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Functions, Measures, and Equipartitioning Convex *k*-Fans

Imre Bárány · Pavle Blagojević ·
Aleksandra Dimitrijević Blagojević

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Abstract A k-fan in the plane is a point $x \in \mathbb{R}^2$ and k halflines starting from x. There are k angular sectors $\sigma_1, \dots, \sigma_k$ between consecutive halflines. The k-fan is convex if every sector is convex. A (nice) probability measure μ is equipartitioned by the k-fan if $\mu(\sigma_i) = 1/k$ for every sector. One of our results: Given a nice probability measure μ and a continuous function f defined on sectors, there is a convex 5-fan equipartitioning μ with $f(\sigma_1) = f(\sigma_2) = f(\sigma_3)$.

Keywords Measures · Convex k-fans · Equipartitions · Functions on sectors

1 Introduction

Let S^2 be the unit sphere in $\mathbb{R}^3$. A k-fan on the sphere S^2 is formed by a point $x \in S^2$ and $k \geq 3$ great semicircles $\ell_1, \dots, \ell_k$, starting from x and ending at $-x$, listed in anticlockwise order when seen from x. The spherical sector σ_i is delimited by ℓ_i and ℓ_{i+1} and its interior is disjoint from all ℓ_j. The k-fan on the sphere is *convex*, by definition, if the angle of each sector is at most π. Given a probability measure μ on S^2, the k-fan $(x; \ell_1, \dots, \ell_k)$ *equipartitions* μ if $\mu(\sigma_i) = 1/k$ for all i.

I. Bárány (✉)
Rényi Institute of Mathematics, Hungarian Academy of Sciences, P.O. Box 127, 1364 Budapest, Hungary
e-mail: barany@renyi.hu

I. Bárány
Department of Mathematics, University College London, Gower Street, London WC1E 6BT, England

P. Blagojević · A. Dimitrijević Blagojević
Institute of Mathematics, Serbian Academy of Sciences and Arts, Knez Mihajlova 36, 11000 Belgrade, Serbia

This paper is a continuation of the one by Bárány, Blagojević and Szűcs [3] which is about the following question of Nandakumar and Ramana Rao [9]. Given an integer $k \geq 2$ and a convex set $K \subset \mathbb{R}^2$ of positive area does there exist a convex k-partition of K such that all pieces have the same area and the same perimeter? The case $k = 2$ is trivial. In [3] the existence of such a partition for $k = 3$ is proved by the following, more general theorem.

Theorem 1.1 *Assume μ is a Borel probability measure on S^2 with $\mu(\ell) = 0$ for all great circles ℓ, and f is a continuous function defined on the sectors in S^2. Then there is a convex 3-fan $(x; \ell_1, \ell_2, \ell_3)$ equipartitioning μ such that*

$$f(\sigma_1) = f(\sigma_2) = f(\sigma_3). \tag{1}$$

We will explain later how this theorem settles the $k = 3$ case of the question of Nandakumar and Ramana Rao. In this paper we prove analogous properties of continuous functions defined on the sectors of equipartitioning convex 4- and 5-fans. Here are our main results.

Theorem 1.2 *Assume μ is a Borel probability measure on S^2 with $\mu(\ell) = 0$ for all great circles ℓ, and f is a continuous function defined on the sectors in S^2. Then*

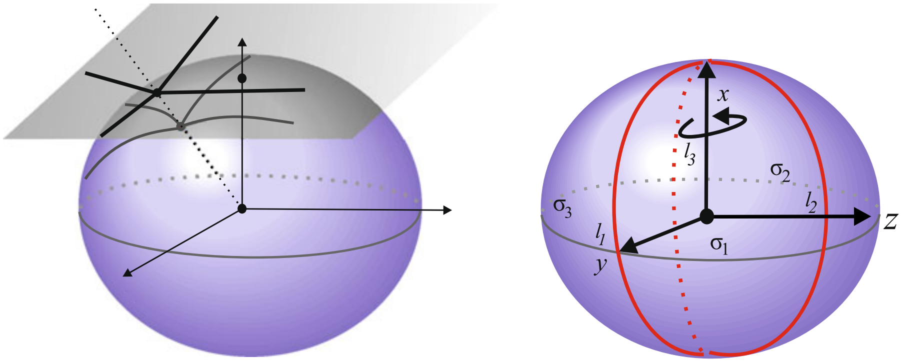
(1) *there is a convex 4-fan equipartitioning μ with $f(\sigma_1) = f(\sigma_3)$, $f(\sigma_2) = f(\sigma_4)$,*
(2) *there is a convex 4-fan equipartitioning μ with $f(\sigma_1) = f(\sigma_2)$, $f(\sigma_3) = f(\sigma_4)$,*
(3) *there is a convex 5-fan equipartitioning μ with $f(\sigma_1) = f(\sigma_2) = f(\sigma_3)$,*
(4) *there is a convex 5-fan equipartitioning μ with $f(\sigma_1) = f(\sigma_2) = f(\sigma_4)$.*

The theorem is proved by the standard configuration space/test map method with some unusual twists. It is carried out in three steps:

- The set of all equipartitioning k-fans is known to be $V_2(\mathbb{R}^3)$ the Stiefel manifold of orthogonal two-frames in $\mathbb{R}^3$. The configuration space V^{conv} is going to be the so called *convex part* of the set of all equipartitioning k-fans. It depends on the measure μ. Its definition and its topological properties will be established in Sect. 2 by geometric methods, similar to the ones used in [3].
- Defining the suitable ($\mathbb{Z}_4$ and $\mathbb{Z}_5$-equivariant) test maps from V^{conv} to the phase space is done in Sect. 3. Some extra care has to be exercised in case (2) of Theorem 1.2. We will show that the non-existence of such a test map implies Theorem 1.2.
- The non-existence of such $\mathbb{Z}_4$ and $\mathbb{Z}_5$-equivariant maps is established in Theorems 5.1 and 6.1 with the help of Serre spectral sequences of Borel constructions. This is in Sects. 5 and 6.

It would be better to show that, under the conditions of Theorem 1.2, there is an equipartitioning convex 4-fan resp, 5-fan with $f(\sigma_1) = f(\sigma_j)$ for all i, j. But this is too much to hope for as the following results show. The examples are in the plane $\mathbb{R}^2$ but they work on S^2 as well (see the remark below).

Theorem 1.3 *There are absolutely continuous measures in the plane μ and ν such that*



(A) (B)

Fig. 1 (**A**) Central projection ρ, (**B**) correspondence $F_k \longleftrightarrow V_2(\mathbb{R}^3)$

(1) *there is no convex* 4*-fan simultaneously equipartitioning* μ *and* ν.
(2) *there is no convex* 5*-fan equipartitioning* μ *such that* $\nu(\sigma_i) = \nu(\sigma_{i+1}) = \nu(\sigma_{i+2}) = \nu(\sigma_{i+3})$ *for some* $i = 1, 2, 3, 4, 5$, *the subscripts are taken* $\mod 5$.
(3) *there is no convex* 4*-fan and no* $t \in (0, 1/3)$ *such that* $\mu(\sigma_i) = \nu(\sigma_i) = t$ *for three subscripts* $i \in \{1, 2, 3, 4\}$.

The first part of the theorem is the result of Bárány and Matoušek [2, Theorem 1.1.(i).(d)]. The proof of their result is repeated in Sect. 7. The same section contains the proof of the second and third parts. In all cases the construction works because the convexity condition reduces the degree of freedom by one.

Remark Here is the short explanation on how Theorem 1.1 answers the question of Nandakumar and Ramana Rao affirmatively. A k-fan in the plane is formed by a point $x \in \mathbb{R}^2$ and k halflines $\ell_1, \dots, \ell_k$, starting from x, listed in anticlockwise order around x. There are k angular sectors $\sigma_1, \dots, \sigma_k$ determined by the fan. Here σ_i is the sector between halflines ℓ_i and ℓ_{i+1}. The k-fan in the plane is *convex* if and only if each of the sectors $\sigma_1, \dots, \sigma_k$ is convex.

It is easier to work with spherical fans than with planar ones mainly because S^2 is compact. The plane $\mathbb{R}^2$ is embedded in $\mathbb{R}^3$ as the tangent plane to S^2 at the point $(0, 0, 1)$. Let $\rho : \{(x_1, x_2, x_3) \in S^2 \mid x_3 > 0\} \to \mathbb{R}^2$ be the central projection. The map ρ lifts any nice measure in the plane to a nice measure on the sphere. Also, a k-fan in the plane lifts to a k-fan on the sphere and a k-fan on the sphere projects to a k-fan in the plane, Fig. 1(A). Also, convexity of the fan is preserved under lifting and projection. Therefore any theorem about fan partitions in the plane is a consequence of a similar and more general theorem about fan partitions on the sphere S^2.

Remark In Theorem 1.2 the measure μ is required to satisfy $\mu(\ell) = 0$ for all great circles ℓ. By a standard compactness argument it suffices to prove the theorem for a dense set of Borel probability measures.

2 Configuration Space of Equipartitioning Convex k-Fans

This section is taken from [3, Sections 3 and 4] with the view towards the high dimensional applications. Here we work with general k -fans for all $k > 3$ although what we have in mind $k = 4, 5$.

