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On the variance of random polygons

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ABSTRACT

A random polygon is the convex hull of uniformly distributed random points in a convex body $K \subset \mathbf{R}^2$. General upper bounds are established for the variance of the area of a random polygon and also for the variance of its number of vertices. The upper bounds have the same order of magnitude as the known lower bounds on variance for these functionals. The results imply a strong law of large numbers for the area and number of vertices of random polygons for all planar convex bodies. Similar results had been known, but only in the special cases when K is a polygon or where K is a smooth convex body. The careful, technical arguments we needed may lead to tools for analogous extensions to general convex bodies in higher dimension. On the other hand one of the main results is a stronger version in dimension $d = 2$ of the *economic cap covering theorem* of Bárány and Larman. It is crucial to our proof, but it does not extend to higher dimension.

1. Introduction and main results

Let $K \subset \mathbf{R}^d$ be a convex set of volume one (we write $V(K) = 1$) and let $x_1, \dots, x_n$ be a random sample of n independent, identically distributed points chosen uniformly from K. The *random polytope* $K_n \equiv [x_1, \dots, x_n]$ is the convex hull of these points. Understanding the asymptotic behaviour of K_n is one of the classical problems in stochastic geometry. Starting with Rényi and Sulanke [8] in 1963, there have been many results concerning the expectation of various functionals of K_n. For instance the expectation of the random variables like the missed volume $V(K \setminus K_n)$, and of $f_0(K_n)$, the number vertices of K_n, have been determined with high precision; see e.g., the book by Schneider and Weil [11].

Determining the variance has proved to be more difficult. For *smooth* convex bodies its order of magnitude was determined by Reitzner [9] and [10]. Schreiber and Yukich [12] have computed the precise asymptotic behaviour of the variance of $f_0(K_n)$ when K is the unit ball, a significant breakthrough. Recently Bárány and Reitzner [3] obtained a lower bound on the variance of $V(K_n)$ and also of $f_\ell(K_n)$ for general convex bodies. Here f_ℓ counts the number of ℓ-dimensional faces.

In order to state the results we need a few definitions. First, $v : K \to \mathbf{R}$ is the function given by

$$v(z) = \min\bigl\{V(K \cap H)\colon\ H \text{ is a halfspace and } z \in H\bigr\}.$$

The *floating body* with parameter t is just the level set $K(v \geqslant t) = \{z \in K\colon\ v(z) \geqslant t\}$, which is clearly convex. The set $K(v \leqslant t)$ is called the *wet part*, that is, where v is at most t. From [3], the general lower bound for variance is

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Proposition 1.1. *Assume $K \subset \mathbf{R}^d$ is a convex body of volume one. Then*

$$n^{-1}V\big(K\big(v \leqslant n^{-1}\big)\big) \ll \operatorname{Var} V(K \setminus K_n),$$
$$nV\big(K\big(v \leqslant n^{-1}\big)\big) \ll \operatorname{Var} f_\ell(K_n).$$

We use Vinogradov's $f(n) \ll g(n)$ notation which means that there are constants n_0 and $c_0 > 0$ (depending possibly on d but not on K) such that $f(n) \leqslant c_0 g(n)$ for every $n \geqslant n_0$.

The main contribution of the present paper is a matching upper bound for the planar case $d = 2$.

Theorem 1.2. *Assume $K \subset \mathbf{R}^2$ is a convex body of area one. Then*

$$\operatorname{Var} V(K \setminus K_n) \ll n^{-1}V\big(K\big(v \leqslant n^{-1}\big)\big),$$
$$\operatorname{Var} f_0(K_n) \ll nV\big(K\big(v \leqslant n^{-1}\big)\big).$$

Note that the constants implied by the $\ll$ notation are universal because $d = 2$. An advantage of this kind of result is that it is usually much easier to compute the volume of the wet part than the variance of K_n.

Theorem 1.2 is the first nontrivial case of the conjecture from [3] that asserted the same upper bounds in all dimensions. It was already known to be true for smooth convex bodies (see Reitzner [10]), and a slightly weaker upper bound is proved in [3] for the cases where K is a polytope. Independently of our work, John Pardon [7] has obtained the same upper bound on the variance. Actually he proved much more, namely, the central limit theorem in the planar case.

Statements similar to Proposition 1.1 and Theorem 1.2 are known for the expectations (see [2]), for instance

$$V\big(K\big(v \leqslant n^{-1}\big)\big) \ll \mathbb{E}V(K \setminus K_n) \ll V\big(K\big(v \leqslant n^{-1}\big)\big).$$

This fact and variance bound in Theorem 1.2 combine to imply a strong law of large numbers for $V(K_n)$ and for $f_0(K_n)$ in the plane.

