * \cdots * X_{n+1}$.*

It is possible to reduce the Carathéodory number $n + 1$ assuming the $(n - k)$-convexity of $X_i$, thus generalizing Corollary 2.4:

**Theorem 4.3** *Let $0 \le k \le n$. If $X_1, \ldots, X_{k+1} \subset \mathbb{R}^n$ are $(n - k)$-convex compacta and $0 \in \mathrm{conv}\, X_i$ for every $i$, then $0 \in X_1 * X_2 * \cdots * X_{k+1}$.*

*Proof* We use the classical scheme [2] along with the degree reasoning used in [1, 4, 8, 19] in the proof of different generalizations of the colorful Carathéodory theorem.

First consider the case $k = n - 1$. In this case we have $n$ sets and 1-convexity. Let $x_1, \ldots, x_n$ be the system of representatives of $X_1, \ldots, X_n$ such that the distance from $S = \text{conv}\{x_1, \ldots, x_n\}$ to the origin is minimal. If this distance is zero, then we are done. Otherwise, assume that $z \in S$ minimizes the distance.

Let $z = t_1 x_1 + \cdots + t_n x_n$, a convex combination of the $x_i$s. If $t_i = 0$, then we observe that $0 \in \text{conv } X_i$, and we can replace $x_i$ by another $x_i'$ so that new simplex $S' = \text{conv}\{x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n\}$ is closer to the origin than $S$. So we may assume that all the coefficients $t_i$ are positive and $z$ is in the relative interior of $S$. This also implies that $S$ is $(n - 1)$-dimensional, i.e., there is a unique hyperplane containing $S$.

Consider the hyperplane $h \ni 0$ parallel to $S$. Applying the definition of 1-convexity to the projection along $h$, we obtain that there exists a system of representatives $y_i \in X_i \cap h$. The set

$$f(B) = \{x_1, y_1\} * \{x_2, y_2\} * \cdots * \{x_n, y_n\}$$

is a piecewise linear image of the boundary of a crosspolytope, which we denote by $B$. Note that for every facet $F$ of $B$, the vertices of the simplex $f(F)$ form a system of representatives for $\{X_1, \ldots, X_n\}$. In particular, $S = f(F)$ for some facet $F$ of $B$. The line $\ell$ through the origin and $z$ intersects the simplex $S = f(F)$ transversally, and so it must intersect some other $f(F')$ (where $F' \neq F$ is a facet of $B$) because of the parity of the intersection index. The intersection $\ell \cap f(F')$ is on the segment $[0, z]$ and cannot coincide with $z$. Therefore, $f(F')$ is closer to the origin than $S$. This is a contradiction with the choice of $S$. Thus, the case $k = n - 1$ is done.

The case $k = 0$ of this theorem is trivial by definition, and the case $k = n$ corresponds to the colorful Carathéodory theorem. Now let $0 < k < n - 1$. Consider again a system of representatives $x_1, \ldots, x_{k+1}$ minimizing the distance $\text{dist}(0, \text{conv}\{x_1, \ldots, x_{k+1}\})$. Put $S = \text{conv}\{x_1, \ldots, x_{k+1}\}$. As above, the closest to the origin point $z \in S$ must lie in the relative interior of $S$ if $z \neq 0$.

Let $L \subset \mathbb{R}^n$ be the $k$-dimensional linear subspace parallel to $S$. As in the first proof, using $(n - k)$-convexity, we select $y_i \in L \cap X_i$. Then we map naturally the boundary $B$ of a $(k + 1)$-dimensional crosspolytope to the geometric join

$$f(B) = \{x_1, y_1\} * \{x_2, y_2\} * \cdots * \{x_{k+1}, y_{k+1}\}.$$

Note that $f(B)$ is contained in the $(k + 1)$-dimensional linear span of $S$ and $L$, so by the parity argument as above the image under $f$ of some face of $B$ must be closer to the origin than $S$. □

*Remark 4.4* In this proof in the case $k < n - 1$ we can choose some $(k + 1)$-dimensional subspace $M \subset \mathbb{R}^n$ and a system of representatives $\{x_1, \ldots, x_{k+1}\}$ for $M \cap X_1, \ldots, M \cap X_{k+1}$. Then we can make the steps reducing $\text{dist}(0, \text{conv}\{x_1, \ldots, x_{k+1}\})$ so that the system of representatives always remains in $M$.