Let μ be an absolutely continuous (with respect to the Lebesgue measure) Borel probability measure on S^2 such that $\mu(\ell) = 0$ for all great circles ℓ. For $k \geq 3$ consider the following family of k-fans on S^2:

$$F_k = \left\{(x; \ell_1, \ldots, \ell_k) \mid \mu(\sigma_1) = \cdots = \mu(\sigma_k) = \frac{1}{k}\right\}.$$

For $(x; \ell_1, \ldots, \ell_k) \in F_k$, let $y = \ell_1 \cap (\mathrm{span}\{x\})^\perp \in S^2$ and $z \in (\mathrm{span}\{x\})^\perp \cap (\mathrm{span}\{y\})^\perp \cap S^2$ be such that the base (y, z) of the linear space $(\mathrm{span}\{x\})^\perp$ induce the orientation given by ordering of great semicircles $(\ell_1, \ldots, \ell_k)$, Fig. 1(B). Thus, $z = x \times y$, where $\times$ denotes the cross product. The correspondence $(x; \ell_1, \ldots, \ell_k) \longmapsto (x, y)$ induces a homeomorphism between the family of fans F_k and the Stiefel manifold $V_2(\mathbb{R}^3)$. Let $\mathbb{Z}_k = \langle \varepsilon \rangle$ be a cyclic group. There is natural free $\mathbb{Z}_k$-action on F_k given by

$$\varepsilon \cdot (x; \ell_1, \ldots, \ell_k) = (x; \ell_2, \ldots, \ell_k, \ell_1).$$

The main objective of this section is to describe the subfamily of all convex k-fans contained in F_k as a $\mathbb{Z}_k$-invariant subspace.

Let $p : (F_k = V_2(\mathbb{R}^3)) \to S^2$ denotes the S^1 fibration given by $(x; \ell_1, \ldots, \ell_k) = (x, y) \longmapsto z = x \times y$ and let $h : S^2 \to \mathbb{R}$ be the function defined by $h(z) = \mu(H(z))$, where $H(z) = \{v \in S^2 \mid v \cdot z \leq 0\}$ is the lower hemisphere with respect to z. As shown in [3], one can assume that the composition $h : S^2 \to \mathbb{R}$ is a smooth map and that has a regular value at the point $\frac{1}{k}$, i.e., $h^{-1}(\{\frac{1}{k}\})$ is an 1-dimensional embedded submanifold of S^2, [5, Corollary 7.4, p. 84].

Lemma 2.1 *For the fan* $(x; \ell_1, \ldots, \ell_k) = (x; \sigma_1, \ldots, \sigma_k) = (x, y)$, $z = x \times y = p(x, y)$, *the sector* σ_k *is not convex if*

$$(h \circ p)(x; \sigma_1, \ldots, \sigma_k) = h(z) < \frac{1}{k}.$$

Proof Since $\mu(\sigma_k) = \frac{1}{k}$ and $\mu(H(z)) < \frac{1}{k}$, then σ_k properly contains the hemisphere $H(z)$ and therefore is not convex. □

Direct consequence of the previous lemma is the characterization of the (non)convex k-fans:

$$(x; \sigma_1, \ldots, \sigma_k) \quad \text{is convex} \quad \Longleftrightarrow \quad (\forall i \geq 0)(h \circ p)\big(\varepsilon^i(x; \sigma_1, \ldots, \sigma_k)\big) \geq \frac{1}{k}$$

or

$$(x; \sigma_1, \ldots, \sigma_k) \quad \text{is not convex} \quad \Longleftrightarrow \quad (\exists i \geq 0)(h \circ p)\big(\varepsilon^i(x; \sigma_1, \ldots, \sigma_k)\big) < \frac{1}{k}.$$

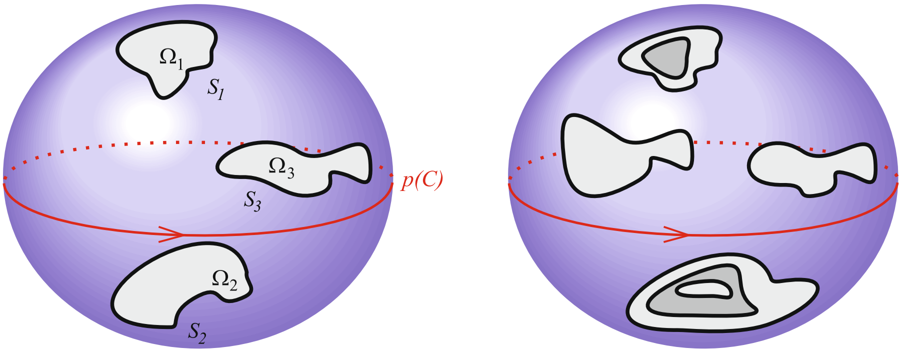


Fig. 2 The cycles S_i and discs Ω_i

Lemma 2.2 *After a possible rotation of the measure* μ, *the circle*

$$C = \{(e_3, y) \in V_2(\mathbb{R}^3) \mid y \in S((\mathrm{span}\{e_3\})^{\perp})\} \subset V_2(\mathbb{R}^3)$$

is invariant under the $\mathbb{Z}_k$*-action and every point* $(e_3, y) \in C$ *defines a convex* k*-fan.*

Proof The following result of Dolnikov [7] and Živaljević, Vrećica [10] is needed.

For $n \le d$ probability measures in $\mathbb{R}^d$, there exists a $(n-1)$-dimensional affine subspace such that the measure of every halfspace containing this affine subspace is at least $\frac{1}{d+2-n}$ in every one of the k measures.

We use it with $d = 3$ and $n = 2$. Let the first measure be μ and the second one concentrated at the origin. Then the affine space is a line passing through the origin. We may assume, by rotating S^2 if necessary, that the line passes through e_3. Since $k > 3$, then $h(e_3 \times y) \ge \frac{1}{3} > \frac{1}{k}$ for every $y \in S((\mathrm{span}\{e_3\})^{\perp})$. Thus, the circle C is invariant under the $\mathbb{Z}_k$-action and each $(e_3, y) \in C$ defines a convex k-fan. □

The point $\frac{1}{k}$ is the regular value of the function h. Thus, $h^{-1}(\{\frac{1}{k}\})$ is an 1-dimensional embedded submanifold of S^2, i.e., union of disjoint cycles S_i, $i \in [m] = \{1, \dots, m\}$. The image $p(C)$ is the equator of the sphere S^2 and $h(e_3 \times y) > \frac{1}{k}$ for every $y \in p(C)$. Therefore, every cycle S_i is disjoint from the equator $p(C)$ and so belongs to the upper or lower hemisphere.

Let Ω_i denote the closed disc bounded by S_i, $\partial\Omega_i = S_i$, and not containing $p(C)$. Notice that also $p(C) \cap \Omega_i = \emptyset$. Let $U_i = p^{-1}(\Omega_i)$ and $T_i = p^{-1}(S_i)$. The fibrations $p : U_i \to \Omega_i$ is the fibration over the contractible space Ω_i and therefore homeomorphic to the trivial fibration. Thus $U_i \approx S^1 \times \Omega_i$ is a solid torus and its boundary $T_i \approx S^1 \times S_i \approx S^1 \times S^1$ is an ordinary torus.

The $\mathbb{Z}_k$-action is given by the homeomorphism $\varepsilon : V_2(\mathbb{R}^3) \to V_2(\mathbb{R}^3)$. Hence $U_i, \varepsilon \cdot U_i, \dots, \varepsilon^{k-1} \cdot U_i$ are solid tori and $T_i, \varepsilon \cdot T_i, \dots, \varepsilon^{k-1} \cdot T_i$ are ordinary tori for every $i \in [m]$. The relationships between these tori are described in the following proposition which is just the modification of [3, Claim 3.7, 3.8, 3.9].

Proposition 2.3

(1) *The cycle C is disjoint from all solid tori $U_i, \varepsilon \cdot U_i, \dots, \varepsilon^{k-1} \cdot U_i$, $i \in [m]$.*
(2) $\varepsilon^{\alpha} \cdot T_i \cap \varepsilon^{\beta} \cdot T_j \neq \emptyset \implies i = j$ *and* $\alpha = \beta$.
(3) *The tori $U_i, \varepsilon \cdot U_i, \dots, \varepsilon^{k-1} \cdot U_i$ are pairwise disjoint*, $i \in [m]$.

Proof

(1) Let us assume that $C \cap \varepsilon^{\alpha} \cdot U_i \neq \emptyset$. Since $\varepsilon \cdot C = C$, we have

$$C \cap U_i \neq \emptyset \quad \Longrightarrow \quad p(C) \cap p(U_i) \neq \emptyset \quad \Longrightarrow \quad p(C) \cap \Omega_i \neq \emptyset.$$

Contradiction with definition of Ω_i.

(2) Let $\alpha = \beta$ and $\varepsilon^{\alpha} \cdot T_i \cap \varepsilon^{\alpha} \cdot T_j \neq \emptyset$. Then

$$T_i \cap T_j \neq \emptyset \Longrightarrow p(T_i) \cap p(T_j) \neq \emptyset \quad \Longrightarrow \quad S_i \cap S_j \neq \emptyset \quad \Longrightarrow \quad i = j.$$

Let $0 \leq \alpha < \beta \leq k$. Without losing the generality, we can assume that $\alpha = 0$. Let $(x; \ell_1, \dots, \ell_k) \in T_i \cap \varepsilon^{\beta} \cdot T_j \neq \emptyset$. Then $(x; \ell_1, \dots, \ell_k) \in T_i$, $\varepsilon^{-\beta} \cdot (x; \ell_1, \dots, \ell_k) = (x; \ell_{k-\beta+1}, \dots, \ell_{k-\beta}) \in T_j$ and consequently σ_k and $\sigma_{k-\beta}$ are hemispheres. This cannot be: a contradiction.