Corollary 1.3. *Assume $K \subset \mathbf{R}^2$ is a convex body of area one and let K_n be the random polygon generated by a uniform sample of n points from K. Then*

$$Prob\big\{n^{2/3}V(K \setminus K_n) \to c_1\big\} = 1,$$
$$Prob\big\{n^{-1/3}f_0(K_n) \to c_2\big\} = 1,$$

where c_1 and c_2 are constants depending on K.

The constants c_1, c_2 are different from zero whenever the boundary of K has a positive measure subset where the curvature exists and is positive. Reitzner [9] obtained similar statements for the case of smooth convex bodies by appealing to Tchebycheff, the Borel–Cantelli lemma, and an argument about convergence of subsequences. Corollary 1.3 actually follows in a simple way from the above bounds of [2] and our variance bound by virtue of the complete convergence theorem of Hsu and Robbins [6].

Theorem 1.2 is a direct consequence of a strengthened version of the economic cap covering theorem of Bárány [1] and Bárány and Larman [2] that holds in dimension 2, and is of independent interest. Specifically we prove

Theorem 1.4. *Let $K \subset \mathbf{R}^2$ be a convex body of area 1. There are numbers $T_0 > 0$ and $q \in (0, 1)$ such that for all $T \in (0, T_0]$ and for all $t \in (0, qT]$ the following holds. For every cap D of K of area T and for every cap covering $C_1, C_2, \dots, C_m$ of $K(v \leqslant t)$,*

$$V\big(K(v \leqslant t) \cap D\big) \ll \sum_{i=1}^{m} V(C_i \cap D) \ll V\big(K(v \leqslant t) \cap D\big).$$

In the next section we will explain "cap" and "cap covering", and show how Theorem 1.4 implies the truth of an earlier conjecture of Bárány and Reitzner [3], one that had already been shown to imply Theorem 1.2. Then, in the remaining sections, we present the proof of Theorem 1.4 and thereby, of Theorem 1.2. As a final point we mention that the present paper is a full version of [4], which appeared in the 2010 Canadian Conference on Computational Geometry.

2. Economic cap coverings

We fix the convex body $K \subset \mathbf{R}^d$ of volume one. A cap C of K is the intersection of K with a closed halfspace H. The *center* of C is a point $x \in C$ (not necessarily unique) with maximal distance from the bounding hyperplane, L, of H. The *width* of C, $w(C)$, is just the distance between x and L. For $\lambda > 0$ let H^λ be the halfspace containing H for which the width of the cap $C^\lambda = K \cap H^\lambda$ is λ times the width of C. Observe that for $\lambda \geqslant 1$, $C^\lambda \subset x + \lambda(C - x)$ implying that $V(C^\lambda) \leqslant \lambda^d V(C)$ if $\lambda \geqslant 1$.

The *minimal cap* of $z \in K$ is a cap $C(z)$ containing z such that $v(z) = V(C(z))$. Again, it need not be unique.

The *Macbeath region*, *or M-region*, for short, with center z and factor $\lambda > 0$ is

$$M(z,\lambda) = M_K(z,\lambda) = z + \lambda\big[(K - z) \cap (z - K)\big].$$

The M-region with $\lambda = 1$ is just the intersection of K and K reflected with respect to z. Thus $M(z,1)$ is convex and centrally symmetric with center z, and $M(z,\lambda)$ is a homothetic copy of $M(z,1)$ with center z and factor of homothety λ. The following lemma, originally from [5], is crucial.

Lemma 2.1. *If $M(x,1/2) \cap M(y,1/2) \neq \emptyset$, then $M(x,1) \subset M(y,5)$.*

Set

$$t_0 = (16d)^{-2d}. \tag{2.1}$$

The boundary of $K(v \geqslant t)$ is clearly $K(v = t)$. Assume $t \leqslant t_0$ and choose a maximal system of points $X = \{x_1,\dots,x_m\}$ on $K(v = t)$ having pairwise disjoint M-regions $M(x_i,1/2)$. Such a system will be called *saturated*. Note that X (and even m) is not defined uniquely. Evidently, $V(C(x_i)) = t$. Set

$$K_i = M(x_i,1/2) \cap C(x_i) \text{ and } C_i = C^{16}(x_i),$$

where $C^{16}(x_i)$ is just $(C(x_i))^\lambda$ with $\lambda = 16$. We write $[m]$ for $\{1,2,\dots,m\}$. The following result, the so-called *economic cap covering theorem*, comes from Theorem 6 in [2] and Theorem 7 in [1]. The present form is copied here from [3].