## 5 A Topological Approach to Theorem 4.3

Theorem 4.3 can also be deduced from the following lemma:

**Lemma 5.1** *Let $\xi : E(\xi) \to X$ be a $k$-dimensional vector bundle over a compact metric space $X$. Let $Y_1, \ldots, Y_{k+1}$ be closed subspaces of $E(\xi)$ such that for every $i$, the projection $\xi|_{Y_i} : Y_i \to X$ is surjective. If $e(\xi) \neq 0$, then for some fiber $V = \xi^{-1}(x)$, the geometric join*

$$(Y_1 \cap V) * \cdots * (Y_{k+1} \cap V)$$

*contains $0 \in V$.*

*Remark 5.2* The Euler class here may be considered in integral cohomology or in the cohomology mod 2. The proof passes in both cases, so we omit the coefficients from the notation.

*Reduction of Theorem 4.3 to Lemma 5.1 for $k < n$* Take a linear subspace $M \subseteq \mathbb{R}^n$ of dimension $k + 1$. For every $k$-dimensional linear subspace $L \subset M$, all the intersections $L \cap X_i$ are nonempty. All such $L$ constitute the canonical bundle $\gamma$ over $G_{k+1}^k = \mathbb{R}P^k$ with nonzero Euler class by Lemma 3.1. For any fixed $i$, the union of sets $L \cap X_i$ constitutes a closed subset of $E(\gamma)$ that we denote by $Y_i$. By Lemma 5.1, for some $L$, the join

$$(Y_1 \cap L) * \cdots * (Y_{k+1} \cap L) = (X_1 \cap L) * \cdots * (X_{k+1} \cap L)$$

must contain the origin.                                                                           $\square$

*Now we prove Lemma 5.1* The proof has much in common with the results of [16]. The main idea is that fiberwise acyclic (up to some dimension) subsets of the total space of a vector bundle behave like sections of that vector bundle.

Let $Y = Y_1 *_X * \cdots *_X Y_{k+1}$ be the *abstract fiberwise join* over $X$, that is, the set of all formal convex combinations

$$t_1 y_1 + t_2 y_2 + \cdots + t_{k+1} y_{k+1},$$

where $t_i$ are nonnegative reals with unit sum, and $y_i \in Y_i$ are points such that

$$\xi(y_1) = \cdots = \xi(y_{k+1}).$$

Denote the natural projection $\eta : Y \to X$. Any formal convex combination $y \in Y$ defines a corresponding "geometric" convex combination $f(y)$ in the fiber $\xi^{-1}(\eta(y))$ depending continuously on $y$. It is easy to check that $f(y)$ can be considered as a section of the pullback vector bundle $\eta^*(\xi)$ over $Y$.

For any point $x \in X$, its preimage under $\eta$ is a join of $(k + 1)$ nonempty sets

$$\left(Y_1 \cap \xi^{-1}(x)\right) * \cdots * \left(Y_{k+1} \cap \xi^{-1}(x)\right),$$

and therefore $\eta^{-1}(x)$ is $(k-1)$-connected. Hence, the Leray spectral sequence for the Čech cohomology $H^*(Y)$ with $E_2^{*,*} = H^*(X; \mathcal{H}^*(\eta^{-1}(x)))$ (the coefficient sheaf is the direct image of the homology of the total space) has empty rows number $1, \ldots, k-1$, and its differentials cannot kill the image of $e(\xi)$ in $E_r^{k,0}$. Hence, $\eta^*(e(\xi)) = e(\eta^*(\xi))$ remains nonzero over $Y$, and by the standard property of the Euler class, for some $y \in Y$, the section $f(y)$ must be zero.                          $\square$

*Remark 5.3* In this proof we essentially use the inequality $k < n$. So the colorful Carathéodory theorem is *not* a consequence of Lemma 5.1, at least in our present state of knowledge.

The subsets $Y_i$ in Lemma 5.1 can be considered as *set-valued* sections. The same technique proves the following:

**Theorem 5.4** *Let $B$ be an $n$-dimensional ball, and $f_i : B \to 2^B \setminus \emptyset$ for $i = 1, \ldots, n+1$ be set-valued maps with closed graphs (in $B \times B$). Then for some $x \in B$, we have the inclusion*

$$x \in f_1(x) * \cdots * f_{n+1}(x).$$

*Proof* We may assume that all sets $f_i(x)$ are in the interior of $B$, because the general case is reduced to this one by composing $f_i$ with a homothety with scale $1 - \varepsilon$ and going to the limit as $\varepsilon \to +0$.

