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# Longest convex lattice chains

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ABSTRACT

Let $T$ be a triangle with two specified vertices $v_0, v_1 \in \mathbb{Z}^2$. A convex lattice chain in $T$ from $v_0$ to $v_1$ is defined naturally (see the next paragraph). In this paper we prove what the maximal length of a convex lattice chain is if the area of $T$ is fixed (and large). It is also shown that the solution is unique apart from lattice preserving affine transformations.


## 1. Introduction and main result

Given a convex body $K \subset \mathbb{R}^2$ and $t > 0$, let $n$ be the largest possible number of vertices that a convex lattice polygon contained in $tK$ can have. In [2], I. Bárány and M. Prodromou study the number $n$ and determine its asymptotic behaviour as $t \to \infty$. In order to do this, they define $m(T)$ as the maximum number of vertices that a *convex lattice chain* within a triangle $T$ can have (see [2] for precise definitions). The behaviour of $m(tT)$ is described in terms of the area of $T$ as $t \to \infty$. We ask a similar question here, but remove the factor $t$.

Define $\mathcal{G}$ to be the set of triangles $T$ in the plane with two specified vertices, $v_0$ and $v_1$, belonging to $\mathbb{Z}^2$, the integer lattice. Distinct points $p_0, p_1, \dots, p_n \in \mathbb{Z}^2 \cap T$ form a convex lattice chain in $T$ (from $v_0$ to $v_1$) if $p_0 = v_0$ and $p_n = v_1$ and the convex hull of $\{p_0, \dots, p_n\}$ has exactly $n+1$ vertices, namely $p_0, \dots, p_n$. The *length* of this convex chain is $n$. Let $\ell(T)$ denote the largest $n$ such that $T$ contains a convex lattice chain of length $n$ (from $v_0$ to $v_1$). This paper is about the maximal value of $\ell(T)$ when the area, $|T|$, of $T$ is fixed. Here is our main result, which is made more precise in Theorem 2.1 in Section 2 below.

**Theorem 1.1.** *There is $t_0 > 0$ such that for all triangles $T \in \mathcal{G}$ with $|T| > t_0$*

$$\frac{1}{8}\big(\ell(T) - 1\big)\ell(T)^2 \leqslant |T|,$$

*and this estimate cannot be improved.*

A few things have been known about $\ell(T)$. Andrews [1] showed in 1963 that the area of a convex lattice $n$-gon is at least constant times $n^3$. Andrews's result is in fact more general and applies in any dimension. It has been proved in [6] and [3] that the value of the constant is at least $1/(8\pi^2)$, implying in our case that

$$|T| \geqslant \big|\operatorname{conv}\{p_0, p_1, \dots, p_n\}\big| \geqslant \frac{(n+1)^3}{8\pi^2}.$$

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Consequently

$$\frac{1}{8\pi^2}\ell(T)^3 \leqslant |T|.$$

Another, and simpler, bound on $\ell(T)$ comes when using the *lattice width*, $w(T)$, of $T$. This is defined (cf. [5] or [3]) more generally for a compact convex set $K \subset \mathbb{R}^2$ as

$$w(K) = \min\{w(K,z)\colon z \in \mathbb{Z}^2,\ z \neq (0,0)\}, \quad \text{where}$$
$$w(K,z) = \max\{z(x-y)\colon x, y \in K\}.$$

A vector $z \in \mathbb{Z}^2$ for which the minimum is attained is called the *lattice width direction* of $K$. It is clear that at most $\lfloor w(K)+1\rfloor$ consecutive lattice lines orthogonal to $z$ intersect $K$. As every lattice line contains at most two points from a convex lattice chain, the bound

$$\ell(T) \leqslant 2w(T) + 1$$

is immediate.

We mention that $\ell(T)$, just like $|T|$ and $w(T)$, is invariant under lattice preserving affine transformations. Thus the use of the lattice width is very natural here. This invariance is important and will be used later. For instance, we assume from now on (and can do so without loss of generality), that one specified vertex of $T$, namely $v_0$, coincides with the origin.

Here is another result from [2] concerning the typical behaviour of $\ell(T)$. Let $T \in \mathcal{G}$ (with $v_0 = (0,0)$ now) and assume $\lambda \to \infty$ so that $\lambda v_1 \in \mathbb{Z}^2$. Theorem 4.1 from [2] says that

$$\lim_{\lambda\to\infty} \lambda^{-2/3}\ell(\lambda T) = \frac{6}{(2\pi)^{2/3}}\sqrt[3]{|T|}.$$

This result can be strengthened. For $T \in \mathcal{G}$ define $\overline{w}(T) = w(T)|T|^{-1/3}$.

**Theorem 1.2.** *There are constants $C, D > 0$ such that if $T \in \mathcal{G}$ with $\overline{w}(T) > C$, then*

$$\left|\ell(T) - \frac{6}{(2\pi)^{2/3}}\sqrt[3]{|T|}\right| \leqslant D\frac{\log \overline{w}(T)}{\overline{w}(T)}.$$

This determines the behaviour of $\ell(T)$ when $w(T) > C\sqrt[3]{|T|}$. Note that for a typical "fat" triangle $T$, $w(T)$ is of order $\sqrt{|T|}$. For the rest $\ell(T)$ is of order $w(T)$. The proof of Theorem 1.2 is almost identical with that of Theorem 4.1 in [2] and is therefore omitted.

In [3] another extremal problem is considered, namely, the determination of the minimal area that a convex lattice $n$-gon can have. Although our question is different, the proof and methods show some similarity.

## 2. Reformulation

We can turn around the question by asking the following minimization problem, to be called Min($n$):

$$\text{minimize}|T| \quad \text{subject to} \quad T \in \mathcal{G},\ \ell(T) = n.$$

Let $T_n$ ($n \geqslant 3$) be the triangle with vertices $v_0 = p_0 = (0,0)$, $v_1 = (\frac{1}{2}n(n-1), n)$ and $v_2 = (\frac{1}{2}n(n-1), \frac{1}{2}n)$. Thus $|T_n| = \frac{1}{8}(n-1)n^2$. For a fixed $n$ define $p_k = (\frac{1}{2}n(n-1) - \frac{1}{2}(n-k)(n-k-1), k)$ for $k = 0, \ldots, n$. It is easy to check that $p_0, \ldots, p_n$ is a convex lattice chain of length $n$ in $T_n$ from $v_0$ to $v_1$. Note that $w(T_n) = n$ so this is the range where the lattice width bound and the area bound from the previous section are about equal. Here comes the more precise form of Theorem 1.1.