(3) In this part of the proof we use the Generalized Jordan Curve theorem [5, Corollary 8.8, p. 353]. Since $H_1(V_2(\mathbb{R}^3), \mathbb{Z}) = 0$, every torus $\varepsilon^{\alpha} \cdot T_i$ splits $V_2(\mathbb{R}^3)$ into two disjoint parts. Let us assume that $\varepsilon^{\alpha} \cdot U_i \cap \varepsilon^{\beta} \cdot U_i \neq \emptyset$, $0 \leq \alpha < \beta \leq k$. Again, it is enough to consider the case $\alpha = 0$. Since $T_i \cap \varepsilon^{\beta} \cdot T_i = \emptyset$ and $H_2(V_2(\mathbb{R}^3), \mathbb{Z}) = 0$ then the complement $V_2(\mathbb{R}^3) \backslash (T_i \cup \varepsilon^{\beta} \cdot T_i)$ has three components. The intersection $U_i \cap \varepsilon^{\beta} \cdot U_i$ is one of these three components with the boundary T_i or $\varepsilon^{\beta} \cdot T_i$ or $T_i \cup \varepsilon^{\beta} \cdot T_i$. We discuss these three cases separately.

   (a) Let $\partial(U_i \cap \varepsilon^{\beta} \cdot U_i) = T_i \subset U_i$. Then $U_i \subseteq \varepsilon^{\beta} \cdot U_i$ and consequently

$$U_i \subseteq \varepsilon^{\beta} \cdot U_i \subseteq \varepsilon^{2\beta} \cdot U_i \subseteq \cdots \subseteq U_i.$$

   Thus, $U_i = \varepsilon^{\beta} \cdot U_i$ and so $T_i = \varepsilon^{\beta} \cdot T_i$, contradiction.

   (b) Let $\partial(U_i \cap \varepsilon^{\beta} \cdot U_i) = \varepsilon^{\beta} \cdot T_i \subset \varepsilon^{\beta} \cdot U_i$. Then $\varepsilon^{\beta} \cdot U_i \subseteq U_i$ and consequently $\varepsilon^{\beta} \cdot U_i = U_i$. Thus $\varepsilon^{\beta} \cdot T_i = T_i$ gives the contradiction.

   (c) Let $\partial(U_i \cap \varepsilon^{\beta} \cdot U_i) = T_i \cup \varepsilon^{\beta} \cdot T_i$. Then $\varepsilon^{\beta} \cdot T_i \subseteq U_i$ and so $\varepsilon^{\beta} \cdot U_i$ is contained in U_i or its complement $(\varepsilon^{\beta} \cdot U_i)^c$ is contained in U_i. Thus either $U_i \cup \varepsilon^{\beta} \cdot U_i = U_i$ or $U_i \cup \varepsilon^{\beta} \cdot U_i \supseteq (\varepsilon^{\beta} \cdot U_i)^c \cup \varepsilon^{\beta} \cdot U_i = V_2(\mathbb{R}^3)$. The later is not possible since C is disjoint from both U_i and $\varepsilon^{\beta} \cdot U_i$. Therefore $\varepsilon^{\beta} \cdot U_i \subseteq U_i$ and consequently $\varepsilon^{\beta} \cdot U_i = U_i$ and $\varepsilon^{\beta} \cdot T_i = T_i$, contradiction. □

Since the cycles S_i, $i \in [m]$ are pairwise disjoint (Fig. 2), the discs Ω_i and Ω_j are either disjoint or one is contained in the other. Consider the discs Ω_i that are not contained in any other disc Ω_j. They are the maximal elements among the Ω_i with the respect to inclusion. For simpler writing we assume that these disks are Ω_i with $i \in [r]$ where, of course, $1 \leq r \leq m$. Consequently, the related U_i, $i \in [r]$, are also maximal between U_i with the respect to inclusion. Let us denote the $\mathbb{Z}_k$ orbit of U_i by $\mathcal{O}(U_i) := U_i \cup (\varepsilon \cdot U_i) \cup \cdots \cup (\varepsilon^{k-1} \cdot U_i)$.

Lemma 2.4 *For distinct $i, j \in [r]$, the orbits $\mathcal{O}(U_i)$ and $\mathcal{O}(U_j)$ are either disjoint or one is contained in the other.*

Proof Let $\mathcal{O}(U_i) \cap \mathcal{O}(U_j) \neq \emptyset$. Then there are α and β, $\alpha \leq \beta$, such that $\varepsilon^\alpha \cdot U_{k(i)} \cap \varepsilon^\beta \cdot U_{k(j)} \neq \emptyset$. Without losing the generality we can assume that $\alpha = 0$. There are two separate cases:

(1) Let $\beta = 0$. Then

$$\begin{aligned} U_i \cap U_j \neq \emptyset \quad &\Longrightarrow \quad \Omega_i \cap \Omega_j \neq \emptyset \quad \Longrightarrow \quad \Omega_i \subset \Omega_j \text{ or } \Omega_j \subset \Omega_i \\ &\Longrightarrow \quad U_i \subset U_j \text{ or } U_j \subset U_i \\ &\Longrightarrow \quad \mathcal{O}(U_i) \subset \mathcal{O}(U_j) \text{ or } \mathcal{O}(U_j) \subset \mathcal{O}(U_i). \end{aligned}$$

(2) Let $\beta \neq 0$. Since $T_i \cap \varepsilon^\beta \cdot T_{k(j)} = \emptyset$ we see that the complement $V_2(\mathbb{R}^3)\backslash(T_i \cup \varepsilon^\beta \cdot T_j)$ has three components. One of them is $U_i \cap \varepsilon^\beta \cdot U_j$ with boundary either T_j or $\varepsilon^\beta \cdot T_j$ or $T_i \cup \varepsilon^\beta \cdot T_j$. We discuss all three possibilities:
   (a) Let $\partial(U_i \cap \varepsilon^\beta \cdot U_j) = T_i \subset U_i$. Then $U_i \subseteq \varepsilon^\beta \cdot U_j$ and consequently the orbit $\mathcal{O}(U_i)$ is contained in the orbit $\mathcal{O}(U_j)$.
   (b) Let $\partial(U_i \cap \varepsilon^\beta \cdot U_j) = \varepsilon^\beta \cdot T_j \subset \varepsilon^\beta \cdot U_{k(j)}$. Then $\varepsilon^\beta \cdot U_j \subseteq U_i$ and consequently the orbit $\mathcal{O}(U_j)$ is contained in the orbit $\mathcal{O}(U_i)$.
   (c) Let $\partial(U_j \cap \varepsilon^\beta \cdot U_j) = T_i \cup \varepsilon^\beta \cdot T_j$. Consequently $T_i \subset \varepsilon^\beta \cdot U_j$ and $\varepsilon^\beta \cdot T_j \subset U_i$. Therefore $\varepsilon^\beta \cdot U_j \subseteq U_i$ or $(\varepsilon^\beta \cdot U_j)^c \subseteq U_i$. Since $(\varepsilon^\beta \cdot U_j)^c \subseteq U_i$ implies that $U_i \cup \varepsilon^\beta \cdot U_j = V_2(\mathbb{R}^3)$, and this is not possible, we conclude that $\varepsilon^\beta \cdot U_j \subseteq U_i$ and consequently the orbit $\mathcal{O}(U_j)$ is contained in the orbit $\mathcal{O}(U_i)$. □

Consider the following subset of the family of all equipartitioning k-fans on the sphere S^2:

$$V^{\text{conv}} = V_2(\mathbb{R}^3)\backslash\left(\bigcup_{i\in[r]} \bigcup_{\alpha\in\{0,\dots,k-1\}} \varepsilon^\alpha \cdot U_i\right).$$

The previous results imply that

(1) $\varepsilon^\alpha \cdot U_i$, for all $i \in [r]$ and $\alpha \in \{0, \dots, k-1\}$, are pairwise disjoint closed solid tori,
(2) every $(x; \ell_1, \dots, \ell_k) = (x, y) \in V^{\text{conv}}$ is a convex k-fan,
(3) $C \subset V^{\text{conv}}$ and V^{conv} are $\mathbb{Z}_k$-invariant subspaces of $V_2(\mathbb{R}^3)$.

Therefore the set V^{conv} will be called the *convex part* of $V_2(\mathbb{R}^3)$. Notice that, as Fig. 2 indicates, there might be some convex k-fans that are not contained in the convex part V^{conv}.

3 Test Maps

In this section we describe four similar test map schemes associated with the parts of Theorem 1.2. Let $\mathbb{Z}_k = \langle\varepsilon\rangle$ denotes the usual cyclic group of order k.

