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Tetrahedra passing through a triangular hole, and tetrahedra fixed by a planar frame

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ABSTRACT

We show that a convex body can pass through a triangular hole iff it can do so by a translation along a line perpendicular to the hole. As an application, we determine the minimum size of an equilateral triangular hole through which a regular tetrahedron with unit edge can pass. The minimum edge length of the hole is $(1+\sqrt{2})/\sqrt{6} \approx 0.9856$. One of the key facts for the proof is that no triangular frame can hold a convex body. On the other hand, we also show that every non-triangular frame can fix some tetrahedron.

1. Introduction

Let Ω be a compact convex disk in a plane. By a *frame* we mean the boundary $\partial\Omega$ of Ω. Suppose that the frame $\partial\Omega$ is *attached to* a convex body $K \subset \mathbb{R}^3$, that is, $K \cap \Omega \neq \emptyset$ and $\mathrm{int}(K) \cap \partial\Omega = \emptyset$, where $\mathrm{int}(K)$ denotes the interior of K. If the frame $\partial\Omega$ can be removed away from K by a continuous rigid motion of $\partial\Omega$ (or K) with keeping $\mathrm{int}(K) \cap \partial\Omega = \emptyset$, then we say $\partial\Omega$ can *slip* out of K, otherwise, we say $\partial\Omega$ *holds* K. A *unit regular tetrahedron* is a regular tetrahedron with unit edges. For example, a circular frame of diameter $1/\sqrt{2}+\varepsilon$ can hold a unit regular tetrahedron if ε is sufficiently small, see Fig. 1.

Zamfirescu [10] proved that most convex bodies can be held by a circular frame. More precisely, the convex bodies in $\mathbb{R}^3$ that cannot be held by any circular frame form a nowhere dense subset of the space of all convex bodies in $\mathbb{R}^3$ with Hausdorff metric. We first show that a triangular frame is quite different from a circular frame as follows.

Theorem 1. *A triangular frame attached to a convex body can always slip out of the convex body. Thus no triangular frame can hold a convex body.*

Regarding a frame as the boundary of a hole in a plane, we may consider whether a given convex body can pass through the hole. Itoh and Zamfirescu [3] studied the size of a hole (diameter and width) through which a regular simplex of unit edges can pass. Itoh, Tanoue, and Zamfirescu [2] determined the smallest circular hole and the smallest square hole through which a unit regular tetrahedron can pass, see also [6] for the problem in higher dimensions. Concerning a triangular hole, we have the following.

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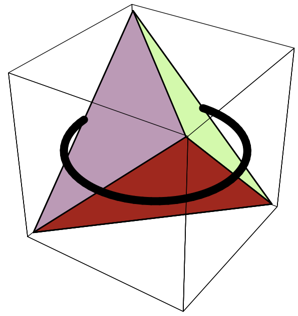

Fig. 1. A circular frame fixes a tetrahedron.

Theorem 2. *A convex body K can pass through a triangular hole Δ iff K can be congruently embedded in a right triangular prism with base Δ.*

Thus, if a convex body can pass through a triangular hole, then it can do so by a continuous translation of the convex body along a line perpendicular to the plane containing the hole. Similar assertion is not true for a circular hole. For example, when a regular tetrahedron passes through a circular hole of the smallest possible size, rotations are necessary, see [2], and [6] for higher dimensional cases.

It is proved in [7] that an equilateral triangular prism can contain a unit regular tetrahedron iff the edge length of the base equilateral triangle of the prism is at least $(1+\sqrt{2})/\sqrt{6}$. Hence we have the following.

Theorem 3. *A unit regular tetrahedron can pass through an equilateral triangular hole iff the edge length of the hole is at least $(1+\sqrt{2})/\sqrt{6}$.*

Finally we consider a fixing problem for non-triangular frames. We say that M_t is a rigid motion if $M_t : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry for each $0 \leqslant t \leqslant 1$ starting with the identity map M_0, and M_t is a continuous function of t for $0 \leqslant t \leqslant 1$. Let P be the xy-plane in $\mathbb{R}^3$, and let $H \subset P$ be a convex disk. We say that H *fixes the convex body* $K \subset \mathbb{R}^3$ if

i. $K \cap P \subset H$, and
ii. if a rigid motion M_t satisfies $(M_t K) \cap P \subset H$ for all $t \in [0,1]$, then $M_t P = P$ for all t.