**Theorem 2.1.** *There is $n_0 > 0$ such that for all $n > n_0$ the following holds. If the triangle $T \in \mathcal{G}$ contains a convex lattice chain of length $n$, then*

$$\frac{1}{8}(n-1)n^2 \leqslant |T|.$$

*Equality holds here iff $T$ is the image of $T_n$ under a lattice preserving affine transformation.*

Almost all the paper is devoted to the proof of this result. The value we obtain for $n_0$ is very large, and can be made explicit, but we have not tried to determine it.

There are two cases we know of where $T_n$ is not the minimizer for Min($n$), namely:

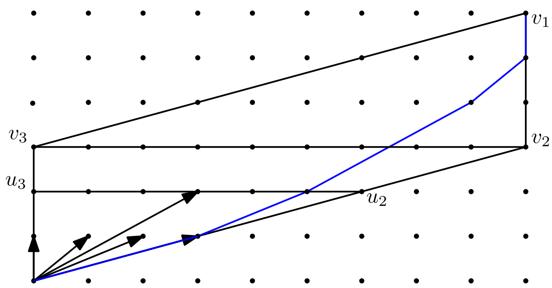


**Fig. 1.** Points for Lemma 3.1.

- When $n=3$. Let $p_0$, $p_1$, $p_2$, $p_3$ be equal to $(0,0)$, $(1,0)$, $(2,1)$, $(2,2)$ respectively. Then $|T|=2$ which is smaller (by $1/4$) than the expected $|T_3|=\frac{9}{4}$.
- When $n=5$. Let $p_0$, $p_1$, $p_2$, $p_3$, $p_4$ and $p_5$ be equal to $(0,0)$ $(1,0)$, $(3,1)$, $(4,2)$, $(6,5)$ and $(7,7)$ respectively, then $|T|=\frac{49}{4}$ which is smaller than $|T_5|=\frac{25}{2}$, again by $1/4$.

## 3. Reduction

We assume from now on that the points $p_0, p_1, \dots, p_n$ lie in this order on the perimeter of their convex hull. Let $z_i = p_i - p_{i-1}$, $i = 1, \dots, n$, these are the edge vectors of the convex lattice chain and determine the convex lattice chain completely. For $T_n$ this is just the vectors $(0,1), (1,1), (2,1), \dots, (n-1,1)$. By ordering the vectors $z_i$ by increasing slope, we can construct a convex lattice chain having them as edge vectors and every convex lattice chain defines the minimal area triangle $T$ that contains it. Define $\mathbb{P}$ to be the set of *primitive vectors* in $\mathbb{Z}^2$, that is $(a,b) \in \mathbb{Z}^2$ is in $\mathbb{P}$ if $a$ and $b$ have no common divisor (apart from $\pm 1$).

For the reduction we consider the set $\mathcal{H}_n$ of triangles $\Delta$ satisfying the conditions

- the origin is a vertex of $\Delta$,
- $|\Delta \cap \mathbb{P}| = n$,
- each side of $\Delta$ contains a point from $\mathbb{P}$.

Every $\Delta \in \mathcal{H}_n$ gives rise to a convex lattice chain with $n$ edges, and every convex lattice chain defines the minimal area triangle that contains it. This way every $\Delta \in \mathcal{H}_n$ gives rise to a triangle $T(\Delta)$. For example, the triangle $\Delta_n = \operatorname{conv}\{(0,0),(0,1),(n-1,1)\}$ gives $T(\Delta_n) = T_n$.

**Lemma 3.1.** *Let $T$ be a minimizer for the problem* $\operatorname{Min}(n)$, *with $\{z_1, \dots, z_n\}$ being the corresponding set of edge vectors. Then there is $\Delta \in \mathcal{H}_n$ with $\Delta \cap \mathbb{P} = \{z_1, \dots, z_n\}$.*

**Proof.** All the edge vectors $z_1, \dots, z_n$ are in $\mathbb{P}$ as otherwise the area of $T$ can be decreased. The vertices of $T$ are $v_0 = 0, v_1, v_2$. Let $P$ be the parallelogram with vertices $v_0, v_2, v_1, v_3 = v_1 - v_2$. (Recall that $T$ contains a convex lattice chain of length $n$ from $v_0 = 0$ to $v_1$.)

Let $C = \operatorname{pos}\{v_2, v_3\}$ be the cone with apex 0 and generators $v_2, v_3$. Then all $z_i \in C$ and also, $C = \operatorname{pos}\{z_1, z_n\}$ clearly. Let $T^*$ be the triangle that is cut off from $C$ by the line $L$ parallel with the one through $v_2$ and $v_3$ that contains $v_1$. We will use a simple fact from elementary plane geometry: If a parallelogram is contained in $T^*$, is different from $P$ and one of its vertices is 0, then its area is less than $|P|$.

Let $u_2$ and $u_3$ be points on the segments $[0, v_2]$ and $[0, v_3]$ respectively so that $[u_2, u_3]$ is parallel with $L$, the triangle $\Delta = \operatorname{conv}\{0, u_2, u_3\}$ contains $z_1, \dots, z_n$, and some edge vector, say $z_i$, is on the segment $[u_2, u_3]$. This segment is unique (see Fig. 1).

We show now that $\Delta \cap \mathbb{P} = \{z_1, \dots, z_n\}$. Assume the contrary, then there is a $z \in \mathbb{P} \cap \Delta$ which is not an edge vector. Replace $z_i$ by $z$. The new edge vectors define a new convex lattice chain that determines (uniquely) a new triangle $W = \operatorname{conv}\{0, w_1, w_2\}$ with minimal area that contains the convex lattice chain from 0 and $w_1$. The parallelogram with vertices $0$, $w_2$, $w_1$, $w_1 - w_2$ is contained in $T^*$. This is very easy when $z_i$ is different from $z_1$ or $z_n$, and not hard to see otherwise. The fact implies then that its area is smaller than $|P|$ which shows, in turn, that $|W| < |T|$. But $W$ contains a convex lattice chain of length $n$ which is impossible as $T$ is a minimizer for $\operatorname{Min}(n)$. So indeed $\Delta \cap \mathbb{P} = \{z_1, \dots, z_n\}$.

