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Homogeneous selections from hyperplanes

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ABSTRACT

Given $d+1$ hyperplanes $h_1, \dots, h_{d+1}$ in general position in $\mathbb{R}^d$, let $\triangle(h_1, \dots, h_{d+1})$ denote the unique bounded simplex enclosed by them. There exists a constant $c(d) > 0$ such that for any finite families $H_1, \dots, H_{d+1}$ of hyperplanes in $\mathbb{R}^d$, there are subfamilies $H_i^* \subset H_i$ with $|H_i^*| \geqslant c(d)|H_i|$ and a point $p \in \mathbb{R}^d$ with the property that $p \in \triangle(h_1, \dots, h_{d+1})$ for all $h_i \in H_i^*$.

1. The main result

Throughout this paper, let $H_1, \dots, H_{d+1}$ be finite families of hyperplanes in $\mathbb{R}^d$ in general position. That is, we assume that (1) no element of $\bigcup_{i=1}^{d+1} H_i$ passes through the origin, (2) any d elements have precisely one point in common, and (3) no $d+1$ of them have a nonempty intersection. A *transversal* to these families is an ordered $(d+1)$-tuple $h = (h_1, \dots, h_{d+1}) \in \prod_{i=1}^{d+1} H_i$, where $h_i \in H_i$ for every i.

Given hyperplanes $h_1, \dots, h_{d+1} \subset \mathbb{R}^d$ in general position in $\mathbb{R}^d$, there is a unique simplex denoted by $\triangle = \triangle(h_1, \dots, h_{d+1})$ whose boundary is contained in $\bigcup_1^{d+1} h_i$. Clearly, this simplex is identical to the convex hull of the points

$$v_i = \bigcap_{j \neq i} h_j, \quad i \in [d+1], \tag{1.1}$$

where, as in the sequel, $[n]$ stands for the set $\{1, 2, \dots, n\}$.

Our main result is the following.

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Theorem 1.1. *For every $d \geqslant 1$ there is a constant $c(d) > 0$ with the following property. Given finite families $H_1, \dots, H_{d+1}$ of hyperplanes in $\mathbb{R}^d$ in general position, there are subfamilies $H_i^* \subset H_i$ with $|H_i^*| \geqslant c(d)|H_i|$ for $i = 1, \dots, d+1$ and a point $p \in \mathbb{R}^d$ such that p is contained in $\triangle(h)$ for every transversal $h \in \prod_{i=1}^{d+1} H_i$.*

It follows from the general position assumption that the simplices $\triangle(h)$ in Theorem 1.1 also have an *interior* point in common.

It will be convenient to use the language of hypergraphs. Let $\mathcal{H} = \mathcal{H}(H_1, \dots, H_{d+1})$ be the complete $(d+1)$-partite hypergraph with vertex classes $H_1, \dots, H_{d+1}$. We refer to $\mathcal{H}$ as the *hyperplane hypergraph*, or *h-hypergraph* associated with the hyperplane families $H_1, \dots, H_{d+1}$. The hyperedges of $\mathcal{H}$ are the transversals of the families $H_1, \dots, H_{d+1}$. Our main result can now be reformulated as follows.

Theorem 1.2. *For every positive integer d, there is a constant $c(d) > 0$ with the following property. Every complete $(d+1)$-partite h-hypergraph $\mathcal{H}(H_1, \dots, H_{d+1})$ contains a complete $(d+1)$-partite h-subhypergraph $\mathcal{H}^*(H_1^*, \dots, H_{d+1}^*)$ such that $|H_i^*| \geqslant c(d)|H_i|$ for all $i \in [d+1]$ and $\bigcap_{h \in \mathcal{H}^*} \triangle(h) \neq \emptyset$.*

In some sense, our theorem extends the following recent and beautiful result of Karasev [7].

Theorem 1.3. *(See [7].) Assume r is a prime power and $t \geqslant 2r - 1$. Let $\mathcal{H}$ be a complete $(d+1)$-partite h-hypergraph with partition classes of size t. Then there are vertex-disjoint hyperedges (transversals) $h^1, \dots, h^r$ of $\mathcal{H}$ such that $\bigcap_{j=1}^{r} \triangle(h^j) \neq \emptyset$.*

Two hyperedges (transversals) h and h' of $\mathcal{H}$ are vertex-disjoint if h_i and h'_i are distinct for each i.