It is known [14] that for a single-valued map $f : B \to \text{int } B$ (considered as a section of the trivial bundle $B \times \mathbb{R}^n \to B$), a fixed point ($x = f(x)$) is guaranteed by the relative Euler class $e(f(x) - x) \in H^n(B, \partial B)$. Then the proof proceeds as in Lemma 5.1 by lifting $e(f(x) - x)$ to the abstract fiberwise join of graphs of $f_i$ over the pair $(B, \partial B)$ and using the properties of the relative Euler class of a section.      $\square$

**Corollary 5.5** *Suppose that $X_1, \ldots, X_{n+1}$ are compacta in $\mathbb{R}^n$ and $\rho$ is a continuous metric on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, denote by $f_i(x)$ the set of farthest from $x$ points in $X_i$ (in the metric $\rho$). Then, for some $x \in \mathbb{R}^n$, we have*

$$x \in f_1(x) * \cdots * f_{n+1}(x).$$

*Remark 5.6* If we denote by $f_i(x)$ the *closets* points in $X_i$, then this assertion becomes almost trivial without using any topology.

## 6 The Carathéodory Number and the Tverberg Property

Tverberg's classical theorem [20] is the following:

**Theorem 6.1** (Tverberg, 1966) *Every set of $(n+1)(r-1)+1$ points in $\mathbb{R}^n$ can be partitioned into $r$ parts $X_1, \ldots, X_r$ so that the convex hulls $\text{conv } X_i$ have a common point.*

From the general position considerations it is clear that the number $(n + 1) \times (r - 1) + 1$ cannot be decreased. But we are going to decrease it after replacing a finite point set by a family of convex compacta. Let us define the Carathéodory number for such families.

**Definition 6.2** Let $\mathcal{F}$ be a family of convex compacta in $\mathbb{R}^n$. The *Carathéodory number* of $\mathcal{F}$ is the least $\kappa$ such that for any subfamily $\mathcal{G} \subseteq \mathcal{F}$,

$$\operatorname{conv} \bigcup \mathcal{G} = \bigcup_{\mathcal{H} \subseteq \mathcal{G}, \ |\mathcal{H}| \leq \kappa} \operatorname{conv} \bigcup \mathcal{H}.$$

We denote the Carathéodory number of $\mathcal{F}$ by $\kappa(\mathcal{F})$.

Again, from the Carathéodory theorem [7] it follows that $\kappa(\mathcal{F}) \leq n + 1$. Another observation is that Corollary 2.4 guarantees that $\kappa(\mathcal{F}) \leq k + 1$ if the union of every subfamily $\mathcal{G} \subseteq \mathcal{F}$ is $(n - k)$-convex.

Now we state the analogue of Tverberg's theorem.

**Theorem 6.3** *Suppose that $\mathcal{F}$ is a family of convex compacta in $\mathbb{R}^n$, $r$ is a positive integer, and*

$$|\mathcal{F}| \geq r\kappa(\mathcal{F}) + 1.$$

*Then $\mathcal{F}$ can be partitioned into $r$ subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_r$ so that*

$$\bigcap_{i=1}^{r} \operatorname{conv} \bigcup \mathcal{F}_i \neq \emptyset.$$

*Remark 6.4* Note the following: If $\kappa(\mathcal{F}) = n + 1$, then taking a system of representatives for $\mathcal{F}$ and applying the Tverberg theorem, we obtain a weaker condition: $|\mathcal{F}| \geq (r - 1)(n + 1) + 1$.

*Remark 6.5* This theorem originated in discussions with Andreas Holmsen, who established the same result in the special case $n = 2$, $\kappa(\mathcal{F}) = 2$, and with $|\mathcal{F}| \geq 2r$ (*not* $2r + 1$), see [13]. Together with the previous remark, this shows that the condition on $|\mathcal{F}|$ may be not tight, though we have no idea how to improve it in the general case.

*Proof of Theorem 6.3* We again use a minimization argument, combined with Sarkaria's trick [21] in the more convenient tensor form, which is from [3].