The last thing to check is that every side of $\Delta$ contains some $z_j$. This follows from $z_1 \in [0, v_2]$ and $z_n \in [0, v_3]$, therefore $\Delta \in \mathcal{H}_n$. $\square$

Consider now the following problem, to be called $\operatorname{Red}(n)$:

$$\text{minimize} \left|T(\Delta)\right| \quad \text{subject to} \quad \Delta \in \mathcal{H}_n.$$

**Theorem 3.1.** *For $n > n_0$ the triangle $\Delta_n$ is a solution to* $\mathrm{Red}(n)$*. This solution is unique apart from a lattice preserving affine transformation.*

It suffices to prove this theorem only. The plan for the proof is given next.

## 4. Plan of proof

First we bring $\Delta \in \mathcal{H}_n$ into *standard position* by a lattice preserving affine transformation as follows. Set $w(\Delta) = w$ and choose a lattice preserving affine transformation so that the lattice width direction of $\Delta$ is $(0, 1)$. Let $(0, 0)$, $(e, a)$, $(c, b)$ be the vertices of $\Delta$. We can assume that $0 \leqslant e \leqslant a$, $|b| \leqslant a$ and $ac - be = 2|\Delta| > 0$, by applying a suitable lattice preserving affine transformation.

Let $h$ be the length of the longest horizontal chord, $H$, of $\Delta$. Then $|\Delta| = \frac{1}{2}wh$. We have to consider two separate cases.

**Case 1.** When $b \geqslant 0$. Then $w = a$ and $(c, b)$ is an endpoint of $H$.

**Case 2.** When $b < 0$. Then $w = a - b$ and $(0, 0)$ is an endpoint of $H$.

It is not hard to see in both cases that $c/2 \leqslant h \leqslant c$.

For later use we record the inequality

$$\frac{cw}{4} \leqslant |\Delta| \leqslant \frac{cw}{2}. \tag{4.1}$$

Now let $S = \sum_{z \in \Delta \cap \mathbb{P}} z$. The area of $T = T(\Delta)$ can be determined in terms of $\Delta$ by

$$|T| = \frac{\det((c, b), S)\det(S, (e, a))}{2\det((c, b), (e, a))} = \frac{(cS_y - bS_x)(aS_x - eS_y)}{4|\Delta|}. \tag{4.2}$$

It is well known that the density of $\mathbb{P}$ in $\mathbb{Z}^2$ is $\frac{6}{\pi^2}$ (e.g. Theorem 459 in [4]). So in a typical triangle $\Delta$, we expect the number of primitive lattice points in $\Delta$ to be close to $\frac{6}{\pi^2}|\Delta|$ and their sum $S$ to be close to $\frac{6}{\pi^2}|\Delta|g$, where $g$ is the centre of gravity of $\Delta$. If this were the case, it follows from (4.2) that $|T|$ is close to $\frac{4|\Delta|^3}{\pi^4}$ and $\frac{|T|}{n^3} \approx \frac{\pi^2}{54} > \frac{1}{8}$.

In the first step of the proof we formalize this argument for triangles with large lattice width. Namely, we show the existence of a finite $w_0$ such that for $w > w_0$ the inequality $\frac{|T|}{n^3} > \frac{1}{8}$ holds.

In the second step we assume that $w \leqslant w_0$, and show, by subtle though lengthy and technical estimates, that $\frac{|T|}{n^3} > \frac{1}{8}$ for $w \geqslant 250$ if $n$, and then $c$, are large enough.

After this we are left with finitely many cases, roughly $250^2$ of them. Here we suppose again that $c$ is large enough. In each case the limit of $T(\Delta)/n^3$ can be exactly expressed as a rational function of the parameters $a, b$. The third step of the proof is carried out by a computer using Mathematica [7], and consists of careful checking of these cases. The outcome is, again, that $\frac{|T|}{n^3} > \frac{1}{8}$, apart from 3 special cases that are treated in the last step of the proof separately.

## 5. Large lattice width

Here we prove that $\Delta \in \mathcal{H}_n$ does not solve $\mathrm{Red}(n)$ if the lattice width of $\Delta$ is large enough.

**Lemma 5.1.** *There is $w_0 > 0$ so that if $\Delta \in \mathcal{H}_n$ and $w(\Delta) > w_0$ then $|T| = |T(\Delta)| > \frac{1}{8}n^3$.*

**Proof.** We assume that $w = w(\Delta)$ is large. In this section we use Vinogradov's convenient $f(c, w) \ll g(c, w)$ notation meaning, in our case, the existence of constants $D_1, D_2 > 0$ such that $f(c, w) \leqslant D_1 g(c, w)$ for all $c \geqslant w \geqslant D_2$ (here $c \geqslant w$ follows from $w(\Delta) = w$). For instance $\sum_{d=1}^{w} \frac{|\mu(d)|}{d} \ll \log w$, since $\sum_{d=1}^{w} \frac{|\mu(d)|}{d} < 1 + \log w \ll \log w$.

We apply a commonly used method involving the Möbius function.

$$n = \sum_{z \in T \cap \mathbb{P}} 1 = \sum_{z \in \Delta \cap \mathbb{Z}^2} \sum_{d|z} \mu(d) = \sum_{d=1}^{w} \mu(d)\#\left(\mathbb{Z}^2 \cap \frac{1}{d}\Delta\right).$$

Here the term $\#(\mathbb{Z}^2 \cap \frac{1}{d}\Delta)$ is approximately equal to $|\frac{1}{d}\Delta| = \frac{1}{d^2}|\Delta|$, so we may write $|\mathbb{Z}^2 \cap \frac{1}{d}\Delta| = \frac{1}{d^2}|\Delta| + E(d)$ where $E(d)$ is an error term. Then

$$n = |\Delta| \sum_{d=1}^{w} \frac{\mu(d)}{d^2} + \sum_{d=1}^{w} \mu(d)E(d). \tag{5.1}$$

The target is to estimate the error term $E=\sum_{d=1}^{w}\mu(d)E(d)$. To this end for every $z\in\mathbb{Z}^2$, define $Q_z$ to be the square of side length 1 with centre $z$ and sides parallel to the axis. We define the sets

$$\Gamma_d^+=\left\{z\in\mathbb{Z}^2\colon\ z\notin\frac{1}{d}\Delta \text{ and } Q_z\cap\frac{1}{d}\Delta\neq\emptyset\right\},$$
$$\Gamma_d^-=\left\{z\in\mathbb{Z}^2\colon\ z\in\frac{1}{d}\Delta \text{ and } Q_z\setminus\frac{1}{d}\Delta\neq\emptyset\right\}.$$

Thus $\Gamma_d^+$ and $\Gamma_d^-$ are the centres of the *boundary squares* $Q_z$ the ones that intersect the boundary of $\frac{1}{d}\Delta$.