Configuration Spaces Consider as the configuration spaces the spaces of equipartitioning convex 4- and 5-fans described in the previous section. Let us denote these spaces by $V_4^{\text{conv}} = B_4 \backslash A_4$ and $V_5^{\text{conv}} = B_5 \backslash A_5$, where $B_4 = B_5 = V_2(\mathbb{R}^3)$ and

$$A_4 = \Bigg(\bigcup_{i \in [r_4]} \bigcup_{\alpha \in \{0,1,2,3\}} \varepsilon^\alpha \cdot U_{k(i)} \Bigg) \quad \text{and} \quad A_5 = \Bigg(\bigcup_{i \in [r_5]} \bigcup_{\alpha \in \{0,1,2,3,4\}} \varepsilon^\alpha \cdot U_{k(i)} \Bigg).$$

Notice that both spaces A_4 and A_5 are homotopy equivalent to disjoint unions of 1-dimensional spheres.

Some Real $\mathbb{Z}_4$ and $\mathbb{Z}_5$-Representations Let $\mathbb{R}^4$ be a real $\mathbb{Z}_4$ -representation equipped with the following $\mathbb{Z}_4$-action $\varepsilon \cdot (x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, x_1)$. The subspaces

$$\begin{aligned} W_4 &= \big\{ (x_1, x_2, x_3, x_4) \mid x_1 + x_2 + x_3 + x_4 = 0 \big\} \subset \mathbb{R}^4, \\ U &= \operatorname{span}\big\{ (1, -1, 1, -1) \big\} = \big\{ (x_1, x_2, x_3, x_4) \in W_4 \mid x_1 = x_3, x_2 = x_4 \big\} \subset W_4, \\ V &= \operatorname{span}\big\{ (1, 0, -1, 0), (0, 1, 0, -1) \big\} \\ &= \big\{ (x_1, x_2, x_3, x_4) \in W_4 \mid x_1 - x_2 + x_3 - x_4 = 0 \big\} \subset W_4. \end{aligned}$$

are $\mathbb{Z}_4$-invariant subspace or real $\mathbb{Z}_4$-representations. It is not hard to prove that there is an isomorphism of real $\mathbb{Z}_4$-representations $W_4 \cong_{\mathbb{R}} U \oplus V$.

Similarly, consider $\mathbb{R}^5$ as a real $\mathbb{Z}_5$ -representation via the action $\varepsilon \cdot (x_1, x_2, x_3, x_4, x_5) = (x_2, x_3, x_4, x_5, x_1)$. The subspace

$$W_5 = \big\{ (x_1, x_2, x_3, x_4, x_5) \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0 \big\} \subset \mathbb{R}^5$$

is $\mathbb{Z}_5$-invariant and therefore a real subrepresentation.

Test Space and Test Map for 4-Fans Let f and $g : V_4^{\text{conv}} \to \mathbb{R}$ be continuous function on the sectors of the convex 4-fan. Consider two test maps $\tau_1 : V_4^{\text{conv}} \to W_4$ and $\tau_2 : V_4^{\text{conv}} \to W_4 \oplus W_4$ given by

$$\tau_1(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \big(f(\sigma_1) - \Delta_f, f(\sigma_2) - \Delta_f, f(\sigma_3) - \Delta_f, f(\sigma_4) - \Delta_f \big)$$

where $\Delta_f = f(\sigma_1) + f(\sigma_2) + f(\sigma_3) + f(\sigma_4)$ and

$$\begin{aligned} \tau_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \big(& f(\sigma_1) - \Delta_f, f(\sigma_2) - \Delta_f, f(\sigma_3) - \Delta_f, f(\sigma_4) - \Delta_f, \\ & g(\sigma_1) - \Delta_g, g(\sigma_2) - \Delta_g, g(\sigma_3) - \Delta_g, g(\sigma_4) - \Delta_g \big) \end{aligned}$$

where, similarly, $\Delta_g = g(\sigma_1) + g(\sigma_2) + g(\sigma_3) + g(\sigma_4)$. Having in mind that for g one can take for example function f^2.

There are two test spaces of interest

$$\begin{aligned} T_1 &= \big\{ (x_1, x_2, x_3, x_4) \in W_4 \mid x_1 = x_3, x_2 = x_4 \big\} = U, \\ T_2 &= \big\{ (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in W_4 \oplus W_4 \mid x_1 - x_2 + x_3 - x_4 \\ &\quad = y_1 - y_2 + y_3 - y_4 = 0 \big\} \cong V \oplus V. \end{aligned} \tag{2}$$

Proposition 3.1

(1) *If there is no* $\mathbb{Z}_4$*-equivariant map* $V_4^{\mathrm{conv}} \to W_4 \backslash T_1$ *and* $V_4^{\mathrm{conv}} \to (W_4 \oplus W_4)\backslash T_2$, *then parts* 1 *and* 2 *of Theorem* 1.2 *hold.*
(2) *If there is no* $\mathbb{Z}_4$*-equivariant map* $V_4^{\mathrm{conv}} \to S(V)$ *and* $V_4^{\mathrm{conv}} \to S(U \oplus U)$, *then parts* 1 *and* 2 *of Theorem* 1.2 *hold.*

Proof

(1) If there is no $\mathbb{Z}_4$-equivariant map $V_4^{\mathrm{conv}} \to W_4\backslash T_1$, then there exists a convex 4-fan with sectors $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ such that $\tau_1(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \cap T_1 \neq \emptyset$ and consequently

$$f(\sigma_1) = f(\sigma_3) \quad \text{and} \quad f(\sigma_2) = f(\sigma_4).$$

If there is no $\mathbb{Z}_4$-equivariant map $V_4^{\mathrm{conv}} \to (W_4 \oplus W_4)\backslash T_2$, then there is a convex 4-fan with sectors $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ such that $\tau_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \cap T_1 \neq \emptyset$. Taking for $g = f^2$ we get

$$f(\sigma_1) + f(\sigma_3) = f(\sigma_2) + f(\sigma_4) \quad \text{and} \quad f(\sigma_1)^2 + f^2(\sigma_3) = f^2(\sigma_2) + f^2(\sigma_4).$$

This implies that either $f(\sigma_1) = f(\sigma_2)$, $f(\sigma_3) = f(\sigma_4)$ or $f(\sigma_1) = f(\sigma_4)$, $f(\sigma_2) = f(\sigma_3)$.
(2) The existence of $\mathbb{Z}_4$-homotopies,

$$\begin{aligned} W_4\backslash T_1 &= W_4\backslash U \simeq U^{\perp}\backslash\big\{(0,0,0,0)\big\} = V\backslash\big\{(0,0,0,0)\big\} \simeq S(V), \\ (W_4 \oplus W_4)\backslash T_2 &= W_4\backslash(V \oplus V) \simeq (V \oplus V)^{\perp}\backslash\big\{(0,0,0,0) \oplus (0,0,0,0)\big\} \\ &= (U \oplus U)\backslash\big\{(0,0,0,0) \oplus (0,0,0,0)\big\} \simeq S(U \oplus U), \end{aligned}$$

and (1) imply the claim (2). □

Test Space and Test Map for 5*-Fans* Let $h : V_5^{\mathrm{conv}} \to \mathbb{R}$ be a continuous function on the sectors of 5-fans. Consider the test map $\tau_3 : V_5^{\mathrm{conv}} \to W_5$ given by

$$\begin{aligned} &\tau_3(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \\ &\quad = \big(f(\sigma_1) - \Delta_f, f(\sigma_2) - \Delta_f, f(\sigma_3) - \Delta_f, f(\sigma_4) - \Delta_f, f(\sigma_5) - \Delta_f\big), \end{aligned}$$

where $\Delta_f = f(\sigma_1) + f(\sigma_2) + f(\sigma_3) + f(\sigma_4) + f(\sigma_5)$. Here $W_5 = \{(x_1, \dots, x_5) | x_1 + \cdots + x_5 = 0\} \subseteq \mathbb{R}^5$.

There are two test spaces T_3 and T_4 we are interested in. They are unions of the minimal $\mathbb{Z}_5$-invariant arrangements $\mathcal{A}_3$ and $\mathcal{A}_4$ containing the linear subspace $L_3 \subset W_5$ and $L_4 \subset W_5$, respectively, given by

$$L_3 = \big\{(x_1, x_2, x_3, x_4, x_5) \in W_5 \mid x_1 = x_2 = x_3\big\}, \tag{3}$$

$$L_4 = \big\{(x_1, x_2, x_3, x_4, x_5) \in W_5 \mid x_1 = x_2 = x_4\big\}. \tag{4}$$

The intersection posets of the arrangements $\mathcal{A}_3$ and $\mathcal{A}_4$ are isomorphic, Fig. 3.

The basic property of the test map scheme follows directly.

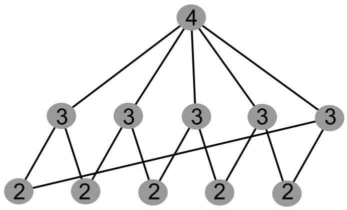


Fig. 3 Hasse diagram of the intersection posets of the arrangement $\mathcal{A}_3$ and $\mathcal{A}_4$ with codimensions in W_5

Proposition 3.2

(1) *If there is no $\mathbb{Z}_5$-equivariant map $V_5^{\text{conv}} \to W_5 \backslash T_3$, then part* 3 *of Theorem* 1.2 *holds.*
(2) *If there is no $\mathbb{Z}_5$-equivariant map $V_5^{\text{conv}} \to W_5 \backslash T_4$, then part* 4 *of Theorem* 1.2 *holds.*

4 Cohomology of the Configuration Spaces as an $R[\mathbb{Z}_n]$-Module

In this section we study the cohomology of the configuration spaces V_4^{conv} and V_5^{conv} as $\mathbb{Z}[\mathbb{Z}_4]$ and $\mathbb{F}_5[\mathbb{Z}_5]$-module, respectively. This will turn out to be an important step in the proof of the non-existence of the appropriate $\mathbb{Z}_4$ and $\mathbb{Z}_5$-equivariant maps, Sects. 5 and 6.