Proposition 2.2. *Suppose $t \in (0,t_0]$, $K \subset \mathbf{R}^d$ is a convex body of volume one, and $X = \{x_1,\dots,x_m\}$ is a saturated system on $K(v = t)$. Then, with C_i and K_i as defined above, the following holds*

(i) $\bigcup_1^m K_i \subset K(v \leqslant t) \subset \bigcup_1^m C_i$,
(ii) $t \leqslant V(C_i) \leqslant 16^d t$, *for* $i \in [m]$,
(iii) $(6d)^{-d} t \leqslant V(K_i) \leqslant 2^{-d} t$, *for* $i \in [m]$,
(iv) *every C with $V(C) \leqslant t$ is contained in some C_i with $i \in [m]$.*

The sets $C_1,\dots,C_m$ from this construction will be called an *economic cap covering* of $K(v \leqslant t)$.

The following conjecture is stated in [3].

Conjecture 2.3. *For every $d \geqslant 2$ there are numbers $T_0 > 0$ and $q \in (0,1)$ such that for all convex bodies $K \subset \mathbf{R}^d$ of volume one, and for all $T \in (0,T_0]$, and for all $t \in (0,qT]$ the following holds. Let $D_1,\dots,D_{m(T)}$, resp. $C_1,\dots,C_{m(t)}$ be the covering caps for $K(v \leqslant T)$ and $K(v \leqslant t)$ from Proposition 2.2. Then*

$$\sum_{i=1}^{m(T)} V\big(K(v \leqslant t) \cap D_i\big) \ll \sum_{i=1}^{m(T)}\sum_{j=1}^{m(t)} V(C_j \cap D_i) \ll \sum_{i=1}^{m(T)} V\big(K(v \leqslant t) \cap D_i\big).$$

In [3] it was shown that this conjecture implies the general upper bound, of the same order as in the lower bounds in Proposition 1.1 on the variances of the random variables $V(K \setminus K_n)$ and $f_\ell(K_n)$. The second main result of the present paper is that this conjecture is true in dimension 2. To see this, simply apply the inequalities of Theorem 1.4 to each D_i and sum the results. The left-hand side inequality is a direct consequence of the cap covering theorem.

Theorem 1.4 is a strengthening, in dimension 2, of the above conjecture. Unfortunately, the theorem does not remain true in higher dimensions. For example let $K \subset \mathbf{R}^d$ $(d \geqslant 3)$ be the truncated cone $x_1^2 + \cdots x_{d-1}^2 \leqslant x_d \leqslant 1$, the cap D is cut off from K by the hyperplane $x_d \leqslant h$ for small $h > 0$ and the cap coverings of $K(v \leqslant t)$ go with small $t > 0$. Direct computation shows that $\sum_1^m V(C_i \cap D)$ is not smaller than constant times $V(K(v \leqslant t) \cap D)$. Details are left to the reader.

If the conjecture holds for $d \geqslant 3$, a different proof idea is needed.

3. Auxiliary lemmas and preparations

Since $d = 2$ we use Area instead of V. A few properties of the M-regions and minimal caps will be needed. Some of them come from previous works, and some are going to be established here. We assume that $t \leqslant t_0$ where $t_0 = 32^{-4}$, but certainly t_0 could be taken much larger, for instance $t_0 = 1/8$. Here and in what follows we make no effort to minimize constants.

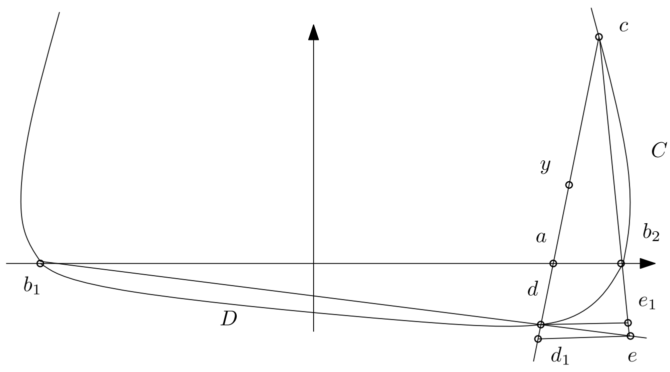

Fig. 1. The caps D, C and the quadrilateral Q.