This, of course, means that the frame ∂H holds K because then no rigid motion can move K away from P. In this definition one cannot require that M_t equals the identity. This is shown by the example in Fig. 1: if $\varepsilon = 0$, then the regular tetrahedron is fixed by the circle but it can clearly be rotated.

Theorem 4. *Every non-triangular frame fixes some tetrahedron.*

2. A convex body through a triangular hole

Proof of Theorem 1. Suppose that the boundary $\partial\Delta$ is a triangular frame attached to a convex body K. Let $\partial\Delta = a \cup b \cup c$ with three edges a, b, c. The triangle Δ divides K into two parts K^+ and K^-. Let H^a be a supporting plane of K containing the edge a. Then, $a \subset H^a$ and $\mathrm{int}(K) \cap H^a = \emptyset$. Define H^b similarly. Let H be the plane containing c and parallel to the line $\ell := H^a \cap H^b$. Then H^a, H^b, H determine a prism $\mathcal{P}$. One of K^+, K^- is contained in $\mathcal{P}$. (For otherwise, we can find a point $p \in K^+$ and a point $q \in K^-$ both lying in the same side of H opposite to the prism $\mathcal{P}$. Then the line segment pq does not intersects Δ, contradicting that Δ cuts the convex body K.) If $K^+ \subset \mathcal{P}$ (resp. $K^- \subset \mathcal{P}$), then K can slip out of the frame $\partial\Delta$ by moving parallel to the line ℓ towards K^- (resp. K^+) side. □

Let P be the xy-plane in $\mathbb{R}^3$. For a convex disk $\Omega \subset P$, the right Ω-prism (denoted by $\Omega \times \mathbb{R}$) is the set obtained as the union of those lines that intersect Ω perpendicularly. The set Ω is called the *base* of $\Omega \times \mathbb{R}$. If Ω is an equilateral triangle of edge length t, then the prism is called an *equilateral triangular prism of size* t.

Lemma 1. *Let $\Omega \subset P$ be a convex disk, and let $\mathcal{P} = \Omega \times \mathbb{R}$. Then, for any convex disk $\tilde{\Omega}$ obtained as a section of $\mathcal{P}$ by a plane, Ω can be congruently embedded in $\tilde{\Omega}$.*

Lemma 1 is a result due to Kovalyov [5] (answering a question of Zalgaller [9]), and independently, Debrunner and Mani-Levitska [1] (answering a question of Pach [8]), see also Kós and Törőcsik [4].

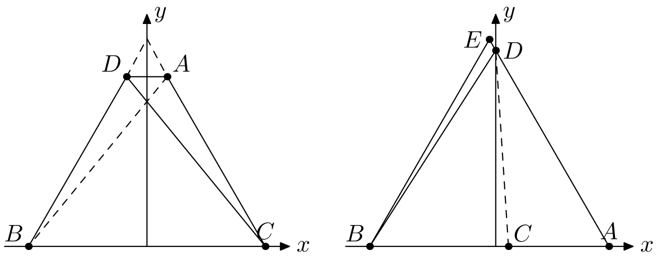

Fig. 2. Top views.

Now, let us regard a triangle $\Delta \subset P$ as a *hole*.

Proof of Theorem 2. If K is congruently embedded in $\Delta \times \mathbb{R}$, then K can pass through Δ by a translation parallel to the z-axis.

Suppose that K can pass through the hole Δ. Let $\partial\Delta = a \cup b \cup c$. Suppose that K can go through the hole Δ from the upper half space $[z \geqslant 0]$ into the lower half space $[z \leqslant 0]$. Let $K_t, 0 \leqslant t \leqslant 1$, denote the continuously moving body congruent with K, passing through the hole Δ from $[z \geqslant 0]$ to $[z \leqslant 0]$; $K_0 \subset [z \geqslant 0]$, $K_1 \subset [z \leqslant 0]$. For each $t \in [0,1]$, the plane P divides K_t into two parts, $K_t^+ = K \cap [z \geqslant 0]$ and $K_t^- = K \cap [z \leqslant 0]$. Let H_t^a be a supporting plane of B_t containing the edge a. Then this is a continuously moving plane such that $a \subset H_t^a$ and $H_t^a \cap \operatorname{int}(K_t) = \emptyset$. Define H_t^b similarly. Let H_t be the plane containing c and parallel to the line $L_t := H_t^a \cap H_t^b$. Then H_t^a, H_t^b, H_t determine a continuously moving triangular prism $\mathcal{P}_t$. Note that $\emptyset = K_0^- \subset \mathcal{P}_0$, and $\emptyset = K_1^+ \subset \mathcal{P}_1$. Furthermore, for each $t \in [0,1]$, one of K_t^+, K_t^- is contained in $\mathcal{P}_t$ as in the proof of Theorem 1. Let $\alpha = \sup\{t \in [0,1]\colon\ K_t^- \subset \mathcal{P}_t\}$. Then, there is a monotone increasing sequence $0, t_1, t_2, t_3, \ldots$ such that $K_{t_n}^- \subset \mathcal{P}_{t_n}$ and $\lim_{n\to\infty} t_n = \alpha$. Hence, by the continuity, we have $K_\alpha^- \subset \mathcal{P}_\alpha$. Similarly, since $t > \alpha$ implies $K_t^+ \subset \mathcal{P}_t$, we have $K_\alpha^+ \subset \mathcal{P}_\alpha$. Therefore, $K_\alpha \subset \mathcal{P}_\alpha$.