Let $|\mathcal{F}| = m$, $\kappa = \kappa(\mathcal{F})$, and

$$\mathcal{F} = \{C_1, C_2, \ldots, C_m\}.$$

Put the space $\mathbb{R}^n$ to $A = \mathbb{R}^{n+1}$ as a hyperplane given by the equation $x_{n+1} = 1$. Consider a set $S$ of vertices of a regular $(r - 1)$-simplex in some $(r - 1)$-dimensional space $V$ and assume that $S$ is centered at the origin.

Now define the subsets of $V \otimes A$ by

$$X_i = S \otimes C_i$$

and consider a system of representatives $(x_1, x_2, \ldots, x_m)$ for the family of sets $\mathcal{G} = \{X_1, X_2, \ldots, X_m\}$. Such a system gives rise to a partition $\{P_s : s \in S\}$ of $\{1, \ldots, m\}$ in the following way. For $s \in S$, define

$$P_s = \left\{ i \in \{1, \ldots, m\} : x_i = s \otimes c_i \text{ for some } c_i \in C_i \right\}.$$

Like in [1, Lemma 2], we observe that $0 \in \text{conv}\{x_1, \ldots, x_m\}$ if and only if $\bigcap_{s \in S} \text{conv}\{c_i : i \in P_s\} \neq \emptyset$. Based on this, we choose a system of representatives $(x_1, \ldots, x_m)$ of $\mathcal{G}$ so that the distance between 0 and $\text{conv}\{x_1, x_2, \ldots, x_m\}$ is minimal. If this distance is zero, then the required partition of $\mathcal{F}$ is given by the sets $\{C_i \in \mathcal{F} : i \in P_s\}$, $s \in S$.

Assume that the minimal distance is not zero. Then it is attained on some convex combination

$$x_0 = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m.$$

We *claim* that $\alpha_i > 0$ for all $i \in \{1, \ldots, m\}$. Assume, for instance, that $\alpha_1 = 0$ and $x_1 = s \otimes c_1$ for some $c_1 \in C_1$ and $s \in S$. Now $x_1$ can be replaced by $t \otimes c_1$ for any $t \in S$ as such a change does not influence $x_0$. The distance minimality condition implies that all the points $t \otimes c_1$ are separated from the origin by a hyperplane in $V \otimes A$, which is the support hyperplane for the ball, centered at the origin and touching $\text{conv}\{x_1, \ldots, x_m\}$. Obviously,

$$\sum_{t \in S} t \otimes c_i = 0,$$

so the points $t \otimes c_i$, $t \in S$, are not separated from the origin. This contradiction completes the proof of the claim.

The above convex combination representing $x_0$ can be written as

$$x_0 = \sum_{s \in S} s \otimes \left( \sum_{i \in P_s} \alpha_i c_i \right).$$

Assume first that none of $P_s$ is the empty set. Define $c(s) = \sum_{i \in P_s} \alpha_i c_i$ and $\alpha(s) = \sum_{i \in P_s} \alpha_i > 0$. Then $c(s)/\alpha(s)$ is a convex combination of elements $c_i \in C_i$, $i \in P_s$. Thus, $c(s)/\alpha(s) \in \text{conv} \bigcup_{i \in P_s} C_i$. According to the definition of the Carathéodory number, there is a subset $P_s' \subset P_s$, of size at most $\kappa$, such that $c(s)/\alpha(s) \in \text{conv} \bigcup_{i \in P_s'} C_i$ for every $s \in S$. This means that there are $c_i' \in C_i$ for all $i \in P_s'$ such that $c(s)/\alpha(s) \in \text{conv}\{c_i' : i \in P_s'\}$; in other words, $c(s) = \sum_{i \in P_s'} \alpha_i' c_i'$ with positive $\alpha_i'$ satisfying $\sum_{i \in P_s'} \alpha_i' = \alpha(s)$. Thus,

$$x_0 = \sum_{s \in S} s \otimes \left( \sum_{i \in P_s'} \alpha_i' c_i \right).$$

In this case the minimum distance is attained on the convex hull of no more that $r\kappa$ elements as each $|P_s'| \leq \kappa$. But $m > r\kappa$, contradicting the claim.

Finally, we have to deal with the (easy) case where some $P_s = \emptyset$. The above argument works, with no change at all, for the nonempty $P_s$, implying that $x_0$ can be written as a convex combination of at most $(r-1)\kappa$ elements. Again, $m > (r-1)\kappa$, and the same contradiction finishes the proof. □

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