The floating body $K(v \geqslant t)$ is convex. It was shown in [1] that its boundary $K(v = t)$ contains no line segment. This implies that if C is a cap with $\operatorname{Area} C \leqslant t_0$, then $\max\{v(x) : x \in C\}$ is reached at a unique point $z \in C$. Actually, with $v(z) = t$, $C \cap K(v \geqslant t) = \{z\}$. So z lies on the bounding segment, $[a, b]$ of C. The convex curve $K(v = t)$ has unique left and right tangents at z that cut off caps C^{left} and C^{right} from K. The next result is from [1].

Lemma 3.1. *$t = \operatorname{Area} C^{left} = \operatorname{Area} C^{right}$ and $C \subset C^{left} \cup C^{right}$. In particular, $t \leqslant \operatorname{Area} C \leqslant 2t$.*

If the left and right tangents to $K(v = t)$ at z coincide, then C is the minimal cap of z, and z is the midpoint of the bounding segment $[a, b]$ of C.

Lemma 3.2. *With the above notation $|a - z| \leqslant 2|b - z|$.*

We omit the simple proof which is based on the fact that $|a - z| = |b - z|$ when the tangents coincide.

The function $u : K \to \mathbf{R}$ is defined by $u(x) = \operatorname{Area} M(x, 1)$. Many things are known about $u(x)$. In particular it is shown in [2] that $u(x)$ and $v(x)$ are very close to each other near the boundary of K:

Lemma 3.3. *For every $x \in K$, $u(x) \leqslant 2v(x)$. If $v(x) \leqslant t_0$ or if $u(x) \leqslant t_0$, then $v(x) \leqslant 16u(x)$.*

We place the coordinate system so that the bounding segment $[b_1, b_2]$, of D lies on the x-axis, and the origin is the point where $v(x)$ takes its maximal value on D as in Fig. 1. Lemma 3.1 shows that $v(0) \leqslant T \leqslant 2v(0)$. In the next lemma, b denotes either one of the points b_1, b_2.

Lemma 3.4. *Suppose $a \in [0, b]$ with $|b| \leqslant 3^k|b - a|$ for some $k = 1, 2, \ldots$. Then $u(0) < 5^{2k}u(a)$ and $v(0) < 2^5 \cdot 5^{2k}v(a)$.*

Proof. It suffices to prove the first inequality since it implies the second via Lemma 3.3. Set $a_0 = a$ and define a_i recursively by $|b - a_i| = 3|b - a_{i-1}|$ and $[b, a_{i-1}] \subset [b, a_i]$ for $i = 1, 2, \ldots, k$ where we stop when $0 \in [a_k, b]$. By the construction neither $M(a_i, 1/2) \cap M(a_{i-1}, 1/2)$ nor $M(0, 1/2) \cap M(a_{k-1}, 1/2)$ is empty. So Lemma 2.1 gives $M(a_i, 1) \subset M(a_{i-1}, 5)$ and $M(0, 1) \subset M(a_{k-1}, 5)$.

The following statement is proved in [1].

Fact. Assume A and B are centrally symmetric convex sets with centre a and b respectively. If $B \subset A$ and $\lambda \geqslant 1$, then $b + \lambda(B - b) \subset a + \lambda(A - a)$. Applying this to the sequence $M(a_i, 1) \subset M(a_{i-1}, 5)$ we see that $M(0, 1) \subset M(a, 5^k)$ showing that $u(0) < 5^{2k}u(a)$. $\square$

Corollary 3.5. *If $a \in [0, b]$ and $v(a) \leqslant 2^{-6}5^{-2k}T$, then $|b - a| \leqslant 3^{-k}|b|$.*

To see this observe that $\operatorname{Area} D = T \leqslant 2v(0)$. Now fix the constants T_0 and q: $T_0 = t_0 = 32^{-4}$ and suppose from now on that $T \leqslant T_0$. There will be an intermediate t^* satisfying

$$t^* = 2^9 t \quad \text{and} \quad t^* \leqslant 2^{-6}5^{-4}T \tag{3.1}$$

so $q = 2^{-15}5^{-4}$.

Next let C be a cap with bounding segment $[c, d]$. Denote by y the point on $[c, d]$ where $v(x)$ reaches its maximal value on $x \in C$. Assume $[c, d]$ intersects $[b_1, b_2]$ in a point $a \in [0, b]$ where, again, b denotes either one of the points b_1 or b_2. We use the notation of Fig. 1 (where $b = b_2$). The figure is distorted since Area C should be much smaller than Area D. We write Q for the quadrilateral with vertices a, b, e, d_1.