Thus K can be congruently embedded in a triangular prism $\mathcal{P}_\alpha$ with $\mathcal{P}_\alpha \cap P = \Delta$. By Lemma 1, $\mathcal{P}_\alpha$ is congruently embedded in $\Delta \times \mathbb{R}$. Hence K can be congruently embedded in $\Delta \times \mathbb{R}$. $\square$

Corollary 1. *If a convex body can pass through a triangular hole, then a whole process of passing through the hole can be realized by a translation along a line perpendicular to the plane having the hole.*

Proof of Theorem 3. Let $\Delta(d)$ denote an equilateral triangle with edge length d. Two congruent regular tetrahedra $T_1, T_2 \subset \Delta(d) \times \mathbb{R}$ are said to be equivalent if it is possible to superpose T_1 on T_2 by a continuous rigid motion of T_1 within the prism. Let $\nu(d)$ denote the maximum number of mutually non-equivalent embeddings of a unit regular tetrahedron into $\Delta(d) \times \mathbb{R}$. The following result is proved in [7]:

$$\nu(d) = \begin{cases} 0 & \text{for } d < d_0 := 1 + \sqrt{2}/\sqrt{6} \approx 0.9856, \\ 6 & \text{for } d_0 \leqslant d < d_1 := \sqrt{3} + 3\sqrt{2}/6 \approx 0.9958, \\ 18 & \text{for } d_1 \leqslant d < 1, \\ 1 & \text{for } 1 \leqslant d. \end{cases} \tag{1}$$

By (1) we have $\nu(d) \neq 0$ iff $d \geqslant (1+\sqrt{2})/\sqrt{6}$. In other words, a unit regular tetrahedron can be congruently embedded in $\Delta(d) \times \mathbb{R}$ iff $d \geqslant (1+\sqrt{2})/\sqrt{6}$. Combining this result with Theorem 2, we get Theorem 3. $\square$

Here we recall two important embeddings which are essentially used to show (1) in [7]. We are going to embed a unit tetrahedron $T = ABCD$ into $\Delta(d)$-prisms. First, let us consider the case $d = d_0$. Let $h = d_0/2 = (1+\sqrt{2})/\sqrt{24}$, and let $\Delta_0 \subset P$ be the triangle with vertices $(\pm h, 0, 0)$, $(0, \sqrt{3}h, 0)$. Then Δ_0 is an equilateral triangle of edge length d_0. Let $\mathcal{P}$ be the $\Delta(d_0)$-prism. Let $k = (\sqrt{2}-1)/\sqrt{24}$, $\ell = 1/\sqrt{2}$, and define four points A, B, C, D by

$$A = (k, \ell, -h), \qquad B = (-h, 0, -k), \qquad C = (h, 0, k), \qquad D = (-k, \ell, h).$$

Then one can check that these four points span a regular tetrahedron of edge length 1, which is contained in the $\Delta(d_0)$-prism $\mathcal{P}$, see Fig. 2 left.