Our Theorem 1.1 implies a weaker version of Karasev's theorem. Namely, the same conclusion holds with arbitrary r and $t \geqslant r/c(d)$. Since $c(d)$ will turn out to be doubly exponential in d, our result is quantitatively much weaker than the bound $t \geqslant 4r$ that follows from Karasev's theorem for any r.

Karasev's result is a kind of dual to Tverberg's famous theorem [10]. In the same sense, our result is dual to the homogeneous point selection theorem of Pach [9] (see also [8]), which guarantees the existence of an absolute constant $c_d > 0$ with the following property. Let $X_1, \dots, X_{d+1}$ be finite sets of points in general position in $\mathbb{R}^d$ with $|X_i| = n$ for every i. Then there exist subsets $X_i^* \subset X_i$ of size at least $c_d n$ for every $i \in [d+1]$ and a point $p \in \mathbb{R}^d$ such that $p \in \operatorname{conv}\{x_1, \dots, x_{d+1}\}$ for all transversals $(x_1, \dots, x_{d+1}) \in \prod_{i+1}^{d+1} X_i^*$. Here the assumption that the sets X_i are of the same size can be removed (see e.g. [5]).

To establish Theorem 1.2, we need some preparation. Let h and h' be two edges of the hypergraph $\mathcal{H} = \mathcal{H}(H_1, \dots, H_{d+1})$. As in (1.1), let v_i (and v'_i) denote the vertex of $\triangle(h)$ (and $\triangle(h')$) opposite to the facet contained in h_i (and h'_i, respectively). The *edges* h and h' are said to be of the *same type* if, for each $i \in [d+1]$, the vertices v_i and v'_i are not separated by either of the hyperplanes h_i and h'_i. We say that the *h-hypergraph* $\mathcal{H}$ is *homogeneous* if every pair of its edges is of the same type.

The heart of the proof of Theorem 1.2 is the following "same type lemma" for hyperplanes.

Lemma 1.4. *For any $d \geqslant 1$, there exists a constant $b(d) > 0$ with the following property. Every complete $(d+1)$-partite h-hypergraph $\mathcal{H}(H_1, \dots, H_{d+1})$ contains a complete $(d+1)$-partite subhypergraph $\mathcal{H}^*(H_1^*, \dots, H_{d+1}^*)$ with $|H_i^*| \geqslant b(d)|H_i|$ for all $i \in [d+1]$ which is homogeneous.*

The rest of this note is organized as follows. In Section 2, we deduce Theorem 1.2 from Lemma 1.4. In Sections 3 and 4 we present two proofs for Lemma 1.4. The first proof, which provides a better estimate for the value of the constant $b(d)$, uses duality and is based on a same type lemma for points, due to Bárány and Valtr [2] (see also [8]). The second proof is shorter, but it utilizes a far reaching generalization of the same type lemma to semi-algebraic relations of several variables, found by Fox, Gromov, Lafforgue, Naor, and Pach [5], see also Bukh and Hubard [4] for a quantitative form. The same result for binary semi-algebraic relations was first established by Alon, Pach, Pinchasi, Radoičić, and Sharir [1].

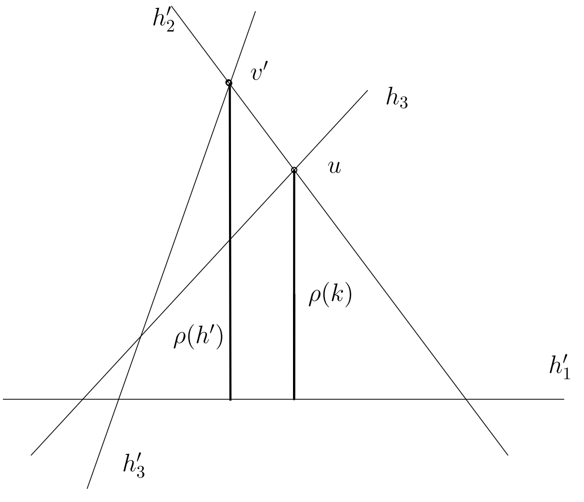

Fig. 1. Illustration for Theorem 1.2, $d = 2$.

2. Proof of Theorem 1.2

In this section, we deduce Theorem 1.2 from Lemma 1.4. The proof of the lemma is postponed to the last two sections.