Lemma 3.6. *If* $|b| > 3|a-b|$, *then* Area $Q \ll$ Area $C \cap D \leqslant$ Area Q.

Proof. The upper bound is trivial since $C \cap D \subset Q$. For the lower bound, let $h(x)$ denote the distance of $x \in \mathbf{R}^2$ from the x axis, and let k be the smallest integer with $|b| \leqslant 3^k|b-a|$. Then $k \geqslant 2$ and $|b| > 3^{k-1}|a-b|$. Lemma 3.2 shows that $|y-d| \leqslant 2|y-c|$ implying $h(d) \leqslant 2h(c)$, and $|d-e_1| \leqslant 3|a-b|$. Then

$$\frac{|d-e_1|}{|b_2-b_1|} = \frac{h(e)-h(d)}{h(e)},$$

implying

$$\frac{h(d)}{h(e)} \geqslant 1 - \frac{3|a-b|}{|b_2-b_1|} \geqslant \beta,$$

where $\beta = 1 - 2\cdot 3^{-k+1} > 0$; this follows from $|a-b| < 3^{-k+1}|b|$ and from $|b_2-b_1| = |b_2|+|b_1| \geqslant |b|+|b|/2$ with Lemma 3.2. Now we have

$$\begin{aligned}
\text{Area } Q &= \left[\left(\frac{h(e)+h(c)}{h(c)}\right)^2 - 1\right]\text{Area}[c,a,b_2] \\
&= \frac{h(e)}{h(c)}\left(\frac{h(e)}{h(c)}+2\right)\frac{1}{2}h(c)|a-b_2| \\
&= \frac{h(e)}{h(d)}\left(\frac{h(e)}{h(c)}+2\right)\frac{1}{2}h(d)|a-b_2| \\
&\leqslant \frac{1}{\beta}\left(\frac{2}{\beta}+2\right)\text{Area}[a,b,d] \leqslant \frac{1}{\beta}\left(\frac{2}{\beta}+2\right)\text{Area } C \cap D.
\end{aligned}$$

Note that β is increasing with k and $\beta \geqslant 1/3$ for all $k \geqslant 2$. Thus Area $Q \leqslant 24$ Area $C \cap D$. □

Remark. The lemma holds even if $a = y$, that is, when $v(x)$ reaches its maximal value on C at $x = a$. It is this form that is going to be used in Lemma 5.1.

4. The proof – part one

We define $I_0 = \{i \in [m]:\ x_i \in D\}$.

Lemma 4.1. $\sum_{i\in I_0} \text{Area } C_i \cap D \ll \text{Area } K(v \leqslant t) \cap D$.

Proof. We assume $I_0 \neq \emptyset$ as otherwise, there is nothing to prove. Using Area $C_i \leqslant 2t$ from Lemma 3.1,

$$\sum_{i\in I_0} \text{Area } C_i \cap D \leqslant \sum_{i\in I_0} \text{Area } C_i \leqslant 2t|I_0| \ll \sum_{i\in I_0} \text{Area } M(x_i, 1/2)$$

where the last inequality holds since $I_0 \neq \emptyset$ and since

$$\text{Area } M(x_i, 1/2) = \frac{1}{4}u(x_i) \geqslant \frac{1}{64}v(x_i) = \frac{1}{64}t$$

by Lemma 3.3. Further, $\sum_{i\in I_0} \text{Area } M(x_i,1/2) = 2\sum_{i\in I_0} \text{Area } K_i$, implying

$$\sum_{i\in I_0} \text{Area } C_i \cap D \ll \sum_{i\in I_0} \text{Area } K_i.$$

The bounding segment $[b_1, b_2]$ of D intersects the boundary of $K(v \geqslant t)$ in the points z_1 and z_2. Let $C(z_1)$ and $C(z_2)$ be the corresponding minimal caps. It is easy to check that, for $i \in I_0$, K_i is contained in the union of $K(v \leqslant t) \cap D$ and $C(z_1) \cup C(z_2)$. As the sets K_i are pairwise disjoint, this implies that

$$\begin{aligned}
\sum_{I_0} \text{Area } K_i &\leqslant \text{Area } K(v \leqslant t) \cap D + \text{Area } C(z_1) + \text{Area } C(z_2) \\
&= \text{Area } K(v \leqslant t) \cap D + 2t \\
&\ll \text{Area } K(v \leqslant t) \cap D
\end{aligned}$$

where the last inequality is a consequence of Area $K(v \leqslant t) \cap D \geqslant t$. □