Next we consider the case $d = d_1$. Let $\Delta_1 \subset P$ be the triangle with vertices

$$A' = \left(\frac{\sqrt{2}}{3}, 0, 0\right), \qquad B' = \left(-\frac{\sqrt{3}+\sqrt{2}}{6}, 0, 0\right), \qquad E = \left(-\frac{\sqrt{3}-\sqrt{2}}{12}, \frac{\sqrt{6}+1}{4}, 0\right).$$

A straightforward calculation shows that Δ_1 is an equilateral triangle with edge length d_1. Let $T = ABCD$ be the tetrahedron with vertices

$$A = \left(\frac{\sqrt{2}}{3}, 0, \frac{1}{3}\right), \qquad B = \left(-\frac{\sqrt{3}+\sqrt{2}}{6}, 0, \frac{\sqrt{6}-1}{6}\right), \qquad C = \left(\frac{\sqrt{3}-\sqrt{2}}{6}, 0, -\frac{\sqrt{6}+1}{6}\right), \qquad D = \left(0, \frac{\sqrt{6}}{3}, 0\right).$$

Then T is a unit regular tetrahedron contained in the Δ_1-prism, see Fig. 2 right.

What is the minimal area of a hole such that a unit regular tetrahedron $ABCD$ can pass through it? This problem is raised in [3]. Let $ABCD$ be a unit regular tetrahedron in $\mathbb{R}^3$ such that the edge AB lies on the z-axis. Then, by projecting $ABCD$ to P, we get an isosceles triangle with sides $1, \sqrt{3}/2, \sqrt{3}/2$, whose area is $1/\sqrt{8}$. Hence $ABCD$ can pass through a triangular hole of area $1/\sqrt{8}$. In fact, this is the minimum area hole that a unit regular tetrahedron can pass through by translation only. So, if we could find a smaller hole by allowing rotation for escape, then the hole would be of non-triangular shape.

Problem 1. Is $1/\sqrt{8}$ the minimal area of a hole through which a unit regular tetrahedron can pass?

In this paper, we have considered problems in $\mathbb{R}^3$. In higher dimensions, the following is proved in [6]. If a regular n-simplex Δ^n in $\mathbb{R}^n$ can pass through a hole of a regular $(n-1)$-simplex with side length ℓ_n, then $\sqrt{1-(1/n)} < \ell_n < 1$.

3. Tetrahedra fixed by a non-triangular frame

Let P be the xy-plane in $\mathbb{R}^3$, and let $H \subset P$ be a convex disk. An alternative description of fixing is the following: H fixes the convex body $K \subset \mathbb{R}^3$ if $K \cap P \subset H$ and if a rigid motion $M_t : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $K \cap (M_t^{-1}P) \subset M_t^{-1}H$ for all $t \in [0, 1]$, then $M_t P = P$ for all t. We need one more definition. A convex disk $C \subset \mathbb{R}^3$ *fits into* H if H contains a congruent copy of C. It is clear that if C fits into H, then the diameter, width, area of C is at most as large as that of H.

We will use two easy facts (Lemmas 2 and 3 below) from elementary plane geometry. Let R be the first quadrant of P. For positive reals p, q and ε, let $D_\varepsilon(p, q)$ be the ε-disk centered at (p, q), that is, $D_\varepsilon(p, q) = \{(x, y)\colon (x-p)^2 + (y-q)^2 < \varepsilon^2\}$.

Lemma 2. *Let $\varepsilon > 0$ and $p_1, q_1 > 2\varepsilon$. Then, for all $(x_1, y_1) \in D_\varepsilon(p_1, q_1) \cap R$, the maximum*

$$\max\bigl\{(x_1 - x)^2 + (y_1 - y)^2\colon (x, y) \in D_\varepsilon(0, 0) \cap R\bigr\}$$

is attained only at $(x, y) = (0, 0)$.

In other words, the origin is the unique farthest point in $D_\varepsilon(0, 0) \cap R$ from any point in $D_\varepsilon(p_1, q_1) \cap R$, which easily follows from the positions of (x, y), (x_1, y_1) and $(0, 0)$.

For $a, b, c \in \mathbb{R}^3$, we write $[a, b]$ for the line segment from a to b, and $\operatorname{dist}(c, [a, b])$ for the distance from c to $[a, b]$.

Lemma 3. *Let $a = (\alpha, 0, 0)$, $b = (\beta, 0, 0)$ and $c = (\gamma, h, 0)$, where $h > 0$. Suppose that the triangle abc has a unique longest side $[a, b]$. Then,*

$$L(c) := \bigl\{(x, y, 0)\colon 0 \leqslant y < h,\ x \in \mathbb{R}\bigr\} \subset P$$

cannot contain a congruent copy of $\triangle abc$.

Proof. The width of $\triangle abc$, that is, the shortest height of the triangle, is $\operatorname{dist}(c, [a, b]) = h$. So, the result follows. □

We also need a stronger version of Lemma 1, namely, the embedding obtained in Lemma 1 is continuous in the sense described below. For an isometry f and a compact set C, let $\|f\|_C := \max_{z \in C} |f(z) - z|$.