Let $\mathcal{H}^*$ denote the complete $(d+1)$-partite subhypergraph of $\mathcal{H}$ whose existence is guaranteed by the lemma. For a fixed $h = (h_1, \dots, h_{d+1}) \in \mathcal{H}^*$, let h_i^+ denote the half-space bounded by h_i that contains vertex v_i of $\triangle(h)$, for $i \in [d+1]$. The lemma implies that, for every hyperedge $k = (k_1, \dots, k_{d+1}) \in \mathcal{H}^*$ and for every i, the half-space h_i^+ contains the vertex u_i of $\triangle(k)$ opposite to hyperplane k_i. To prove the theorem, it suffices to establish the following claim:

$$\bigcap_{h \in \mathcal{H}^*} \triangle(h) \neq \emptyset.$$

For $h = (h_1, \dots, h_{d+1}) \in \mathcal{H}^*$, let $\rho(h)$ denote the distance between h_1 and $v_1 = \bigcap_2^{d+1} h_j$, and let $h' \in \mathcal{H}^*$ be the edge for which $\rho(h)$ is minimal. By the general position assumption, we have $\rho(h') > 0$. Set $v' = \bigcap_2^{d+1} h_j'$. We show that $v' \in \triangle(h)$ for every $h \in \mathcal{H}^*$, which implies the claim. To see this, we have to verify that $v' \in h_i^+$ for every $h \in \mathcal{H}^*$ and for every i.

This is trivial for $i = 1$. Suppose that $i \geqslant 2$. By symmetry, we may assume that $i = d+1$. We have to show that $v' \in h_{d+1}^+$ for every $h_{d+1} \in H_{d+1}^*$.

Assume to the contrary that $v' \notin h_{d+1}^+$ for some $h_{d+1} \in H_{d+1}^*$. Setting $k = (h_1', \dots, h_d', h_{d+1})$, we clearly have $k \in \mathcal{H}^*$. The simplices $\triangle(k)$ and $\triangle(h')$ share the vertex $v_{d+1} = \bigcap_1^d h_i'$. As $v_{d+1} \in h_{d+1}^+$, by the construction, $v' \notin h_{d+1}^+$ implies that h_{d+1} intersects the segment $[v_{d+1}, v']$ in a point u in its relative interior, see Fig. 1. On the other hand, we know that $u = \bigcap_2^{d+1} k_i$ is the vertex of $\triangle(k)$ opposite to $h_1' = k_1$. Thus, u is closer to $h_1 = k_1$ than v' is. Therefore, we obtain that $\rho(k) < \rho(h')$, contradicting the definition of h'.

It follows from the above proof that Theorems 1.2 and 1.1 hold with $c(d) = b(d)$.

3. A same type lemma for hyperplanes — First proof of Lemma 1.4

Before turning to the proof of Lemma 1.4, we need some preparation. A collection of $m \geqslant d+1$ finite sets of points, $X_1, \dots, X_m \subset \mathbb{R}^d$, is said to be *strongly separated* if every hyperplane intersects at most d of the sets $\operatorname{conv} X_i$, $i \in [m]$. This property can be rephrased in several equivalent forms; see, e.g., [6,2,9,8].

Proposition 3.1. *A collection of finite point sets $X_1, \ldots, X_m$ in $\mathbb{R}^d$ with $m \geqslant d+1$ is strongly separated if and only if every $d+1$ of them are strongly separated.*

Proposition 3.2. *A collection of finite sets $X_1, \ldots, X_{d+1}$ in $\mathbb{R}^d$ is strongly separated if and only if for every subset $I \subset [d+1]$ the sets $\bigcup_{i \in I} X_i$ and $\bigcup_{i \in [d+1]\setminus I} X_i$ can be strictly separated by a hyperplane.*

Two transversals $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m) \in \prod_{i=1}^m X_i$ are said to be *of the same type* if the orientations of the simplices $\text{conv}\{x_{i_1}, \ldots, x_{i_{d+1}}\}$ and $\text{conv}\{y_{i_1}, \ldots, y_{i_{d+1}}\}$ are the same for all $1 \leqslant i_1 < i_2 < \cdots < i_{d+1} \leqslant m$. In other words, the signs of the determinants of the matrices

$$\begin{pmatrix} x_{i_1} & x_{i_2} & \ldots & x_{i_{d+1}} \\ 1 & 1 & \ldots & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_{i_1} & y_{i_2} & \ldots & y_{i_{d+1}} \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$

are the same.

Proposition 3.3. *A collection of finite sets $X_1, \ldots, X_m$ in $\mathbb{R}^d$ with $m \geqslant d+1$ is strongly separated if and only if every pair of transversals of X_i ($i \in [m]$) are of the same type.*

As usual, we say that a set of points $X \subset \mathbb{R}^d$ is in *general position* if no $d+1$ elements of X lie on a hyperplane. We need the same type lemma of Bárány and Valtr [2] for points.