4.1 Cohomology of V_4^{conv}

We establish the following isomorphisms of $\mathbb{Z}[\mathbb{Z}_4]$-modules:

$$H^0\big(V_4^{\text{conv}};\mathbb{Z}\big)=\mathbb{Z} \quad \text{and} \quad H^1\big(V_4^{\text{conv}};\mathbb{Z}\big)\cong\big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4}. \tag{5}$$

Proposition 4.1 *The cohomology with the $\mathbb{Z}$ coefficients of the pair (B_4, A_4) is given by*

$$H^i(B_4, A_4; \mathbb{Z}) \cong \begin{cases} (\mathbb{Z}[\mathbb{Z}_4])^{\oplus k}/_{(1+\varepsilon+\varepsilon^2+\varepsilon^3)^{\oplus k}\mathbb{Z}} & \\ \quad\cong (\mathbb{Z}[\mathbb{Z}_4])^{\oplus(k-1)} \oplus (\mathbb{Z}[\mathbb{Z}_4]/_{(1+\varepsilon+\varepsilon^2+\varepsilon^3)\mathbb{Z}}), & i=1,\\ M, & i=2,\\ \mathbb{Z}, & i=3,\\ 0, & \textit{otherwise}. \end{cases}$$

where $\mathbb{Z}[\mathbb{Z}_4]$-module M is a part of the following exact sequence of $\mathbb{Z}[\mathbb{Z}_4]$-modules:

$$0 \longrightarrow \big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus k} \longrightarrow M \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{6}$$

Proof The pare (B_4, A_4) generates the following long exact sequence in cohomology with $\mathbb{Z}$ coefficients:

$$\begin{aligned} 0 \longrightarrow H^0(B_4, A_4) &\longrightarrow H^0(B_4) \xrightarrow{\Phi_0} H^0(A_4)\\ \longrightarrow H^1(B_4, A_4) &\longrightarrow H^1(B_4) \xrightarrow{\Phi_1} H^1(A_4) \end{aligned}$$

$$\longrightarrow H^2(B_4, A_4) \longrightarrow H^2(B_4) \xrightarrow{\Phi_2} H^2(A_4)$$
$$\longrightarrow H^3(B_4, A_4) \longrightarrow H^3(B_4) \xrightarrow{\Phi_3} 0.$$

We know that $H^0(B_4) = \mathbb{Z}$, $H^0(A_4) = (\mathbb{Z}[\mathbb{Z}_4])^{\oplus r_4}$ and

$$\Phi_0(a) = \big(a + \varepsilon \cdot a + \varepsilon^2 \cdot a + \varepsilon^3 \cdot a\big) \oplus \cdots \oplus \big(a + \varepsilon \cdot a + \varepsilon^2 \cdot a + \varepsilon^3 \cdot a\big).$$

Thus Φ_0 is an injection. Since $H^1(B_4) = 0$, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Phi_0} \big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4} \longrightarrow H^1(B_4, A_4) \longrightarrow 0$$

and therefore

$$\begin{aligned} H^1(B_4, A_4) &\cong \big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4} / _{(1+\varepsilon+\varepsilon^2+\varepsilon^3)^{\oplus r_4}\mathbb{Z}} \\ &\cong \big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus (r_4-1)} \oplus \big(\mathbb{Z}[\mathbb{Z}_4] / _{(1+\varepsilon+\varepsilon^2+\varepsilon^3)\mathbb{Z}}\big). \end{aligned}$$

From the fact that $H^1(A_4) = (\mathbb{Z}[\mathbb{Z}_4])^{\oplus r_4}$, $H^2(A_4) = 0$ and $H^2(B_4) = \mathbb{Z}_2$ we obtain an exact sequence

$$0 \longrightarrow \big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4} \longrightarrow H^2(B_4, A_4) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Finally, the fact that $H^3(B_4) = \mathbb{Z}$ gives the exact sequence

$$0 \longrightarrow H^3(B_4, A_4) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and the isomorphism $H^3(B_4, A_4) \cong \mathbb{Z}$. □

Corollary 4.2

$$H_i(B_4 \backslash A_4; \mathbb{Z}) \cong \begin{cases} (\mathbb{Z}[\mathbb{Z}_4])^{\oplus r_4} / _{(1+\varepsilon+\varepsilon^2+\varepsilon^3)^{\oplus r_4}\mathbb{Z}} & \\ \quad \cong (\mathbb{Z}[\mathbb{Z}_4])^{\oplus (r_4-1)} \oplus (\mathbb{Z}[\mathbb{Z}_4] / _{(1+\varepsilon+\varepsilon^2+\varepsilon^3)\mathbb{Z}}), & i = 2, \\ M, & i = 1, \\ \mathbb{Z}, & i = 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof The Poincaré–Lefschetz duality [8, Theorem 70.2, p. 415] applied on the compact manifold B_4 relates the homology of the difference $B_4 \backslash A_4$ with the cohomology of the pair (B_4, A_4), i.e.,

$$H_i(B_4 \backslash A_4; \mathbb{Z}) \cong H^{3-i}(B_4, A_4; \mathbb{Z}).$$

Now the claim follows directly from the previous proposition. □

Proposition 4.3 $\mathrm{Hom}(M, \mathbb{Z}) \cong (\mathbb{Z}[\mathbb{Z}_4])^{\oplus r_4}$.

Proof The $\mathbb{Z}[\mathbb{Z}_4]$-module M seen as an abelian group can be decomposed into the direct sum of the free and the torsion part, $M = \mathrm{Free}(M) \oplus \mathrm{Torsion}(M)$. This is decomposition is a $\mathbb{Z}_4$-invariant. Then $\mathrm{Hom}(M,\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{Free}(M),\mathbb{Z}) \cong \mathrm{Free}(M)$ and therefore $\mathrm{Hom}(M,\mathbb{Z})$ is a free abelian group. The exact sequence (6) implies that $\mathrm{rank}(\mathrm{Hom}(M,\mathbb{Z})) \geq 4r_4$. Application of the Hom functor on the same exact sequence (6) yields the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}) &\longrightarrow \mathrm{Hom}(M,\mathbb{Z}) \longrightarrow \mathrm{Hom}\big(\big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4},\mathbb{Z}\big) \\ \longrightarrow \mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}) &\longrightarrow \mathrm{Ext}(M,\mathbb{Z}) \longrightarrow \mathrm{Ext}\big(\big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4},\mathbb{Z}\big). \end{aligned}$$

Since, as $\mathbb{Z}[\mathbb{Z}_4]$-modules,

$$\begin{aligned} &\mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}) = 0, \qquad &&\mathrm{Hom}\big(\big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4},\mathbb{Z}\big) \cong \big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4}, \\ &\mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}) \cong \mathbb{Z}_2, \qquad &&\mathrm{Ext}\big(\big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4},\mathbb{Z}\big) = 0, \end{aligned}$$

the exact sequence transforms into

$$0 \longrightarrow \mathrm{Hom}(M,\mathbb{Z}) \longrightarrow \big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4} \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Ext}(M,\mathbb{Z}) \longrightarrow 0.$$

First notice that $\mathrm{rank}(\mathrm{Hom}(M,\mathbb{Z})) \leq 4r_4$ and therefore

$$\mathrm{rank}\big(\mathrm{Hom}(M,\mathbb{Z})\big) = \mathrm{rank}\big(\mathrm{Hom}\big(\mathrm{Free}(M),\mathbb{Z}\big)\big) = \mathrm{rank}\big(\mathrm{Free}(M)\big) = 4r_4.$$

Since the exact sequence (6) gives an inclusion of $\mathbb{Z}[\mathbb{Z}_4]$-modules $(\mathbb{Z}[\mathbb{Z}_4])^{\oplus r_4} \longrightarrow \mathrm{Free}(M)$, and $(\mathbb{Z}[\mathbb{Z}_4])^{\oplus r_4}$ is the direct sum of the free $\mathbb{Z}[\mathbb{Z}_4]$-modules we can conclude that $\mathrm{Free}(M) \cong (\mathbb{Z}[\mathbb{Z}_4])^{\oplus r_4}$. Thus we have an isomorphism of $\mathbb{Z}[\mathbb{Z}_4]$-modules

$$\mathrm{Hom}(M,\mathbb{Z}) \cong \mathrm{Hom}\big(\mathrm{Free}(M),\mathbb{Z}\big) \cong \mathrm{Hom}\big(\big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4},\mathbb{Z}\big) \cong \big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4}. \qquad \square$$

Finally, we have to verify the isomorphisms (5) of $\mathbb{Z}[\mathbb{Z}_4]$-modules.