Lemma 4. *Let $\Omega \subset P$ and $\tilde{\Omega}$ be as in Lemma 1. Then, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any rigid motion M_t with $M_1(\tilde{\Omega}) \subset P$ and $\|M_1\|_\Omega < \delta$, one can find an isometry g on P with $g(\Omega) \subset M_1(\tilde{\Omega})$ and $\|g\|_\Omega < \varepsilon$.*

This is an easy consequence of a result from [4]. For convenience we include a sketch of the proof here.

Proof. By choosing a suitable coordinate system on P, we may assume that there exist a $\lambda \geqslant 1$ and a map $p_\lambda : (x, y) \mapsto (x, \lambda y)$ with $p_\lambda(\Omega) = \tilde{\Omega}'$, where $\tilde{\Omega}' \subset P$ is a congruent copy of $\tilde{\Omega}$. It is proved in [4] that there are two points $E, F \in \partial\Omega$ with the following property:

> Let $E' = p_\lambda(E)$ and $F' = p_\lambda(F)$ be points on $\partial\tilde{\Omega}'$. Choose F'' on the line segment $[E', F']$ so that $|E' - F''| = |E - F|$. Let h be the rotation preserving isometry on P sending E and F to E' and F'', respectively. Then, $h(\Omega) \subset \tilde{\Omega}'$.

Let N_t be a rigid motion with $N_1(\tilde{\Omega}) = \tilde{\Omega}'$. Then $g := M_1 \circ N_1^{-1} \circ h$ is the desired isometry. Indeed, $g(\Omega) \subset M_1(\tilde{\Omega})$ follows from the construction. If $\|M_1\|_\Omega$ is small, then we see that $\|N_1\|_\Omega$, $\lambda - 1$, and $\|h\|_\Omega$ are small as well. In fact, by choosing

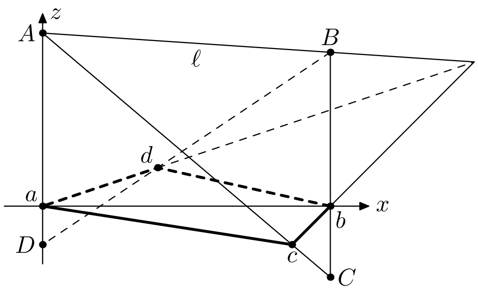

Fig. 3. Case 1. $(a, b) \subset \operatorname{int} H$.

δ sufficiently small, we can guarantee that $\|M_1\|_\Omega < \delta$ implies $\max\{\|M\|_\Omega, \|N_1\|_\Omega, \|h\|_\Omega\} < \varepsilon/3$. So it follows that $\|g\|_\Omega \leqslant \|M_1\|_\Omega + \|N_1\|_\Omega + \|h\|_\Omega < \varepsilon$. $\square$

Proof of Theorem 4. Let $H \subset P$ be a non-triangular convex disk. We construct a tetrahedron T fixed by H. Let $f(x, y) = |x - y|$ be the *distance* function, restricted to $(x, y) \in H \times H$.

Case 1. There is a local maximum of f at (a, b) such that the open segment $(a, b) \subset \operatorname{int} H$.

We may assume that $|a - b| = 1$. So let $a = (0, 0, 0)$ and $b = (1, 0, 0)$. Choose two points $c = (c_x, c_y, 0)$ and $d = (d_x, d_y, 0)$ on ∂H in the opposite side with respect to the x-axis, that is, $c_y d_y < 0$. Let $Q := \operatorname{conv}\{a, c, b, d\} \subset H$ be the convex hull of $\{a, b, c, d\}$. We construct a tetrahedron T fixed by H so that $Q = T \cap P$.

Choose a point A on the z-axis. If the lines ad and bc intersect, then let ℓ be a line passing through the intersection and A, else if $ad \parallel bc$, then let ℓ be a line passing through A and parallel to ad. Let B be the intersection of the line ℓ and the plane $x = |a - b| = 1$. Let D be the intersection of the lines Bd and Aa. Since two lines Ac and Bb intersects by Desargues's theorem, let C be the intersection. Then,

$$ab \perp AD, \; ab \perp BC, \; \text{and two lines } AD \text{ and } BC \text{ are skew}, \tag{2}$$

see Fig. 3.