**Claim 5.1.** $|\Gamma_d^+|+|\Gamma_d^-|\leqslant 2\lceil\frac{c}{d}\rceil+2\lceil\frac{w}{d}\rceil+4\ll\frac{c+w}{d}$.

**Proof.** The sides of the smallest axis parallel rectangle containing $\frac{1}{d}\Delta$ have lengths $\frac{c}{d}$ and $\frac{w}{d}$, which gives the bound on the number of boundary squares. $\square$

Define now $A_d^+$ be the union over $z\in\Gamma_d^+$ of the sets $Q_z\cap\frac{1}{d}\Delta$ and $A_d^-$ be the union over $z\in\Gamma_d^-$ of the sets $Q_z\setminus\frac{1}{d}\Delta$. Clearly $|A_d^+|\leqslant\#\Gamma_d^+$ and $|A_d^-|\leqslant\#\Gamma_d^-$.

Since we have $\#(\mathbb{Z}^2\cap\frac{1}{d}\Delta)=|\frac{1}{d}\Delta|+|A_d^+|-|A_d^-|$, it follows that

$$E(d)=\left|A_d^+\right|-\left|A_d^-\right| \quad\text{and so}\quad \left|E(d)\right|=\left|\left|A_d^+\right|-\left|A_d^-\right|\right|\ll\frac{c+w}{d}. \tag{5.2}$$

Consequently

$$\left|\sum_{d=1}^{w}\mu(d)E(d)\right|\ll(c+w)\log w\ll c\log w.$$

As $cw\ll|\Delta|\ll cw$ this implies that

$$\left|n-\sum_{d=1}^{w}\frac{\mu(d)}{d^2}|\Delta|\right|\ll|\Delta|\frac{\log w}{w}. \tag{5.3}$$

Estimating the sum of the primitive vectors in $\Delta$ is similar, just a little more involved. Let $g=\frac{1}{3}(e+c,a+b)$ be the centre of gravity of $\Delta$. Then

$$S=\sum_{z\in\Delta\cap\mathbb{P}^2}z=\sum_{z\in\Delta\cap\mathbb{Z}^2}\sum_{d|z}\mu(d)z=\sum_{d=1}^{w}d\mu(d)\sum_{z\in\frac{1}{d}\Delta}z=|\Delta|\sum_{d=1}^{w}\frac{\mu(d)}{d^2}g+\sum_{d=1}^{w}d\mu(d)\vec{E}(d), \tag{5.4}$$

where $\vec{E}(d)=(E_x(d),E_y(d))\in\mathbb{R}^2$ represents the error here. Since $\int_{\frac{1}{d}\Delta}z\,dz=\frac{1}{d^2}|\Delta|g$, we have, similarly as in Claim 5.1, that

$$\left|E_x(d)\right|\leqslant\left|\int_{A_d^+}x\,dz-\int_{A_d^-}x\,dz\right|\ll\frac{c(c+w)}{d^2}$$

and

$$\left|E_y(d)\right|\leqslant\left|\int_{A_d^+}y\,dz-\int_{A_d^-}y\,dz\right|\ll\frac{w(c+w)}{d^2}.$$

For simpler writing we define $\sigma_w=\sum_{d=1}^{w}\frac{\mu(d)}{d^2}$, and set $E_x=\sum_1^w d\mu(d)E_x(d)$ and $E_y=\sum_1^w d\mu(d)E_y(d)$. Thus with notation $S=(S_x,S_y)$ and $g=(g_x,g_y)$, $S_x=\sigma_w|\Delta|g_x+E_x$ and $S_y=\sigma_w|\Delta|g_y+E_y$. Then

$$|E_x|=\left|\sum_{d=1}^{w}d\mu(d)E_x(d)\right|\ll c^2\log w \quad\text{and}\quad |E_y|=\left|\sum_{d=1}^{w}d\mu(d)E_y(d)\right|\ll cw\log w.$$

We use (4.2) to compute $|T|$. First

$$\begin{aligned}cS_y-bS_x&=c\big(|\Delta|\sigma_w g_y+E_y\big)-b\big(|\Delta|\sigma_w g_x+E_x\big)\\&=|\Delta|\sigma_w(cg_y-bg_x)+(cE_y-bE_x)=\sigma_w\frac{2}{3}|\Delta|^2+(cE_y-bE_x)\end{aligned}$$

and similarly

$$aS_x - eS_y = a\big(|\Delta|\sigma_w g_x + E_x\big) - e\big(|\Delta|\sigma_w g_y + E_y\big)$$
$$= |\Delta|\sigma_w(ag_x - eg_y) + (aE_x - eE_y) = \sigma_w \frac{2}{3}|\Delta|^2 + (aE_x - eE_y),$$

where we used the fact that $\frac{2}{3}|\Delta| = cg_y - bg_x = ag_x - eg_y$. So we have

$$|T| = |\Delta|^3\left(\frac{1}{3}\sigma_w + \frac{cE_y - bE_x}{2|\Delta|^2}\right)\left(\frac{1}{3}\sigma_w + \frac{aE_x - eE_y}{2|\Delta|^2}\right). \tag{5.5}$$

Here $|cE_y|, |aE_x|, |bE_x| \ll c^2 w \log w$ and $|eE_y| \ll cw^2 \log w$, thus

$$|cE_y - bE_x| \ll c^2 w \log w \quad \text{and} \quad |aE_x - eE_y| \ll c^2 w \log w.$$