Corollary 4.4 *$H^0(B_4\backslash A_4;\mathbb{Z}) = \mathbb{Z}$ and $H^1(B_4\backslash A_4;\mathbb{Z}) \cong \mathrm{Hom}(M,\mathbb{Z}) \cong (\mathbb{Z}[\mathbb{Z}_4])^{\oplus r_4}$.*

Proof The complement $B_4\backslash A_4$ is connected. Therefore the cohomology in dimension zero is $\mathbb{Z}$. The Universal coefficient theorem applied for the first cohomology gives the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}\big(H_0(B_4\backslash A_4;\mathbb{Z}),\mathbb{Z}\big) &\longrightarrow H^1(B_4\backslash A_4;\mathbb{Z}) \\ &\longrightarrow \mathrm{Hom}\big(H_1(B_4\backslash A_4;\mathbb{Z}),\mathbb{Z}\big) \longrightarrow 0. \end{aligned}$$

Since $\mathrm{Ext}(H_0(B_4\backslash A_4;\mathbb{Z}),\mathbb{Z}) = \mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = 0$, the exact sequence gives the isomorphism

$$H^1(B_4\backslash A_4;\mathbb{Z}) \cong \mathrm{Hom}\big(H_1(B_4\backslash A_4;\mathbb{Z}),\mathbb{Z}\big) = \mathrm{Hom}(M,\mathbb{Z}) \cong \big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4}. \qquad \square$$

4.2 Cohomology of V_5^{conv}

Like in [3, Sect. 6], we establish the following isomorphisms of $\mathbb{F}_5[\mathbb{Z}_5]$-modules:

$$H^0\big(V_5^{\text{conv}};\mathbb{F}_5\big)=\mathbb{F}_5 \quad \text{and} \quad H^1\big(V_5^{\text{conv}};\mathbb{F}_5\big)\cong\big(\mathbb{F}_5[\mathbb{Z}_5]\big)^{\oplus r_5}.$$

Proposition 4.5 $H^0(V_5^{\text{conv}};\mathbb{F}_5)=\mathbb{F}_5$ *and* $H^1(V_5^{\text{conv}};\mathbb{F}_5)=\bigoplus_{i=1}^{r_5}\mathbb{F}_5[\mathbb{Z}_5]$.

Proof Since the complement $V_5^{\text{conv}}=B_5\backslash A_5$ is connected, the first claim easily follows. The second claim follows from Poincaré–Lefschetz duality [8, Theorem 70.2, p. 415] and the homology exact sequence of the pair (B_5,A_5) since $H_1(B_5;\mathbb{F}_5)=H_2(B_5;\mathbb{F}_5)=0$. Indeed,

$$H^1\big(V_5^{\text{conv}};\mathbb{F}_5\big)\cong H_2(B_5,A_5;\mathbb{F}_5)\cong H_1(A_5;\mathbb{F}_5)\cong\big(\mathbb{F}_5[\mathbb{Z}_5]\big)^{\oplus r_5}.$$ □

5 Non-existence of the Test Map, Proof of Theorem 1.2(1)–(2)

The first two parts of Theorem 1.2, via Proposition 3.1, are direct consequences of the following theorem.

Theorem 5.1 *There is no* $\mathbb{Z}_4$*-equivariant map*

(i) $V_4^{\text{conv}}\to S(V)$,
(ii) $V_4^{\text{conv}}\to S(U\oplus U)$.

Proof The proof is obtained by studying the morphism of Serre spectral sequences associated with the Borel constructions of $B_4\backslash A_4$, $S(V)$ and $S(U\oplus U)$. We denote the cohomology of the group $\mathbb{Z}_4$ with $\mathbb{Z}$ coefficients by $H^*(\mathbb{Z}_4;\mathbb{Z})$. It is well known that

$$H^*(\mathbb{Z}_4;\mathbb{Z})=\mathbb{Z}[T]/_{\langle 4T\rangle},$$

where $\deg T=2$.

The Serre Spectral Sequence of $V_4^{\text{conv}}\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4$ The E_2-term of the sequence is given by $E_2^{p,q}=H^p(\mathbb{Z}_4,H^q(V_4^{\text{conv}},\mathbb{Z}))$. For $q=1$, from Corollary 4.4 and [6, Example 2, p. 58] we find that the first row is

$$E_2^{p,1}=H^p\big(\mathbb{Z}_4;\big(\mathbb{Z}[\mathbb{Z}_4]\big)^{\oplus r_4}\big)=\begin{cases}\mathbb{Z}^{\oplus r_4}, & p=0,\\ 0, & p\neq 0.\end{cases}$$

Since the differentials in the spectral sequence are $H^*(\mathbb{Z}_4;\mathbb{Z})$-module maps, we have $d_2^{0,1}=0$. This means, in particular, that $T,2T\in H^2(\mathbb{Z}_4;\mathbb{Z})=E_2^{2,0}$ survive to the E_∞-term.

The Serre Spectral Sequence of $S(V)\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4$ The E_2-term of the sequence is given by

$$E_2^{p,q} = H^p\big(\mathbb{Z}_4; H^q\big(S(V);\mathbb{Z}\big)\big) = H^p(\mathbb{Z}_4;\mathbb{Z})\otimes H^q\big(S(V);\mathbb{Z}\big)$$
$$= \begin{cases} H^p(\mathbb{Z}_4;\mathbb{Z}), & q=0,1,\\ 0, & \text{otherwise.}\end{cases}$$

In general, the coefficients should be twisted, but the $\mathbb{Z}_4$ action on $S(V)$ is orientation preserving, hence the coefficients are untwisted. The action of $\mathbb{Z}_4$ on $S(V)\approx S^1$ is free and therefore

$$S(V)\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4 \simeq S^1/\mathbb{Z}_4 \quad\Rightarrow\quad H^i\big(S(V)\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4;\mathbb{Z}\big)=0 \quad \text{for } i>1.$$

The spectral sequence converges to $H^*(S(V)\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4;\mathbb{Z})$ and therefore in the E_∞-term everything in positions $p+q>1$ must vanish. Since our spectral sequence has only two non-zero rows and the only possibly non-zero differential is d_2 it follows that $d_2(1\otimes L)=T\in H^2(\mathbb{Z}_4;\mathbb{Z})=E_2^{2,0}$. Here $L\in H^1(S(V);\mathbb{Z})$ denotes a generator. Therefore, the element $T\in H^2(\mathbb{Z}_4;\mathbb{Z})=E_2^{2,0}$ vanishes in the E_3-term.

The Serre Spectral Sequence of $S(U\oplus U)\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4$ The representation V is the 1-dimensional complex representation of $\mathbb{Z}_4$ induced by $1\mapsto e^{i\pi/2}$. Then $U\oplus U\cong V\otimes_{\mathbb{C}} V$. Following [1, Sect. 8, p. 271 and Appendix, p. 285] we deduce the first Chern class of the $\mathbb{Z}_4$-representation $U\oplus U$

$$c_1(U\oplus U)=c_1(V\otimes_{\mathbb{C}} V)=c_1(V)+c_1(V)=T+T=2T\in H^2(\mathbb{Z}_4;\mathbb{Z}).$$

There by [4, Proposition 3.11] we know that in the E_2-term of the Serre spectral sequence associated to $S(U\oplus U)\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4$ the second $(0,1)$-differential is given by $d_2(1\otimes L)=2T\in H^2(\mathbb{Z}_4;\mathbb{Z})=E_2^{2,0}$. Here again $L\in H^1(S(U\oplus U);\mathbb{Z})$ denotes the generator. Thus the element $2T\in H^2(\mathbb{Z}_4;\mathbb{Z})=E_2^{2,0}$ vanishes in the E_3-term.

The Non-existence of Both $\mathbb{Z}_4$*-Equivariant Maps* Assume that in both cases there exists a $\mathbb{Z}_4$-equivariant,

(i) $f: V_4^{\mathrm{conv}}\to S(V)$,
(ii) $g: V_4^{\mathrm{conv}}\to S(U\oplus U)$.

Then f and g induce maps between

- Borel constructions, $V_4^{\mathrm{conv}}\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4\to S(V)\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4$ and $V_4^{\mathrm{conv}}\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4\to S(U\oplus U)\times_{\mathbb{Z}_4}\mathrm{E}\mathbb{Z}_4$,
- equivariant cohomologies,

$$f^*: H_{\mathbb{Z}_4}\big(S(V);\mathbb{Z}\big)\to H_{\mathbb{Z}_4}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big)$$
$$\text{and}\quad g^*: H_{\mathbb{Z}_4}\big(S(U\oplus U);\mathbb{Z}\big)\to H_{\mathbb{Z}_4}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big),$$

- associated Serre spectral sequences,

$$E_r^{p,q}(f): E_r^{p,q}\big(S(V);\mathbb{Z}\big) \to E_r^{p,q}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big)$$

  and

$$E_r^{p,q}(g): E_r^{p,q}\big(S(U\oplus U);\mathbb{Z}\big) \to E_r^{p,q}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big)$$

  such that in the 0-row of the E_2-term

$$E_2^{p,0}(f): \big(E_2^{p,0}\big(S(V);\mathbb{Z}\big) = H^p(\mathbb{Z}_4;\mathbb{Z})\big) \to \big(E_2^{p,0}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big) = H^p(\mathbb{Z}_4;\mathbb{Z})\big)$$

  and

$$E_2^{p,0}(g): \big(E_2^{p,0}\big(S(U\oplus U);\mathbb{Z}\big) = H^p(\mathbb{Z}_4;\mathbb{Z})\big) \to \big(E_2^{p,0}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big) = H^p(\mathbb{Z}_4;\mathbb{Z})\big)$$

  are identity maps.