Let $T = ABCD$ be our tetrahedron. Now let us verify that the four vertices a, c, b, d are all on the edges of T. To see this, it is enough to check that A and B sit in the same half-space according to the plane P, while C and D are in the other half-space. By direct computation, this is equivalent to the condition that x-coordinates of c and d are in $(0, 1)$, and y-coordinates of c and d have opposite signs. In fact, this property is equivalent to our assumption that (a, b) is a local maximum of f. Consequently, we have $Q = T \cap P$.

We fix the tetrahedron T and we try to move the frame ∂H. If we can move the frame within P only, then, by definition, T is fixed by H. Now suppose that we can move the frame slightly and it is on the plane $\tilde{P} \neq P$. More precisely, we consider a rigid motion M_t such that $T \cap (M_t^{-1} P) \subset M_t^{-1} H$ for all $t \in [0, 1]$ and $M_1^{-1} P = \tilde{P}$. Then, by (2), we have $M_t a = a$ and $M_t b = b$ for all t. So M_t is a rotation around the line ab, and thus $P \cap \tilde{P}$ coincides with the line ab.

Let $\tilde{Q} = \operatorname{conv}\{a, b, \tilde{c}, \tilde{d}\}$ be the section of our tetrahedron by the plane $\tilde{P}$, where $\tilde{c}$ (resp. $\tilde{d}$) is on the edge $[A, C]$ (resp. $[B, D]$), and let $Q' = \operatorname{conv}\{a, b, c', d'\} \subset P$ be the projection of $\tilde{Q}$ to P. Then, c' is on the line ac, because $\tilde{c}$ is on $[A, C]$. On the other hand, $\tilde{c}$ is obtained by rotating c around the line ab, and so c' is an interior point of $\triangle abc$. This contradiction completes the proof of Case 1.

Next we assume that we are not in Case 1, that is, if f has a local maximum at $(a, b) \in H \times H$, then the open segment (a, b) is on the boundary of H. Let $a, b \in H$ and suppose that $[a, b]$ is a diameter of H. Then $[a, b] \subset \partial H$, otherwise we are in Case 1. We may assume that H is contained in the first quadrant of P and $|a - b| = 1$. So put $a = (0, 0, 0)$ and $b = (1, 0, 0)$ on the x-axis. Define a distance function from b by $f_b(x) = |x - b|$ for $x \in H_0 := \partial H \setminus (a, b)$. Then, $f_b(x)$ is monotone increasing as x moves from b to a along H_0. To see this, suppose, to the contrary, that there is $c \in H_0$ such that f_b has a local maximum at $c \in H_0$. Then $[b, c] \subset \partial H$. Since H is not a triangle, we have $(a, c) \subset \operatorname{int} H$. But, by Lemma 2, f has a local maximum at (a, c). This means that we are in Case 1, a contradiction. So f_b is monotone, and similarly $f_a(x) := |x - a|$ for $x \in H_0$ is also monotone.

Case 2. There is a diameter $[a, b] \subset \partial H$ of H, and f_a is monotone.

We will choose $c, d \in H_0$, and a_i, b_i, c_i, d_i $(i = 1, 2)$ from P, see Fig. 4. We start with the following construction.

Lemma 5. *There are points $c, d \in H_0$ and $d_1, c_1, c_2 \in P$ such that c is the midpoint of $[c_1, c_2]$ and $[c_1, c_2] \cap H = \{c\}$, $[d_1, c_1]$ is parallel with $[d, c]$, $[a, d_1] \cap H = [a, d]$, $\operatorname{dist}(c, [a, b]) \geqslant \operatorname{dist}(d, [a, b])$, and the line $c_1 c_2$ intersects the line ab at z with $b \in [a, z]$.*

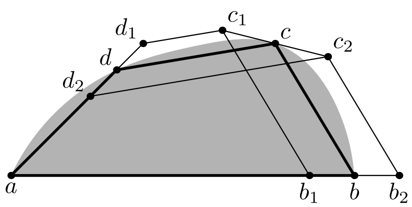

Fig. 4. Case 2. $[a,b] = \operatorname{diam} H$.

Proof. Let v be the farthest point of H from $[a,b]$. Suppose $[b,v] \subset \partial H$. Then v would do for c, we just let $z = 2b - a$ and choose a suitable pair of point c_1, c_2 on the line cz. We find d above the chord $[a,v]$ as follows. Let ℓ be the line parallel with $[a,v]$ and supporting H between a and v. As H is not a triangle, $(a,v) \subset \operatorname{int} H$, and so ℓ is disjoint from the chord $[a,v]$. Let d be the point in $\ell \cap H$ closest to $[a,b]$. The position of d_1 on the line ad is determined by the condition that $[c_1,d_1]$ parallel with $[d,c]$.