Using (4.1) it follows that

$$\left| |T| - \frac{\sigma_w^2}{9}|\Delta|^3 \right| \ll |\Delta|^3\left(\frac{\log w}{w} + \frac{\log^2 w}{w^2}\right) \ll |\Delta|^3 \frac{\log w}{w}.$$

This inequality, together with (5.3) finishes the proof quickly. For suitable positive constants $D_1$, $D_2$, $D_3$ we have

$$\frac{|T|}{n^3} \geqslant \frac{(\frac{\sigma_w^2}{9} - D_1\frac{\log w}{w})|\Delta|^3}{(\sigma_w + D_2\frac{\log w}{w})^3|\Delta|^3} \geqslant \frac{1}{9\sigma_w} - D_3\frac{\log w}{w}.$$

As $\sigma_w$ tends to $\frac{6}{\pi^2}$ as $w \to \infty$, the right-hand here tends to $\frac{\pi^2}{54} = 0.18277\ldots > \frac{1}{8}$. This shows that, indeed, $\frac{|T|}{n^3} > \frac{1}{8}$ if $w$ is large enough. $\square$

**Remark.** This argument can show, with a more precise computation, that $\frac{|T|}{n^3} > \frac{1}{8}$ when $w > 10^4$. But we won't need this explicit bound.

## 6. Auxiliary lemmas

We need some preparations for the case $w \leqslant w_0$. Recall that we keep the parameters $a, b, e$ fixed and wish to show that $\lim |T|/n^3 > 1/8$ as $n \to \infty$, or equivalently, as $c \to \infty$.

First we get rid of the parameter $e$: We simply change the triangle $\Delta$ by replacing its vertex $(e, a)$ by $(0, a)$. It is clear that the change in $\#(\Delta \cap \mathbb{P})$ is at most $w^2$, and the change in $S_x$, $S_y$ resp., is at most $wc$ and $w^2$ which is smaller order than the corresponding error terms (as we shall see). We keep the notation $\Delta$ for the new triangle.

We also have in both Case 1 (when $b \geqslant 0$) and Case 2 (when $b < 0$) that

$$|\Delta| = \frac{ac}{2},$$

which will work better than (4.1).

We show now that $|b| \geqslant 1$. Since the edge vector $z_1 \in \mathbb{Z}^2$ of the convex lattice chain lies on the segment $[(0,0), (c, b)]$, $|b| < 1$ implies $b = 0$, and then $z_1 = (1, 0)$ is the only possibility. Removing this vector from the convex lattice chain can only decrease the limit $\lim |T|/n^3$ and does not affect the lattice width direction, as one can check easily. We assume further that $a - b \geqslant 1$. This is evident in Case 2, and if $a - b < 1$ in Case 1, then one can change $a$ and $b$ a little so that $a - b \geqslant 1$ while $\Delta \cap \mathbb{P}$ remains unchanged.

Recall that in Case 1 $w = a$ and in Case 2 $a < w = a - b \leqslant 2a$ since $|b| \leqslant a$. So $w$ and $a$ are comparable, and in the next section it will be more convenient to work with $a$ instead of $w$.

We will need a simple bound on $\sum_1^a |\mu(d)|$ and on $\sum_1^a \frac{|\mu(d)|}{d}$.

**Lemma 6.1.** *If $a \geqslant 44$ then*

$$\sum_{d=1}^{a} |\mu(d)| \leqslant \frac{2}{3}a,$$

*and if $a \geqslant 126$ then*

$$\sum_{d=1}^{a} \frac{|\mu(d)|}{d} \leqslant \frac{6}{10} + \frac{7}{10}\log a.$$

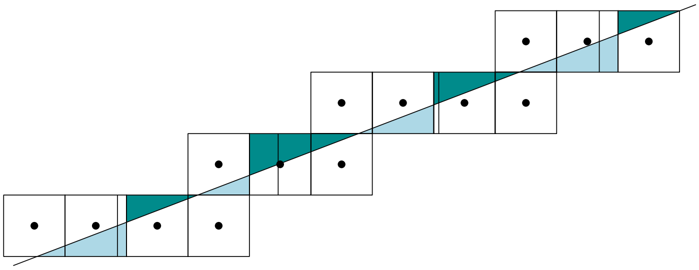

**Fig. 2.** $A_d^+$ and $A_d^-$ near $L$ and correction lines.

**Proof.** Let $I$ be a set consisting of 36 consecutive positive integers and $I'$ be equal to $I$ without the multiples of 4 and 9. Since $\mu(d)=0$ if $d$ is divisible by a square

$$\sum_{d\in I}\left|\mu(d)\right| \leqslant \sum_{d\in I'}1=\frac{2}{3}\sum_{d\in I}1.$$

The rest is a simple checking that the result holds for $a\in[44,80]$.

For the other inequality we first show that

$$\sum_{d\in I'}\frac{1}{d}<\frac{7}{10}\sum_{d\in I}\frac{1}{d}$$

if all the elements of $I$ are larger than 125. Note that this inequality is of the form

$$\sum_{d\in I''}\frac{1}{d+m}<\frac{7}{10}\sum_{d=0}^{35}\frac{1}{d+m},$$

where $m>125$ and $I''\subset[0,35]$ is set $I'$ reduced modulo 36 to the interval $[0,35]$. There are 36 different possibilities for the set $I''$ depending on the value of $m$ mod 36. For each of these cases, the inequality reduces to showing that a polynomial in the variable $m$ of degree at most 36 is positive for $m>125$. After this the only thing left is to verify the original inequality for $a\in[126,162]$. Both can be confirmed easily with the help of a computer. $\square$

## 7. Medium lattice width

Here we prove a strengthening of Lemma 5.1 for the case when $w(\Delta)$ is not too small but at most $w_0$. More precisely we show the following.

**Lemma 7.1.** *There is $n_0>0$ so that if $\Delta\in\mathcal{H}_n$, $n>n_0$ and $a>250$ then $|T|=|T(\Delta)|>\frac{1}{8}n^3$.*

**Proof.** By Lemma 5.1 $w\leqslant w_0$, and so $c\to\infty$ as $n\to\infty$. We show that for some $\epsilon>0$, $\lim_{c\to\infty}\frac{|T(\Delta)|}{n^3}>\frac{1}{8}+\epsilon$ when $a>250$ and $w\leqslant w_0$. Since here both $|T(\Delta)|$ and $n^3$ are of order $c^3$ we can ignore smaller order terms during the computations.