Theorem 3.4. *(See [2].) For every positive integer d and every $m \geqslant d+1$, there is a constant $c(d,m) > 0$ with the following property. Let $X_1, \ldots, X_m$ be a collection of pairwise disjoint finite point sets in $\mathbb{R}^d$ such that their union is in general position. Then there exist subsets $X_i^* \subset X_i$ with $|X_i^*| \geqslant c(d,m)|X_i|$ for all $i \in [m]$ such that the collection $X_1^*, \ldots, X_m^*$ is strongly separated.*

Now we turn to the proof of Lemma 1.4. We use the standard duality between points $a \in \mathbb{R}^d \setminus \{0\}$ and hyperplanes $h \subset \mathbb{R}^d$ with $0 \notin h$. Every hyperplane not passing through the origin 0 is of the form

$$h = \{x \in \mathbb{R}^d \colon a \cdot x = 1\}, \tag{3.1}$$

with a unique $a \in \mathbb{R}^d \setminus \{0\}$. Conversely, every $a \in \mathbb{R}^d \setminus \{0\}$ gives rise to a unique hyperplane h via (3.1). By the general position assumption, no element of $\bigcup_{i=1}^{d+1} H_i$ passes through the origin. For any $i \in [d+1]$, let A_i denote the set of points dual to the hyperplanes in H_i via the standard duality (3.1).

Applying Theorem 3.4 to the sets $A_0 = \{0\}, A_1, \ldots, A_{d+1}$, we obtain a collection of subsets $A_0^* = \{0\}, A_1^* \subset A_1, \ldots, A_{d+1}^* \subset A_{d+1}$ with $|A_i^*| \geqslant c(d, d+2)|A_i|$ for all $i \in [d+1]$ such that all $(d+2)$-transversals of them are of the same type. The sets of hyperplanes dual to the elements of $A_1^*, \ldots, A_{d+1}^*$, denoted by $H_1^*, \ldots, H_{d+1}^*$, form a complete $(d+1)$-partite h-hypergraph $\mathcal{H}^*(H_1^*, \ldots, H_{d+1}^*)$, which is a subhypergraph of the original hypergraph $\mathcal{H}$.

Claim 3.5. *The h-hypergraph $\mathcal{H}^*$ is homogeneous.*

Proof. We show that, given $h_i \in H_i^*$ and $h_j, k_j \in H_j^*$ (where $j \neq i$), h_i does not separate the points $v = \bigcap_{j \neq i} h_j$ and $u = \bigcap_{j \neq i} k_j$. By symmetry, it suffices to prove this in the case $i = d+1$.

Consider the simplices $\triangle_0 = \triangle(h_1, \ldots, h_{d+1})$, $\triangle_1 = \triangle(k_1, h_2, \ldots, h_{d+1})$, $\triangle_2 = \triangle(k_1, k_2, h_3, \ldots, h_{d+1})$, $\ldots$, $\triangle_d = \triangle(k_1, \ldots, k_d, h_{d+1})$. Let u_i be the vertex opposite to h_{d+1} in $\triangle_i$. We have $u_0 = v$ and $u_d = u$. Obviously, it is sufficient to verify that h_{d+1} does not separate u_{i-1} and u_i for $i \in [d]$, see Fig. 2.

Again, by symmetry, it is enough to consider the case $i = 1$. Assume that h_1 and k_1 are given by the equations $a_1 \cdot x = 1$ and $a_1' \cdot x = 1$, respectively. Set $a(t) = (1-t)a_1 + t a_1'$ for $t \in [0,1]$ and let $h(t)$ be the hyperplane with equation $a(t) \cdot x = 1$, and let $\triangle(t)$ be the corresponding simplex (if it exists, which is not entirely clear at the moment) with vertex $u(t)$ opposite to h_{d+1}.

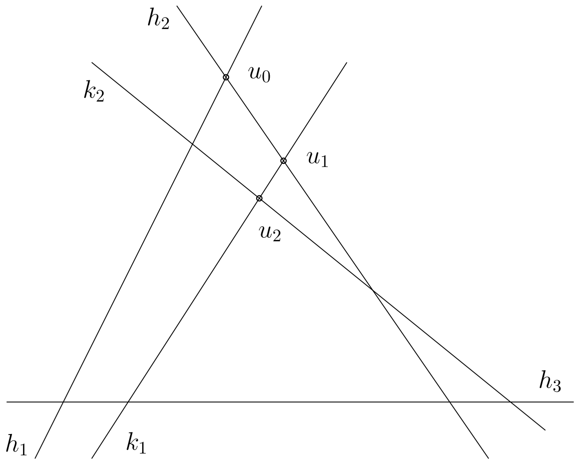

Fig. 2. Illustration for Claim 3.5, case $d = 2$.