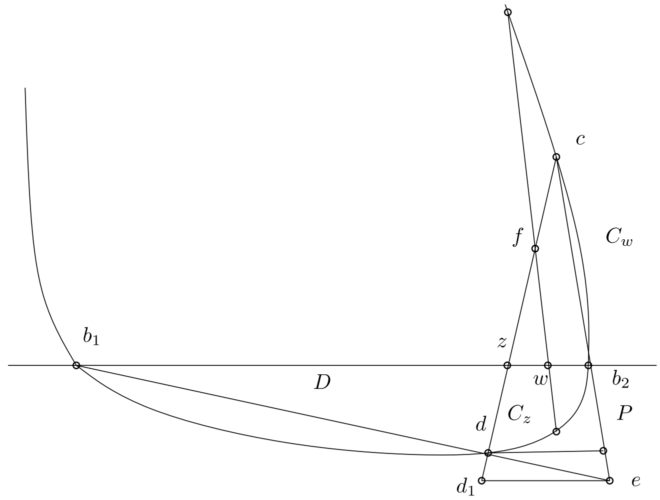

Fig. 2. Proof of Claim 5.2.

For each $x_i \notin D$ we define the cap C_i^* whose bounding segment is parallel with that of C_i so that $C_i^* \cap K(v \geqslant t^*)$ is a single point y_i. Here t^* is given by $2^9 t = t^*$, according to (3.1). We claim that $C_i \subset C_i^*$ for every $i \in [m] \setminus I_0$. Indeed, Area $C_i \leqslant 16^2 t$ because $C_i = C(x_i)^{16}$. So even if C_i is not a minimal cap, it is disjoint from $K(v \geqslant t^*)$ as shown by Lemma 3.1. It is also clear that Area $C_i^* \ll t$. We are going to show that

$$\sum_{i \in [m] \setminus I_0} \operatorname{Area} C_i^* \cap D \ll t, \tag{4.1}$$

which will finish the proof since $C_i \subset C_i^*$.

Remark. This inequality does not hold for the example given at the end of Section 2.

Define $I_1 = \{i \in [m]\colon x_i \notin D \text{ and } y_i \in D\}$ and $I = [m] \setminus (I_0 \cup I_1)$. We show next that the contribution of the terms Area $C_i^* \cap D$ with $i \in I_1$ is not too large. Estimating the contribution of I is more difficult and is done in the last section.

Lemma 4.2. $\sum_{i \in I_1} \operatorname{Area}(C_i^* \cap D) \ll t$.

This **proof** is simpler than the previous one. The wet part $K(v \leqslant t^*)$ intersects $[b_1, b_2]$ in two segments, consider one of them, $[a, b_2]$ say. Let I^* be the set of those $i \in I_1$ for which the bounding segment of C_i intersects $[a, b_2]$. By symmetry it is enough to show that $\sum_{i \in I^*} \operatorname{Area} C_j^* \cap D \ll t$.

Let $j \in I^*$ be the element for which $h(y_j)$ is the smallest. Then all other x_i with $i \in I^*$ lie in $C_j^* \setminus D \subset C_j^*$, and the corresponding K_i are pairwise disjoint, and all of them (except possibly the leftmost) are contained in C_j^*, and $K_j \subset C_j^*$, of course. Thus $|I^*| \ll \frac{1}{t} \operatorname{Area} C_j^* + 1 \ll 1$ which implies the lemma since Area $C_i^* \ll t$. $\square$

5. The proof – part two

The final steps in the proof of Theorem 1.4 bound $\sum_{i \in I} \operatorname{Area} C_i^* \cap D$ when $I = [m] \setminus (I_0 \cap I_1)$. This is more difficult than the cases covered in the previous section.

Lemma 5.1. $\sum_{i \in I} \operatorname{Area} C_i^* \cap D \ll t$.

Consider again the segment $[a, b_2]$ which is one of the two segments whose union is $K(v \leqslant t^*) \cap [b_1, b_2]$. Given $z \in [a, b_2]$ let C_z be the cap containing z on its bounding segment and having $C_z \cap K(v \geqslant t^*) = \{y\}$, a single point which is above the x axis. Similarly, C_w is the same cap with $w = (z + b_2)/2$, see Fig. 2 where we use the same notation as in Fig. 1 with a replaced by z.