We want to have sharper and explicit estimates instead of (5.2) and (5.4). For simpler notation set $\widetilde{\Delta}=\frac{1}{d}\Delta$, let $\widetilde{a}=\frac{a}{d}$, $\widetilde{b}=\frac{b}{d}$, and $\widetilde{c}=\frac{c}{d}$.

We begin with Case 1, so $a=w$. We are going to estimate $E(d)=|A_d^+|-|A_d^-|$ again. The triangle $\widetilde{\Delta}$ has a lower side $L=[(0,0),(0,\widetilde{a})]$ and upper one $U=[(0,0),(\widetilde{c},\widetilde{b})]$. We ignore the boundary cells on its vertical side since they cause only a minimal $(O(1))$ error. Fig. 2 shows that $A_d^+\cup A_d^-$ near $L$ (resp. $U$) consists of triangles, bounded by $L$ (and $U$), and horizontal segments (on the lines $y=m+\frac{1}{2}$, $m$ an integer) and vertical segments of unit length (on the lines $x=m+\frac{1}{2}$, $m$ an integer). These triangle alternately belong to $A_d^+$ and $A_d^-$ and two consecutive triangles have almost the same area. We modify these triangles by moving the unit segment containing their vertical side so that $L$ (resp. $U$) halves the new unit segment. This is called a *correction*. Each correction changes the sum of the signed area of the two triangles it affects by at most $\frac{1}{2}$. After correction consecutive triangles have the same area so they cancel in $E(d)=|A_d^+|-|A_d^-|$.

Even more generally the following holds. Call a *valid period* any vertical strip of width $\frac{\widetilde{c}}{\widetilde{b}}$ between the lines $x=0$ and $x=\widetilde{c}$. Then the sum of the signed areas of the triangles in $A_d^+$ and $A_d^-$ near $L$ in a valid strip equals zero.

Consequently the contribution of $|A_d^+|-|A_d^-|$ from triangles near $L$ is at most the area of one of the triangles, which is $\frac{\widetilde{c}}{8\widetilde{b}}$ if $\widetilde{b}\geqslant 1$. There is no valid period if $\widetilde{b}<1$, and then the contribution of $|A_d^+|-|A_d^-|$ near $L$ is at most $\frac{\widetilde{c}}{2}$.

Similarly, the contribution of $|A_d^+| - |A_d^-|$ near $U$ is at most $\frac{\widetilde{c}}{8(\widetilde{a}-\widetilde{b})}$ if $\widetilde{a} - \widetilde{b} \geqslant 1$ and $\frac{\widetilde{c}}{2}$ if $\widetilde{a} - \widetilde{b} < 1$. Then, ignoring the correction terms,

$$|E| = \left|\sum_1^a \mu(d)E(d)\right| \leqslant \sum_1^a |\mu(d)|\big||A_d^+| - |A_d^-|\big|$$
$$\leqslant \sum_1^b \frac{c}{8b}|\mu(d)| + \sum_b^a \frac{c}{2}\frac{|\mu(d)|}{d} + \sum_1^{a-b} \frac{c}{8(a-b)}|\mu(d)| + \sum_{a-b}^a \frac{c}{2}\frac{|\mu(d)|}{d}$$
$$\leqslant \left[\frac{c}{8}\left(\frac{1}{b}\sum_1^b |\mu(d)| + \frac{1}{a-b}\sum_1^{a-b}|\mu(d)|\right) + \frac{c}{2}\left(\sum_b^a \frac{|\mu(d)|}{d} + \sum_{a-b}^a \frac{|\mu(d)|}{d}\right)\right].$$

Here we can assume by symmetry that $b \leqslant a - b$. The first term in the square brackets is bounded using Lemma 6.1 by $\frac{c}{2}(1 + \frac{2}{3})$. For the second term we can use the method as in the proof of Lemma 6.1 as follows: Let $\mu^*(d) = 0$ if 4 or 9 divides $d$ and $\mu^*(d) = 1$ otherwise. Then

$$\sum_b^a \frac{|\mu(d)|}{d} + \sum_{a-b}^a \frac{|\mu(d)|}{d} \leqslant \sum_b^a \frac{\mu^*(d)}{d} + \sum_{a-b}^a \frac{\mu^*(d)}{d} \leqslant \sum_{b-36m}^a \frac{\mu^*(d)}{d} + \sum_{a-b+36m}^a \frac{\mu^*(d)}{d},$$

for every positive integer $m$ such that $b - 36m > 0$. Choose $m$ so that $1 \leqslant b_0 = b - 36m \leqslant 36$. Then

$$\sum_b^a \frac{|\mu(d)|}{d} + \sum_{a-b}^a \frac{|\mu(d)|}{d} \leqslant \sum_{b_0}^a \frac{\mu^*(d)}{d} + \sum_{a-b_0}^a \frac{\mu^*(d)}{d}$$
$$\leqslant \sum_1^a \frac{\mu^*(d)}{d} + \sum_{a-36}^a \frac{\mu^*(d)}{d} \leqslant \left(\frac{6}{10} + \frac{7}{10}\log a\right) + \frac{24}{250-36}. \tag{7.1}$$

Therefore we have the bound

$$|E| < \frac{c}{2}\left(2.3789 + \frac{7}{10}\log a\right).$$

The same general method applies to $E_y = \sum_1^a d\mu(d)E_y(d)$ where $E_y(d) = \int_{A_d^+} y\,dz - \int_{A_d^-} y\,dz$. The integral on the corrections is $\frac{\widetilde{a}}{2}$, small again. On a valid period the contribution in absolute value of the integral near $L$ is at most $\frac{\widetilde{c}}{6}$ and the contribution of the final part is at most $\frac{\widetilde{c}}{4}$ if $\widetilde{b} \geqslant 1$. The same contribution near $U$ is at most $\frac{\widetilde{c}}{24} + \frac{\widetilde{c}}{8(\widetilde{a}-\widetilde{b})}(\widetilde{b}+1)$ if $\widetilde{a} - \widetilde{b} \geqslant 1$, and is $\frac{1}{2}\widetilde{c}\widetilde{a}$ if $\widetilde{a} - \widetilde{b} < 1$. This way we obtain, ignoring correction terms and using Lemma 6.1 again,