We move h_1 to k_1 by the homotopy $h(t)$ and check how $u(t)$ behaves. The common vertex of $\triangle_0$ and $\triangle_1$ is $z = \bigcap_2^{d+1} h_i$. The segment $[z, u_0]$ is an edge of $\triangle_0$. We define the half-line $L = \{z + \lambda(u_0 - z)\colon \lambda > 0\}$.

We will show that $h(t) \cap L$ is a single point for every $t \in [0, 1]$. This will complete the proof, because $h(0) \cap L = u_0$, $h(1) \cap L = u_1$, and L lies completely on one side of h_{d+1}. Suppose the contrary and let $T \in [0, 1]$ be the smallest $t \in [0, 1]$ such that for all $\tau \in [0, t)$, $h(\tau) \cap L$ is a single point but $h(t) \cap L$ is not. (General position implies that $T > 0$.) This can happen in two different ways: either $h(T)$ contains z or $h(T)$ becomes parallel to L.

Case 1. $z \in h(T)$. Then the equations

$$a(T) \cdot x = 1, \qquad a_2 \cdot x = 1, \qquad \ldots, \qquad a_{d+1} \cdot x = 1$$

have a common solution, namely z. The points $a(T) \in \operatorname{conv} A_1^*, a_2 \in A_2^*, \ldots, a_{d+1} \in \operatorname{conv} A_{d+1}^*$ lie on the same hyperplane, namely on $\{x\colon x \cdot z = 1\}$. But this is impossible, as $A_1^*, \ldots, A_{d+1}^*$ satisfy Theorem 3.4.

Case 2. $h(T)$ is parallel to L or, equivalently, to $u_0 - z$. Then $u_0 - z$ is a solution of the equations

$$a(T) \cdot x = 0, \qquad a_2 \cdot x = 0, \qquad \ldots, \qquad a_d \cdot x = 0,$$

and also to $a_0 \cdot x = 0$ where $a_0 = 0$. Therefore, the points $a_0 \in A_0^*, a(T) \in \operatorname{conv} A_1^*, \ldots, a_d \in \operatorname{conv} A_d^*$ lie on the same hyperplane, namely on the one with equation $x \cdot (u_0 - z) = 0$. This is again impossible. □

In view of the above arguments, in Lemma 1.4 and in Theorems 1.1 and 1.2, one can take $c(d) = b(d) = c(d, d + 2) = 2^{-(d+1)2^d}$, where $c(d, d + 2)$ comes from Theorem 3.4.

4. Semi-algebraic relations — Second proof of Lemma 1.4

A *real semi-algebraic set* in $\mathbb{R}^d$ is the locus of all points that satisfy a given finite Boolean combination of polynomial equations and inequalities in the d coordinates. We say that the *description complexity* of such a set is at most s if in some representation the number of equations and inequalities is at most s and each of them is of degree at most s. Such a representation is usually called *quantifier-free*. Note that semi-algebraic sets can also be defined using quantifiers involving additional variables, but these quantifiers can always be eliminated (see [3]).

Let $H_1, \ldots, H_m$ be families of semi-algebraic sets of constant description complexity, and let R be an m-ary relation on $\prod_1^m H_i$. We assume that R is also semi-algebraic, in the following sense. We associate each $h \in H_i$ with a point $\bar{h} \in \mathbb{R}^{d_i}$ (say, with the point whose coordinates are the coefficients of the monomials in the polynomial inequalities defining h). We say that R is a *semi-algebraic m-ary relation* if its corresponding representation

$$\overline{R} = \left\{(\bar{h}_1, \dots, \bar{h}_m) \in \mathbb{R}^{d_1+\cdots+d_m} \;\middle|\; h_1 \in H_1, \;\dots,\; h_m \in H_m \; (h_1, \dots, h_m) \in R\right\}$$

is a semi-algebraic set.

We need the following result of Fox et al. [5]. Its proof is based on the case $m = 2$, established by Alon et al. [1].