The contradiction is obtained by tracking the behavior of the $E_r^{2,0}(f)$ and $E_r^{2,0}(g)$ images of $T \in H^2(\mathbb{Z}_4;\mathbb{Z})$ and $2T \in H^2(\mathbb{Z}_4;\mathbb{Z})$ as r grows from 2 to 3. Explicitly,

$$E_2^{2,0}\big(S(V);\mathbb{Z}\big) \ni T \overset{E_2^{2,0}(f)}{\longmapsto} T \in E_2^{2,0}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big),$$
$$E_2^{2,0}\big(S(U\oplus U);\mathbb{Z}\big) \ni 2T \overset{E_2^{2,0}(g)}{\longmapsto} 2T \in E_2^{2,0}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big)$$

and

$$E_3^{2,0}\big(S(V);\mathbb{Z}\big) \ni 0 \overset{E_3^{2,0}(f)}{\longmapsto} T \in E_3^{3,0}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big),$$
$$E_3^{2,0}\big(S(U\oplus U);\mathbb{Z}\big) \ni 0 \overset{E_3^{2,0}(f)}{\longmapsto} 2T \in E_3^{3,0}\big(V_4^{\mathrm{conv}};\mathbb{Z}\big).$$

Since the image of zero cannot be different from zero we have reached a contradiction. Thus, there are no $\mathbb{Z}_4$-equivariant maps in both cases:

$$V_4^{\mathrm{conv}} \to S(V), \qquad V_4^{\mathrm{conv}} \to S(U\oplus U).$$

The theorem is proved. □

6 Non-existence of the Test Map, Proof of Theorem 1.2(3)–(4)

We conclude the proof of Theorem 1.2, using Proposition 3.2, by showing the following non-existence theorem.

Theorem 6.1 *There is no $\mathbb{Z}_5$-equivariant map $V_5^{\mathrm{conv}} \to W_5\backslash T_j$, where $j \in \{3,4\}$.*

Proof Again we study the morphism of Serre spectral sequences associated with the Borel constructions of V_5^{conv} and $W_5\backslash T_3$. The cohomology ring of the group $\mathbb{Z}_5$ with $\mathbb{F}_5$ coefficients will be denoted by $H^*(\mathbb{Z}_5;\mathbb{F}_5)$. It is known that

$$H^*(\mathbb{Z}_5;\mathbb{F}_5)=\mathbb{F}_5[t]\otimes\big(\mathbb{F}_5[e]/e^2\big),$$

where $\deg t=2$, $\deg e=1$.

The Serre Spectral Sequence of $V_5^{\text{conv}}\times_{\mathbb{Z}_5}\mathrm{E}\mathbb{Z}_5$ The E_2-term of the sequence is given by $E_2^{p,q}=H^p(\mathbb{Z}_5,H^q(V_5^{\text{conv}},\mathbb{F}_5))$. For $q=1$, from the Proposition 4.5 and [6, Example 2, p. 58] we have

$$E_2^{p,1}=H^p\big(\mathbb{Z}_5;\big(\mathbb{F}_5[\mathbb{Z}_5]\big)^{\oplus r_5}\big)=\begin{cases}\mathbb{F}_5^{\oplus r_5}, & p=0,\\ 0, & p\neq 0.\end{cases}$$

The differentials in the spectral sequence are $H^*(\mathbb{Z}_5;\mathbb{F}_5)$-module maps. Therefore $d_2^{0,1}=0$. In particular, $\alpha t\in H^2(\mathbb{Z}_5;\mathbb{F}_5)=E_2^{2,0}$ survive to the E_∞-term for all $\alpha\in\mathbb{F}_5\backslash\{0\}$.

The Serre Spectral Sequence of $(W_5\backslash T_j)\times_{\mathbb{Z}_5}\mathrm{E}\mathbb{Z}_5$ First, we need to understand the cohomology of $W_5\backslash T_j$ with $\mathbb{F}_5$ coefficients. According to Goresky–MacPherson formula

$$\tilde{H}^i(W_5\backslash T_j;\mathbb{F}_5)\cong\bigoplus_{p\in P_{\mathcal{A}_j}}\tilde{H}_{2-i-\dim p}\big(\Delta\big((P_{\mathcal{A}_j})_{<p}\big);\mathbb{F}_5\big).$$

Here $P_{\mathcal{A}_j}$ is an intersection poset of the arrangement $\mathcal{A}_j$. The intersection posets $P_{\mathcal{A}_3}$ and $P_{\mathcal{A}_4}$ are isomorphic. Since the cohomology of the arrangement complement is completely determined by the intersection poset, we do not need the distinguish between the test spaces T_3 and T_4.

From Hasse diagram of the poset $P_{\mathcal{A}_j}$, Fig. 3, we have

$$H^i(W_5\backslash T_j;\mathbb{F}_5)\cong\begin{cases}\mathbb{F}_5, & i=0,\\ \mathbb{F}_5[\mathbb{Z}_5]\oplus\mathbb{F}_5[\mathbb{Z}_5]\oplus\mathbb{F}_5, & i=1,\\ 0, & i\neq 0,1.\end{cases}$$

Thus the E_2-term of the Serre spectral sequence of $(W_5\backslash T_j)\times_{\mathbb{Z}_5}\mathrm{E}\mathbb{Z}_5$ is

$$\begin{aligned}E_2^{p,q}&=H^p\big(\mathbb{Z}_5;H^q(W_5\backslash T_j;\mathbb{F}_5)\big)\\&=\begin{cases}H^p(\mathbb{Z}_5;\mathbb{F}_5), & q=0,\\ H^p(\mathbb{Z}_5;\mathbb{F}_5[\mathbb{Z}_5])\oplus H^p(\mathbb{Z}_5;\mathbb{F}_5[\mathbb{Z}_5])\oplus H^p(\mathbb{Z}_5;\mathbb{F}_5), & q=1,\\ 0, & \text{otherwise}.\end{cases}\end{aligned}$$

The action of $\mathbb{Z}_5$ on $W_5\backslash T_j$ is free and therefore

$$(W_5\backslash T_j)\times_{\mathbb{Z}_5}\mathrm{E}\mathbb{Z}_5\simeq(W_5\backslash T_j)/\mathbb{Z}_5\quad\Rightarrow\quad H^i\big((W_5\backslash T_j)\times_{\mathbb{Z}_5}\mathrm{E}\mathbb{Z}_5;\mathbb{F}_5\big)=0\quad\text{for } i>1.$$

The spectral sequence converges to $H^*((W_5\backslash T_j)\times_{\mathbb{Z}_5}\mathrm{E}\mathbb{Z}_5;\mathbb{F}_5)$ and so in the E_∞-term everything for $p+q>1$ must vanish. Since our spectral sequence has only two non-zero rows and the only possibly non-zero differential is d_2 it follows that $d_2(x)=t\in H^2(\mathbb{Z}_5;\mathbb{F}_5)=E_2^{2,0}$. Here

$$x\in H^0(\mathbb{Z}_5;\mathbb{F}_5)\subset H^0\big(\mathbb{Z}_5;\mathbb{F}_5[\mathbb{Z}_5]\big)\oplus H^0\big(\mathbb{Z}_5;\mathbb{F}_5[\mathbb{Z}_5]\big)\oplus H^0(\mathbb{Z}_5;\mathbb{F}_5)=E_2^{0,1}$$

denotes a suitably chosen generator. Thus the element $t\in H^2(\mathbb{Z}_4;\mathbb{Z})=E_2^{2,0}$ vanishes in the E_3-term.

The Non-existence of $\mathbb{Z}_5$-Equivariant Maps Assume that there exists a $\mathbb{Z}_5$-equivariant map $f:B_5\backslash A_5\to W_5\backslash T_j$. Then f induces the maps between

- Borel constructions, $(B_5\backslash A_5)\times_{\mathbb{Z}_5}\mathrm{E}\mathbb{Z}_5\to(W_5\backslash T_j)\times_{\mathbb{Z}_5}\mathrm{E}\mathbb{Z}_5$,
- equivariant cohomologies, $f^*:H_{\mathbb{F}_5}(W_5\backslash T_j;\mathbb{F}_5)\to H_{\mathbb{Z}_5}(B_5\backslash A_5;\mathbb{F}_5)$, and
- associated Serre spectral sequences,

$$E_r^{p,q}(f):E_r^{p,q}(W_5\backslash T_j;\mathbb{F}_5)\to E_r^{p,q}(B_5\backslash A_5;\mathbb{F}_5)$$

such that on the 0-row of the E_2-term

$$E_2^{p,0}(f):\big(E_2^{p,0}(W_5\backslash T_j;\mathbb{F}_5)=H^p(\mathbb{Z}_5;\mathbb{F}_5)\big)\\ \to\big(E_2^{p,0}(B_5\backslash A_5;\mathbb{F}_5)=H^p(\mathbb{Z}_5;\mathbb{F}_5)\big)$$

is the identity map.