If both $(a,v), (b,v) \subset \operatorname{int} H$, then let d be the same point as before. We find c above the chord $[b,v]$ just as d was found above $[a,v]$. We assume (by swapping H with its mirror image if necessary), that $\operatorname{dist}(c,[a,b]) \geqslant \operatorname{dist}(d,[a,b])$. It is clear that there is a supporting line ℓ_c to H with $H \cap \ell_c = \{c\}$, and that ℓ_c intersects the line ab at a point z with $b \in [a,z]$. We can choose the points c_1, c_2 on ℓ_c satisfying all the conditions, and then find d_1 on the line ad such that $[c_1,d_1]$ parallel with $[d,c]$. □

Here the segment $[c_1,c_2]$ can be chosen as small as needed. For $i = 1,2$, choose b_i on the line ab so that b_ic_i is parallel to bc, and choose d_2 on the line ad so that c_2d_2 is parallel to cd. By choosing $[c_1,c_2]$ sufficiently short we can make sure that d_2 lies in the interior of the segment $[a,d]$. Let $a_1 = a_2 = a$. Set $Q_i = \operatorname{conv}\{a_i,b_i,c_i,d_i\}$ for $i = 1,2$. Let e be the unit (upward) normal vector of the plane P. Let T be the tetrahedron delimited by the planes $\operatorname{aff}\{a,b,a_1+e\}$, $\operatorname{aff}\{b,c,b_1+e\}$, $\operatorname{aff}\{c,d,c_1+e\}$, and $\operatorname{aff}\{d,a,d_1+e\}$. By the construction, we have

$$T \cap P = Q = \operatorname{conv}\{a,b,c,d\},$$
$$T \cap (P+e) = Q_1 + e = \operatorname{conv}\{a_1+e, b_1+e, c_1+e, d_1+e\},$$
$$T \cap (P-e) = Q_2 - e = \operatorname{conv}\{a_2-e, b_2-e, c_2-e, d_2-e\}.$$

We fix the tetrahedron T and we try to move the frame ∂H. Suppose that we can move the frame slightly and it is on the plane $\tilde{P}$. Namely, we consider a rigid motion M_t such that $T \cap (M_t^{-1}P) \subset M_t^{-1}H$ for all $t \in [0,1]$ and $M_1^{-1}P = \tilde{P}$. Our goal is to show that M_t is the identity, which means T is fixed by H. The plane $\tilde{P}$ intersects the edge $[a_1+e, a_2-e]$ in the point $\tilde{a}$. Define $\tilde{b}, \tilde{c}$ and $\tilde{d}$ similarly. By the construction, we have $T \cap P = Q \subset H$, and $\tilde{Q} := T \cap \tilde{P} = \operatorname{conv}\{\tilde{a},\tilde{b},\tilde{c},\tilde{d}\} \subset M_1^{-1}(H)$ fits into H. Let a' denote the orthogonal projection of $\tilde{a}$ onto the plane P. Define b', c' and d' similarly. Notice that $a' = a$, $b' \in [b_1,b_2]$, $c' \in [c_1,c_2]$, $d' \in [d_1,d_2]$.

Choose $\varepsilon > 0$ so that $6\varepsilon < \min\{c_x,c_y\}$, where $c = (c_x,c_y,0)$. (We will need this to apply Lemma 2 later.) We plug this ε into Lemma 4 to get δ. Assume that Q and $\tilde{Q}$ differ only slightly. More precisely, we assume that

$$|\tilde{c} - c| < \varepsilon/3, \quad \text{and} \quad \|M_1\|_H < \delta/3 < \varepsilon/3.$$