Theorem 4.1. *Let $\alpha > 0$, let $H_1, \dots, H_m$ be finite families of semi-algebraic sets of constant description complexity, and let R be a fixed semi-algebraic m-ary relation on $H_1 \times \cdots \times H_m$ such that the number of m-tuples that are related (resp. unrelated) with respect to R is at least $\alpha \prod_{i=1}^m |H_i|$. Then there exists a constant $c' > 0$, which depends on α, m and on the maximum description complexity of the sets in H_i ($i \in [m]$) and R, and there exist subfamilies $H_i^* \subseteq H_i$ with $|H_i^*| \geqslant c'|H_i|$ ($i \in [m]$) such that $\prod_1^m H_i^* \subset R$ (resp. $\prod_1^m H_i^* \cap R = \emptyset$).*

Proof of Lemma 1.4. We apply Theorem 4.1 with $m = d + 1$ for the families of hyperplanes H_i, $i \in [d+1]$. As in the previous section, we associate each hyperplane $h_i \in H_i$ with its dual vector $a_i \in \mathbb{R}^d \setminus \{0\}$ satisfying

$$h_i = \left\{x \in \mathbb{R}^d \colon a_i \cdot x = 1\right\}.$$

As in (1.1), given a $(d+1)$-tuple of hyperplanes $(h_1, \dots, h_{d+1}) \in \prod_1^{d+1} H_i$, for every $i \in [d+1]$, let $v_i = \bigcap_{j \in [d+1] \setminus \{i\}} h_j$. That is, v_i is the unique solution of the equations $a_j \cdot x = 1$ for $j \in [d+1] \setminus \{i\}$. Using the assumption that the hyperplanes are in general position, we have $v_i \notin h_i$. Therefore, v_i must lie in one of the open half-spaces bounded by h_i, depending on $\operatorname{sign}(a_i \cdot v_i - 1)$.

Define 2^{d+1} different $(d+1)$-ary relations on $\prod_1^{d+1} H_i$, depending on the sign pattern

$$\big(\operatorname{sign}(a_1 \cdot v_1 - 1), \dots, \operatorname{sign}(a_{d+1} \cdot v_{d+1} - 1)\big).$$

For example, one of these relations is the relation R^+, according to which $(h_1, \dots, h_{d+1})$ are related if and only if $\operatorname{sign}(a_i \cdot v_i - 1) > 0$ for all $i \in [d+1]$. Obviously, each $(d+1)$-tuple $(h_1, \dots, h_{d+1}) \in \prod_1^{d+1} H_i$ is related by precisely one of the above relations. Therefore, for at least one relation R, the number of $(d+1)$-tuples related with respect to R is at least $\frac{1}{2^{d+1}} \prod_{i=1}^{d+1} |H_i|$. Hence, if R is a *semi-algebraic relation*, then Lemma 1.4 follows directly from Theorem 4.1.

To see that the above relations are semi-algebraic, it is sufficient to observe the following. Let A be the d by d matrix whose columns are $a_2, \dots, a_{d+1}$, and write A_k for the matrix obtained from A by replacing its kth column by a column whose each entry is 1. Since v_1 is the unique solution of the equations $a_j \cdot x = 1$ for $j \in [d+1] \setminus \{1\}$, by Cramer's rule we obtain that the kth coordinate of $v_1 \in \mathbb{R}^d$ is $\det A_k / \det A$. Thus, we have

$$a_1 \cdot v_1 - 1 = \sum_{k=1}^d a_{1k} \frac{\det A_k}{\det A} - 1,$$

where a_{1k} denotes the kth component of a_1. Consequently,

$$\operatorname{sign}(a_1 \cdot v_1 - 1) = \operatorname{sign}\left[\det A \left(\sum_{k=1}^d a_{1k} \det A_k\right) - (\det A)^2\right].$$

The last expression in square brackets is a polynomial in the variables a_{ik}, $i \in [d+1]$, $k \in [d]$. Analogously, $\operatorname{sign}(a_2 \cdot v_2 - 1), \dots, \operatorname{sign}(a_{d+1} \cdot v_{d+1} - 1)$ can be written as the sign of a polynomial, which implies that the above relations are indeed semi-algebraic. □

This proof gives a weaker constant in Lemma 1.4 and consequently in Theorem 1.1. Namely, using a quantitative version of a weaker form of Theorem 4.1 obtained by Bukh and Hubard [4], we obtain

$c(d) = b(d) = 3^{-(d+1)3^{d^2+d+1}}$. Note that Fox et al. [5] used Theorem 4.1 to establish a much stronger structure theorem for semi-algebraic relations.

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