$$|E_y| = \left|\sum_1^a d\mu(d)E_y(d)\right| \leqslant \frac{c}{6}\sum_1^b |\mu(d)| + \frac{c}{4}\sum_b^a |\mu(d)|$$
$$+ \left(\frac{c}{24} + \frac{bc}{8(a-b)}\right)\sum_1^{a-b}|\mu(d)| + \frac{c}{8(a-b)}\sum_1^{a-b} d|\mu(d)| + \frac{ca}{2}\sum_{a-b}^a \frac{|\mu(d)|}{d}$$
$$< \frac{c}{2}\left(\frac{a}{3} + \frac{a+2b}{18} + \frac{a-b}{6} + a\sum_{a-b}^a \frac{|\mu(d)|}{d}\right) < \frac{c}{2}\left(1.1556a + \frac{7}{10}a\log a\right).$$

In this last part we used $b > 0$ and Lemma 6.1.

The estimate for $E_x$ is similar. The correction term is $O(c)$ this time. For the integral on the triangles near $L$ on a given valid period we get the bound $\frac{\widetilde{c}^2}{6\widetilde{b}}$ if $\widetilde{b} \geqslant 1$ and $\frac{\widetilde{c}^2}{6}$ if $\widetilde{b} < 1$. For those near $U$ we have $\frac{\widetilde{c}^2}{6(\widetilde{a}-\widetilde{b})}$ if $\widetilde{a} - \widetilde{b} \geqslant 1$ and $\frac{\widetilde{c}^2}{6}$ if $\widetilde{a} - \widetilde{b} < 1$. This gives, ignoring the correction terms again,

$$|E_x| \leqslant \frac{c^2}{6b}\sum_1^b |\mu(d)| + \frac{c^2}{6}\sum_b^a \frac{|\mu(d)|}{d} + \frac{c^2}{6(b-a)}\sum_1^{a-b}|\mu(d)| + \frac{c^2}{6}\sum_{a-b}^a \frac{|\mu(d)|}{d}$$
$$\leqslant \frac{c^2}{2}\left[\frac{2}{9} + \frac{1}{3}\sum_b^a \frac{|\mu(d)|}{d} + \frac{2}{9} + \frac{1}{3}\sum_{a-b}^a \frac{|\mu(d)|}{d}\right]$$
$$< \frac{c^2}{2}\left(0.6819 + \frac{7}{30}\log a\right).$$

Here we used Lemma 6.1 and (7.1).

Recall that $\sigma_a = \sum_1^a \frac{\mu(d)}{d^2}$. We use Eq. (5.5), which is simpler this time as $e = 0$:

$$|T| = |\Delta|^3 \left(\frac{1}{3}\sigma_a + \frac{cE_y - bE_x}{2|\Delta|^2}\right)\left(\frac{1}{3}\sigma_a + \frac{aE_x}{2|\Delta|^2}\right). \tag{7.2}$$

If $a \geqslant 250$ then $|\sigma_a - \frac{6}{\pi^2}| < \sum_{250}^{\infty} \frac{1}{d^2} < 0.004$. Finally we use $|cE_y - bE_x| \leqslant |cE_y| + |bE_x|$ to obtain

$$\lim_{n\to\infty} \frac{|T|}{n^3} \geqslant \frac{(\frac{1}{3}\sigma_a - \frac{1.8374+0.9334\log a}{a})(\frac{1}{3}\sigma_a - \frac{0.6819+0.2334\log a}{a})}{(\sigma_a + \frac{2.3789+0.7\log a}{a})^3} > \frac{1}{8} + 10^{-5}$$

when $a \geqslant 250$. This finishes the proof of Case 1.

The proof in Case 2 is very similar. There are some necessary changes, but no new idea or method. This time $b$ is negative, so $w = a - b$ and $a \geqslant -b \geqslant 1$. This means that in (5.1) for instance, $d$ runs from 1 to $a$ instead of $w$ and $\widetilde{a} - \widetilde{b}$ is never smaller than 1. It is easy to see that in this case we can obtain smaller bounds for $E$, $E_y$ and $E_x$ than in Case 1 and so $\lim \frac{|T|}{n^2} > \frac{1}{8} + 10^{-5}$ when $a \geqslant 250$. $\square$

## 8. Small lattice width

We have reduced the problem to a relatively small amount of cases, to deal with them all we use a computer. We assume again that $e = 0$. We use the Euler totient function $\varphi$ to compute $n$ and $S$.

We determine $n$ in Case 1 the following way. Given an integer $k \in [1, b]$, the number of primitive points on the line $y = k$ in $\Delta$ is $\varphi(k)\frac{b}{c} + O(k)$. The same number for an integer $k \in (b, a]$ is $\frac{\varphi(k)}{k}\frac{ca-ck}{a-b} + O(k)$. The $O(k)$ terms are small, and so is their sum.

$$n = \sum_{k=\lfloor b+1 \rfloor}^{a} \frac{\varphi(k)}{k}\left(\frac{ca - ck}{a - b} + O(k)\right) + \sum_{k=1}^{b} \frac{\varphi(k)}{k}\left(\frac{c}{b}k + O(k)\right),$$

$$\frac{n}{c} = \frac{a}{a-b}\sum_{k=b}^{a} \frac{\varphi(k)}{k} - \frac{1}{a-b}\sum_{k=b}^{a} \varphi(k) + \frac{1}{b}\sum_{k=1}^{b} \varphi(k) + O\left(\frac{1}{c}\right).$$

The computation for $S_x$, $S_y$ is similar:

$$(S_x, S_y) = \sum_{k=\lfloor b+1 \rfloor}^{a} \frac{\varphi(k)}{2k}\left(\frac{ca - ck}{a - b} + O(k)\right)\left(\frac{ca - ck}{a - b} + O(k), 2k\right) + \sum_{k=1}^{b} \frac{\varphi(k)}{2k}\left(\frac{c}{b}k + O(k)\right)\left(\frac{c}{b}k + O(k), 2k\right),$$

$$\frac{S_x}{c^2} = \frac{a^2}{2(a-b)^2}\sum_{k=b}^{a} \frac{\varphi(k)}{k} - \frac{a}{(a-b)^2}\sum_{k=b}^{a} \varphi(k) + \frac{1}{2(a-b)^2}\sum_{k=1}^{a} k\varphi(k) + \frac{1}{2b^2}\sum_{k=1}^{b} k\varphi(k) + O\left(\frac{1}{c}\right),$$

$$\frac{S_y}{c} = \frac{a}{a-b}\sum_{k=b}^{a} \varphi(k) - \frac{1}{a-b}\sum_{k=b}^{a} k\varphi(k) + \frac{1}{b}\sum_{k=1}^{b} k\varphi(k) + O\left(\frac{1}{c}\right).$$