The contradiction is obtained by tracking the image of $t\in H^2(\mathbb{Z}_5;\mathbb{F}_5)$ mapped by $E_r^{2,0}(f)$ as r grows from 2 to 3. Explicitly,

$$E_2^{2,0}(W_5\backslash T_3;\mathbb{F}_5)\ni t\ \overset{E_2^{2,0}(f)}{\longmapsto}\ t\in E_2^{2,0}(B_5\backslash A_5;\mathbb{F}_5)\\ E_3^{2,0}(W_5\backslash T_3;\mathbb{F}_5)\ni 0\ \overset{E_3^{2,0}(f)}{\longmapsto}\ t\in E_3^{3,0}(B_5\backslash A_5;\mathbb{F}_5).$$

The image of zero cannot be different from zero, thus we have reached a contradiction. There is no $\mathbb{Z}_5$-equivariant map $V_5^{\mathrm{conv}}\to W_5\backslash T_j$ and the theorem is proved. □

7 Counter Examples, Proof of Theorem 1.3

7.1 Proof of Theorem 1.3(1)

We will prove more, namely, that given $\alpha_i>0$ $(i=1,2,3,4)$ with $\sum_1^4\alpha_i=1$, there are two probability measures μ and ν on $\mathbb{R}^2$ such that no convex 4-fan satisfies the conditions $\mu(\sigma_i)=\nu(\sigma_i)=\alpha_i$ for all $i=1,2,3,4$.

This construction is from [2, Theorem 1.1.(i).(d)]. Let Q resp. T be the segment $[(-2,0),(2,0)]$ and $[(-1,1),(1,1)]$, and let ν be the uniform (probability) measure on Q. Also, let μ be the uniform (probability) measure on T for the time being. It will be modified soon. Assume there is a convex 4-fan α-partitioning both measures. Then three consecutive rays intersect both Q and T and so the center of the 4-fan cannot lie between the lines containing Q and T. It cannot be below the line containing T as otherwise one sector would meet Q in an interval too short to have the prescribed ν measure. The only way to make the 4-fan convex is that there are three downward rays and the fourth ray points upward. The three downward rays split Q, resp. T into four intervals of ν- and μ-measure $\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}$ in this order for some $i = 1, 2, 3, 4$ (the subscripts are meant modulo 4). Thus the lengths of these intervals are $4\alpha_i, 4\alpha_{i+1}, 4\alpha_{i+2}, 4\alpha_{i+3}$ on Q and $2\alpha_i, 2\alpha_{i+1}, 2\alpha_{i+2}, 2\alpha_{i+3}$ on T. So given α, the three downward rays, together with the center, are uniquely determined by the index i specifying that the starting interval is of length $4\alpha_i$ on Q. Let $(z_i, 1)$ be the point where the middle downward ray intersects T. This is four points corresponding to the four possible cases. Now we modify the measure μ a little. We move a small mass of μ from the left of $(z_i, 1)$ to the right, for each $i = 1, 2, 3, 4$. Each moving takes place in a very small neighborhood of $(z_i, 1)$. This changes only the position of the middle downward ray (in the modified measure μ), and the new ray will not pass through the intersection of the other two. We need to check that the four modifications are compatible. This is clearly the case when all the z_i are distinct if the mass that has been moved is close enough to the corresponding $(z_i, 1)$. If two or more z_i coincide, then the modification for one i will do for the others as well.

7.2 Proof of Theorem 1.3(2)

This construction is similar to the previous one. This time μ is the uniform measure on the interval $T = [(-1, 1), (1, 1)]$, but Q, the support of $\nu = \nu_h$, is the whole x axis and the distribution function of ν_h, $F = F_h$, which depends on a parameter $h \in (0, 1)$, is given explicitly as

$$F_h(x) = \begin{cases} he^x & \text{if } x \le 0, \\ 1 - (1-h)e^{-x} & \text{if } x \ge 0. \end{cases}$$

Note that F_h is concave resp. convex on $[0, \infty)$ and $(-\infty, 0]$. The following properties of F_h are easily checked:

(i) no line intersects the graph of F_h in more than three points,
(ii) no line intersects the graph of the convex (concave) part of F_h in more than two points,
(iii) F_h is symmetric, in the sense that $F_{1-h}(-x) = 1 - F_h(x)$ for all h and x.

We are going to show that, for some $h \in (0, 1)$, the measures μ and ν_h satisfy the requirements.

Assume that this is false, that is, for each $h \in (0, 1)$ there is a convex 5-fan equipartitioning μ and $\nu(\sigma_i) = \nu(\sigma_{i+1}) = \nu(\sigma_{i+2}) = \nu(\sigma_{i+3}) > 0$. As we have seen before, the center of the 5-fan cannot lie between Q and T. Consequently four consecutive rays intersect T at points $(-0.6, 1), (-0.2, 1), (0.2, 1), (0.6, 1)$ and then intersect Q at points $x, x + y, x + 2y, x + 3y$, say. These four points split Q into five intervals $I_1 = (-\infty, x),\ I_2 = (x, x + y),\ I_3 = (x + y, x + 2y),\ I_4 = (x + 2y, x + 3y),\ I_5 = (x + 3y, \infty)$. Because of symmetry (iii) it suffices to consider three cases:

Case 1 when $\nu_h(I_1)=\nu_h(I_2)=\nu_h(I_3)=\nu_h(I_4)$,
Case 2 when $\nu_h(I_1)=\nu_h(I_2)=\nu_h(I_3)=\nu_h(I_5)$,
Case 3 when $\nu_h(I_1)=\nu_h(I_2)=\nu_h(I_4)=\nu_h(I_5)$.

We show that there is a small $h_0>0$ such that all three cases fail for $h\in(0,h_0)$ and for $h\in(1-h_0,1)$. This is needed because of symmetry.

Case 1 This case is the simplest: the points $(x+iy, F(x+iy))$, $i=0,1,2,3$ are on the same line contradicting property (i).

Case 2 Now $x<0$ as otherwise the points $(x+iy, F(x+iy))$, $i=0,1,2$ would be on the same line contradicting property (ii). Similarly $x+2y>0$. The conditions say that $2F_h(x)=F_h(x+y)$, $3F_h(x)=F_h(x+2y)$ and $F_h(x)=1-F_h(x+3y)$. If $0\in(x,x+y]$, then we have

$$6he^x=3-3(1-h)e^{-x-y}=2-2(1-h)e^{-x-2y}=6(1-h)e^{-x-3y}.$$

Here the middle equation fails to hold when h is close to 1. When h is close to 0, then $x+y$ and $x+2y$ have to be close to 0; consequently $x+3y$ is also close to 0. But then $F_h(x)$ is close to 0 and $1-F_h(x+3y)$ is close to 1 so they cannot be equal.

If $0\in(x+y,x+2y)$, then we have

$$6he^x=3he^{x+y}=2-2(1-h)e^{-x-2y}=6(1-h)e^{-x-3y}.$$

The first equation shows that $e^y=2$. Then the last equation fails to hold when h is close to 1. We also have $he^x=(1-h)e^{-x}/8$, or $8he^{2x}=1-h$ which cannot hold when h is close to 0.

Case 3 Again, $x<0$ and $x+3y>0$ follow from (ii). By (iii) it suffices to consider the case $0\in[x+y,x+3y]$. Then $2F_h(x)=F_h(x+y)$ implies, again, that $y=\log 2$. Then, just as before, $F_h(x)=1-F_h(x+3y)$ gives $8he^{2x}=1-h$. This cannot hold for h close to 0. When h is close to 1, then $x\to-\infty$ and $x+3y>0$ is not possible since $y=\ln 2$.

7.3 Proof of Theorem 1.3(3)

We construct two probability measures μ and ν on $\mathbb{R}^2$ such that there is no $t\in(0,1/3)$ and no convex 4-fan in $\mathbb{R}^2$ satisfy the conditions $\mu(\sigma_i)=\nu(\sigma_i)=t$ for three consecutive subscripts.

This is similar to the example in Sect. 7.1. T is the same as there, μ is the uniform measure on T, and Q is again the interval $[(-2,0),(2,0)]$. But this time the measure ν has a continuous distribution function $F(x)$, defined on $x\in[-2,2]$. We assume that $F(x)$ is a strictly concave function with $F(-2)=0$ and $F(2)=1$ (of course). This implies that no line intersects the graph of F in more than two points. Assume there is $t>0$ and a convex 4-fan with $\mu(\sigma_i)=\nu(\sigma_i)=t$ for three subscripts i. Then for the fourth subscript j, $\mu(\sigma_j)=\nu(\sigma_j)=1-3t$.

As we have seen above, the center of the 4-fan cannot be between the lines of T and Q. Consequently three consecutive rays intersect both Q and T. Let x,y,z be the intersection points of these rays with Q in this order from left to right. The conditions on μ and ν imply that either $y-x=\lambda t$, $z-y=\lambda t$ and $F(y)-F(x)=t$, $F(z)-F(y)=t$, or $y-x=\lambda t$, $z-y=\lambda(1-3t)$ and $F(y)-F(x)=t$, $F(z)-F(y)=1-3t$, or $y-x=\lambda(1-3t)$, $z-y=\lambda t$ and $F(y)-F(x)=1-3t$, $F(z)-F(y)=t$ with a suitable positive λ. In all three cases

$$\frac{F(y)-F(x)}{y-x}=\frac{F(z)-F(y)}{z-y}=\frac{1}{\lambda}.$$

So the points $(x,F(x)),(y,F(y)),(z,F(z))$ from the graph of F are on the same line, contrary to the assumption of concavity of F.

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