By Lemma 1, $a'b'c'd'$ also fits into H, and moreover, by Lemma 4, we can find an embedding close to the original position, that is, there is an isometry $g: P \to P$ satisfying $a''b''c''d'' := g(a'b'c'd') \subset H$ and $\|g\|_H < \varepsilon/3$. Then we have $|c''-c'| = |g(c')-c'| \leqslant \|g\|_H < \varepsilon/3$, $|c'-\tilde{c}| \leqslant \|M_1\|_H < \varepsilon/3$, and $|\tilde{c}-c| < \varepsilon/3$. Thus we get $|c''-c| \leqslant |c''-c'| + |c'-\tilde{c}| + |\tilde{c}-c| < \varepsilon$. Similarly, we get $|M_1\tilde{c} - c| \leqslant |M_1\tilde{c} - \tilde{c}| + |\tilde{c}-c| \leqslant \|M_1\|_H + \varepsilon/3 < 2\varepsilon/3$. In summary, we have

$$\{c'', M_1\tilde{c}\} \subset D_\varepsilon(c). \tag{3}$$

Since $c'' \in D_\varepsilon(c)$ by (3), we can apply Lemma 2 to get

$$|c''-a''| \leqslant |c''-a'|.$$

By Lemma 3, $\triangle a'b'c'$ does not fit into $L(c')$. The same is true for $\triangle a''b''c''$ ($\equiv \triangle a'b'c'$). So we have $c'' \in H \setminus L(c')$. Let c'_H (resp. c''_H) be the intersection of ∂H and the line ac' (resp. ac''), see Fig. 5.

Since $c'' \in H \setminus L(c')$, using the monotonicity of f_a, we have

$$|c''_H - a'| \leqslant |c'_H - a'|.$$

Therefore we have

$$|c''-a''| \leqslant |c''-a'| \leqslant |c''_H - a'| \leqslant |c'_H - a'| \leqslant |c'-a'| = |c''-a''|,$$

and thus $|c''-a''| = |c''-a'| = |c'-a'|$. Then, by Lemma 2, $|c''-a''| = |c''-a'|$ gives $(a=)a' = a''$. Also $c'' \in H \setminus L(c')$ and $|c''-a'| = |c'-a'|$ give $c' = c''$, which is only possible if $c' = c'' = c = \tilde{c}$.

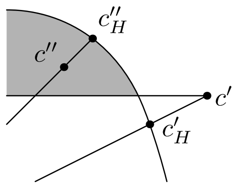

Fig. 5. $c'_H, c''_H \in \partial H$.

We will show that $a = \tilde{a}$. Observe that $M_1(\tilde{Q}) \subset H$ and

$$\operatorname{dist}\big(M_1\tilde{c}, M_1[\tilde{a}, \tilde{b}]\big) = \operatorname{dist}\big(\tilde{c}, [\tilde{a}, \tilde{b}]\big) = \operatorname{dist}\big(c, [\tilde{a}, \tilde{b}]\big) \geqslant \operatorname{dist}\big(c, [a, b]\big),$$

where the last inequality follows from the fact that $[\tilde{a}, \tilde{b}]$ is contained in the plane $y = 0$, namely, the plane whose distance to c equals $\operatorname{dist}(c, [a, b])$. So, by Lemma 3, the triangle $M_1(\triangle\tilde{a}\tilde{b}\tilde{c})$ does not fit into $L(c)$, and thus $M_1\tilde{c} \in H \setminus L(c)$. Then we have

$$|M_1\tilde{a} - M_1\tilde{c}| \leqslant |a - M_1\tilde{c}| \leqslant |a - c|,$$

where we use $M_1\tilde{c} \in D_\varepsilon(c)$ from (3) to apply Lemma 2 for the first inequality, and we use the monotonicity of f_a for the second inequality. On the other hand $|M_1\tilde{a} - M_1\tilde{c}| = |\tilde{a} - \tilde{c}| = |\tilde{a} - c| \geqslant |a - c|$ where the last inequality follows from the construction. Thus $|M_1\tilde{a} - M_1\tilde{c}| = |\tilde{a} - c| = |a - c|$ and then $\tilde{a} = a$ follows.

Now it follows from $\tilde{a} = a$ and $\tilde{c} = c$ that M_t is a rotation around the line ac. Thus $\tilde{b}$ is obtained by rotating b around ac. In this case, $b \neq \tilde{b}$ is impossible because $bb' \not\perp ac$. Therefore we have $\tilde{a} = a$, $\tilde{b} = b$ and $\tilde{c} = c$. Thus $\tilde{P} = P$ and M_t is the identity. This completes the proof of Case 2 and also of the theorem. □

Similarly to the proof of Theorem 4, one can show the following: for every convex quadrilateral $H \subset P$, there is a tetrahedron T such that T is fixed by H and $H = T \cap P$. Conversely, if we are given a tetrahedron first, then can we find such a quadrilateral frame?

Problem 2. Let T be a tetrahedron. Is it true that there is a plane P such that $H := T \cap P$ fixes T?

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