The area of $T$ is $\frac{(\frac{c}{b}S_y - S_x)S_x}{2c/b} = \frac{1}{2}(S_y - \frac{b}{c}S_x)S_x$, so we want to bound

$$F_1 = \frac{(\frac{S_y}{c} - b\frac{S_x}{c^2})\frac{S_x}{c^2}}{2(\frac{n}{c})^3}$$

from below.

In Case 2, $b$ is negative, but we change its sign and work with it. So $(c, -b)$ is a vertex of $\Delta$ and $1 \leqslant b \leqslant a$. Doing a similar computation as in Case 1 we obtain

$$\frac{n}{c} = \frac{a}{a+b}\sum_{k=1}^{a} \frac{\varphi(k)}{k} - \frac{1}{a+b}\sum_{k=1}^{a} \varphi(k) + \frac{a}{a+b}\sum_{k=1}^{b} \frac{\varphi(k)}{k} - \left(\frac{1}{b} - \frac{1}{a+b}\right)\sum_{k=1}^{b} \varphi(k) + O\left(\frac{1}{c}\right),$$

$$\frac{S_x}{c^2} = \frac{a^2}{2(a+b)^2}\sum_{k=1}^{a} \frac{\varphi(k)}{k} - \frac{a}{(a+b)^2}\sum_{k=1}^{a} \varphi(k) + \frac{1}{2(a+b)^2}\sum_{k=1}^{a} k\varphi(k) + \frac{a^2}{2(a+b)^2}\sum_{k=1}^{b} \frac{\varphi(k)}{k}$$
$$+ \frac{a}{(a+b)^2}\sum_{k=1}^{b} \varphi(k) + \left(\frac{1}{2(a+b)^2} - \frac{1}{2b^2}\right)\sum_{k=1}^{b} k\varphi(k) + O\left(\frac{1}{c}\right),$$

$$\frac{S_y}{c} = \frac{a}{a+b}\sum_{k=1}^{a}\varphi(k) - \frac{1}{a+b}\sum_{k=1}^{a}k\varphi(k) - \frac{a}{a+b}\sum_{k=1}^{b}\varphi(k) + \left(\frac{1}{b} - \frac{1}{a+b}\right)\sum_{k=1}^{b}k\varphi(k) + O\left(\frac{1}{c}\right).$$

The area of $T$ is $\frac{(\frac{c}{b}S_y+S_x)S_x}{2c/b} = \frac{1}{2}(S_y + \frac{b}{c}S_x)S_x$, so we want to bound

$$F_2 = \frac{(\frac{S_y}{c} + b\frac{S_x}{c^2})\frac{S_x}{c^2}}{2(\frac{n}{c})^3}$$

from below.

As $c \to \infty$ we can ignore the terms $O(\frac{1}{c})$ and fix the values $\bar{a} = \lfloor a \rfloor$ and $\bar{b} = \lfloor b \rfloor$. Then for $i = 1, 2$, $F_i$ is a rational function of the variables $a$ and $b$. Not all pairs $(a, b)$ of real numbers come from one of these triangles, but we treat $F_i$ as a function defined on all real numbers. The infimum of $F_i$ in the square $(a, b) \in [\bar{a}, \bar{a}+1] \times [\bar{b}, \bar{b}+1]$ can be computed exactly using the Mathematica function `MinValue`.

If the infimum of $F_i$ is larger than $\frac{1}{8}$ and the infimum of $\frac{n}{c}$ is positive, then it follows that $F_i$ is larger than $\frac{1}{8}$ for all $a \in [\bar{a}, \bar{a}+1]$, $b \in [\bar{b}, \bar{b}+1]$ when $c$ is large enough.

This was verified for all but three of the pairs $(\bar{a}, \bar{b})$ determined by triangles $\Delta$ in standard position with $a \leqslant 250$ and $1 \leqslant |b| \leqslant a$. The pairs on which this could not be verified are $(\bar{a}, \bar{b}) = (1, 1), (2, 1)$ in Case 1 and $(\bar{a}, \bar{b}) = (1, 1)$ in Case 2. We now deal with these last three pairs.

If $(\bar{a}, \bar{b}) = (1, 1)$ in Case 1, then $\Delta \cap \mathbb{P}$ consists of the vectors $(k, 1)$ for $k = 0, \ldots, n-1$. This is the only example for which $\frac{|T|}{n^2(n-1)} = \frac{1}{8}$.

If $(\bar{a}, \bar{b}) = (2, 1)$ in Case 1, let $(k, 1)$ be the rightmost point on the line $y = 1$ and let $(l, 2)$ be the rightmost point on the line $y = 2$. Note that $l$ must be an odd integer and that $2k > l$. Then,

$$\frac{|T|}{n^2(n-1)} = \frac{(2k(1+k) + (1+l)^2)(2k^2 - (1+l)^2 + k(6+4l))}{4k(1+2k+l)(3+2k+l)^2}$$

and it can be verified with the Mathematica function `Reduce` that under these conditions this is larger than $\frac{1}{8}$ if $n \geqslant 6$.

In Case 2, when $(\bar{a}, \bar{b}) = (1, 1)$, let $(k, 1)$ be the rightmost lattice point of $\Delta$ on the line $y = 1$ and let $(m, -1)$ and $(m+l, -1)$ be the first and last lattice points in $\Delta$ on the line $y = -1$. Then,

$$\frac{|T|}{n^3} = \frac{(2+k+k^2+l+l^2+2(1+k)m)(2+k+k^2+l+l^2+2(1+l)m)}{8(3+k+l)^3 m}$$

which can be verified, again with Mathematica, to be larger than $\frac{1}{8}$ if $n \geqslant 9$.

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