For simpler writing set $b = b_2$. Let k be the smallest integer with $|b| < 3^k |z - b|$. The choice of t^* in (3.1) and Corollary 3.5 imply that $3|a - b| \leqslant |b|$. Then $3|z - b| < |b|$ as well, so $k \geqslant 2$. Recall that $\beta = 1 - 2 \cdot 3^{-k+1}$ and set $\gamma = \frac{1}{2}[1 + (2/\beta)]^{-2}$.

Claim 5.2.

$$\operatorname{Area} C_z \cap C_w > \gamma t^*.$$

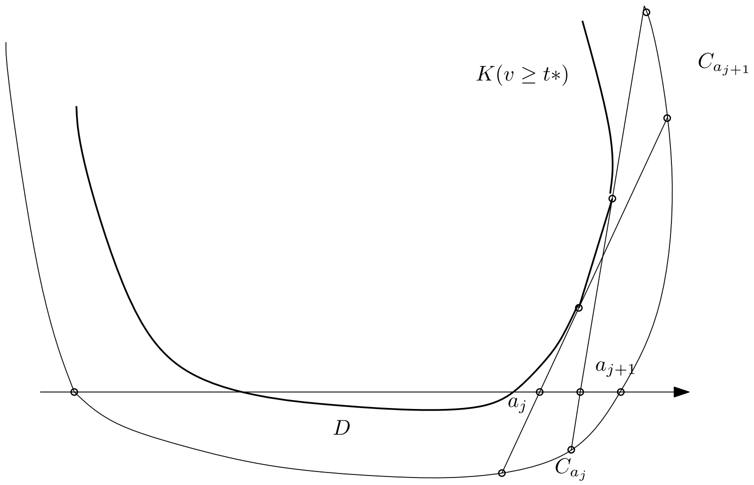

Fig. 3. Proof of Claim 5.3.

Proof. We are going to use the method of Lemma 3.6. Suppose, on the contrary, that Area $C_z \cap C_w \leqslant \gamma t^*$. Then, of course, Area$[w, b, f] <$ Area $C_z \cap C_w \leqslant \gamma t^*$ and consequently Area$[z, w, f] =$ Area$[w, b, f] \leqslant \gamma t^*$. Then Area$[z, b, c] \leqslant$ Area $C_z \cap C_w +$ Area$[z, w, f] \leqslant 2\gamma t^*$. As C_z has a single common point with $K(v \geqslant t^*)$, Area $C_z \geqslant t^*$ by Lemma 3.1, and so

$$\text{Area}\, C_z \cap D > \text{Area}\, C_z - \text{Area}\, C_z \cap C_w - \text{Area}[z, w, f] \geqslant (1 - 2\gamma)t^*.$$

Let P be the quadrilateral $[z, b_2, e, d_1]$.

$$\begin{aligned}(1-2\gamma)t^* &< \text{Area}\, C_z \cap D \leqslant \text{Area}\, P \\ &= \text{Area}[z, b, c]\left[\left(\frac{h(e)+h(c)}{h(c)}\right)^2 - 1\right] \\ &\leqslant 2\gamma t^* \frac{h(e)}{h(c)}\left(\frac{h(e)}{h(c)} + 2\right) \leqslant 2\gamma t^* \frac{2}{\beta}\left(\frac{2}{\beta} + 2\right),\end{aligned}$$

where we have $h(e)/h(c) \leqslant 2h(e)/h(d) \leqslant 2/\beta$ from the proof of Lemma 3.6. Simplifying by t^* we get

$$1 - 2\gamma < 2\gamma \frac{2}{\beta}\left(\frac{2}{\beta} + 2\right).$$

It is easy to check that this contradicts the choice of γ. $\square$

Since $\gamma = \frac{1}{2}[1 + (2/\beta)]^{-2}$ increases with k, $\gamma \geqslant 1/98$ and the Claim implies that

$$\text{Area}\, C_z \cap C_w > \frac{t^*}{98}. \tag{5.1}$$

Set $a_0 = a$ and let $a_{i+1} \in [a_i, b_2]$ be given by $|a_{i+1} - b_2| = 2|a_i - b_2|$ for $i = 0, 1, \ldots$. Define $I(j)$ as the set of those $i \in I$ for which the bounding segment of C_i^* intersects $[a, b_2]$ on the segment $[a_j, a_{j+1}]$.

Claim 5.3. $I(j) \ll 1$.

Proof. Assume $i \in I(j)$. By construction $v(x_i) = t$ and $x_i \in C_i^* \subset \bigcup\{C_z\colon\ z \in [a_j, a_{j+1}]\}$. We claim that $x_i \in C_{a_j} \cup C_{a_{j+1}}$. If not, then $C_{a_j} \cap C_{a_{j+1}} \subset C(x_i)$, see Fig. 3. But this is impossible: Area $C(x_i) = v(x_i) = t$ and Area $C_{a_j} \cap C_{a_{j+1}} > t^*/98 = 2^9 t/98$ by (5.1). Thus $x_i \in C_{a_j} \cap C_{a_{j+1}}$ for all $i \in I(j)$ implying that $K_i \subset C_{a_j} \cap C_{a_{j+1}}$. The sets K_i are disjoint and have Area $K_i \geqslant 12^{-2}t$ and the usual argument gives

$$\left|I(j)\right| \leqslant \frac{\text{Area}\, C_{a_j} \cap C_{a_{j+1}}}{12^{-2}t} \ll 1. \qquad \square$$

We are nearly finished. For all $i \in I(j)$ $\operatorname{Area} C_i^* \leqslant \operatorname{Area} C_{a_j}$ and so

$$\sum_{I(j)} \operatorname{Area} C_i^* \cap D \leqslant \left|I(j)\right| \operatorname{Area} C_{a_j} \cap D \ll \operatorname{Area} C_{a_j} \cap D.$$

We have to estimate $\sum_0^\infty \operatorname{Area} C_{a_j} \cap D$.

Define $Q = [a, b_2, e, d_1]$ as in Lemma 3.6, this time the chord $[c, d]$ touches $K(v \geqslant t^*)$ as mentioned in the Remark at the end of Section 4.

Let W_j denote the (convex) cone delimited by the halflines starting at c in direction a_j and b_2. Then $C_{a_j} \cap D \subset Q \cap W_j$ and clearly $\operatorname{Area} Q \cap W_j = 2^{-j} \operatorname{Area} Q$. Thus $\operatorname{Area} C_{a_j} \cap D \leqslant 2^{-j} \operatorname{Area} Q$. Now, using Lemma 3.6,

$$\sum_0^\infty \operatorname{Area} C_{a_j} \cap D \leqslant \sum_0^\infty 2^{-j} \operatorname{Area} Q = 2 \operatorname{Area} Q \ll \operatorname{Area} C_a \cap D \leqslant 2t^*.$$

By symmetry, the same estimate applies when we consider the other segment in $K(v \leqslant t^*) \cap [b_1, b_2]$. $\square$

References

[1] I. Bárány, Intrinsic volumes and f-vectors of random polytopes, Math. Ann. 285 (1989) 671–699.

[2] I. Bárány, D.G. Larman, Convex bodies, economic cap coverings, random polytopes, Mathematika 35 (1988) 274–291.

[3] I. Bárány, M. Reitzner, On the variation of random polytopes, Advances in Math. 225 (2010) 1986–2001.

[4] I. Bárány, W. Steiger, On the Variance of Random Polygons, in: Proceedings of the Canadian Conference on Computational Geometry, Winnipeg, Canada, 2010, pp. 211–214.

[5] G. Ewald, D.G. Larman, C.A. Rogers, The directions of the line segments and of the r-dimensional balls on the boundary of a convex body in Euclidean space, Mathematika 17 (1970) 1–20.

[6] P. Hsu, H. Robbins, Complete convergence and the law of large numbers, Proceedings of the National Academy of Sciences USA 33 (1947) 25–31.

[7] J. Pardon, Central limit theorems for random polygons in an arbitrary convex set, Annals of Probability 39 (2011) 881–903.

[8] A. Rényi, R. Sulanke, Über die konvexe Hülle von n zufällig gewählten Punkten, Z. Wahrsch. Verw. Geb. 2 (1963) 75–84.

[9] M. Reitzner, Random polytopes and the Efron–Stein jackknife inequality, Annals of Probability 31 (2003) 2136–2166.

[10] M. Reitzner, Central limit theorems for random polytopes, Probab. Theory Relat. Fields 133 (2005) 483–507.

[11] R. Schneider, W. Weil, Stochastic and Integral Geometry, Probability and its Applications, Springer-Verlag, New York–Berlin, 2008.

[12] T. Schreiber, J.E. Yukich, Variance asymptotics and central limit theorems for generalized growth processes with applications to convex hulls and maximal points, Annals of Probability 36 (2008) 363–396.