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Jarník's convex lattice n-gon for non-symmetric norms

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Abstract What is the minimum perimeter of a convex lattice n-gon? This question was answered by Jarník in 1926. We solve the same question, and prove a limit shape result, in the case when perimeter is measured by a (not necessarily symmetric) norm.

Keywords Convex lattice polygon · Primitive vectors · Isoperimetric problem

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1 Introduction

What is the minimal perimeter L_n that a convex lattice polygon with n vertices can have? In 1926 Jarník [4] proved that $L_n = \frac{\sqrt{6\pi}}{9} n^{3/2} + O(n^{3/4})$. The aim of this paper is to extend this result to all, not necessarily symmetric, norms in the plane. As usual, such a norm is defined by a convex compact set $D \subset \mathbf{R}^2$ with $0 \in \text{int}\, D$, and the norm of $x \in \mathbf{R}^2$ is

$$||x|| = ||x||_D = \min\{t \geq 0 : x \in tD\}.$$

Let $\mathbf{Z}^2$ be the lattice of integer points in $\mathbf{R}^2$, and write $\mathcal{P}_n$ ($n \geq 3$) for the set of all convex lattice n-gons in $\mathbf{R}^2$, that is, $P \in \mathcal{P}_n$ if $P = \text{conv}\,\{z_1, \ldots, z_n\}$ where $z_1, \ldots, z_n \in \mathbf{Z}^2$ are the

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vertices, in anticlockwise order, of P. The D-perimeter of P is defined by

$$\text{Per } P = \text{Per}_D P = \sum_{i=1}^{n} ||z_{i+1} - z_i||_D$$

where $z_{n+1} = z_1$ by convention. Note that for a non-symmetric D, $\text{Per}_D P$ depends on the orientation of P as well. Define now

$$L_n = L_n(D) = \min\{\text{Per}_D P : P \in \mathcal{P}_n\} \tag{1.1}$$

and $P_n \in \mathcal{P}_n$ a (not necessarily unique) minimizer satisfying $\text{Per}_D P_n = L_n(D)$.

In this paper we determine the asymptotic behaviour of $L_n(D)$ for all norms and show, further, that, after suitable scaling, the minimizing polygons have a limiting shape. Similar results were proved by Maria Prodromou [5] in 2005 in the case when D is symmetric, that is, $D = -D$.

An important part of the proof is a centering procedure of the original norm which yields a new norm whose unit ball has its center of gravity at the origin. Moreover, the perimeter of a polygon using this new norm is the same as the perimeter computed with the original norm. This procedure is described in Sect. 3.

Once this new norm is defined, we follow Jarník's idea in Sect. 6, by using the n shortest primitive vectors to estimate the asymptotics of L_n.

The existence of a limit shape for P_n is proved in Sect. 9 and requires substantial additional efforts which are not encountered in the centrally symmetric case of [5].

2 Results and notations

Assume that the vertices of a minimizer $P_n \in \mathcal{P}_n$ are $z_1, \ldots, z_n$ in anticlockwise order (which is the orientation giving the minimal D-perimeter). Then $E_n = \{z_2 - z_1, \ldots, z_n - z_{n-1}, z_1 - z_n\}$ is the edge set of P_n. Define $C_n = \text{conv } E_n$. Note that E_n determines P_n uniquely (up to translation). Even more generally, the following is true.

Proposition 2.1 *Suppose $V \subset \mathbf{R}^2$ is a finite set of vectors whose sum is zero. Assume further that $u, v \in V$, $u = \lambda v$ with $\lambda > 0$ implies that $u = v$. Then there is a unique (up to translation) convex polygon whose edge set is equal to V.*

Proof This is very simple. One has to order (cyclically) the vectors in V by increasing slope as $v_1, \ldots, v_n, v_1$. Then the polygonal path through the points $0, v_1, v_1 + v_2, v_1 + v_2 + v_3, \ldots, v_1 + \cdots + v_n = 0$ in this order is a convex polygon with edge set V. Uniqueness is clear. □

We call this construction the *increasing slope construction*. Here come our main results. We let $\mathcal{K}$ denote the family of all convex compact sets in $\mathbf{R}^2$ with non-empty interior. For $K, L \in \mathcal{K}$, dist (K, L) denotes their Hausdorff distance.

Theorem 2.2 *There is a unique $C \in \mathcal{K}$ such that* $\lim \text{dist}\,((\text{Area } C_n)^{-1/2} C_n, C) = 0$. *Moreover, $g(C) = 0$ and* $\lim n^{-3/2} L_n(D)$ *exists and equals*

$$\alpha(D) = \frac{\pi}{\sqrt{6}} \int_C ||x|| dx.$$

The two parts of this statement are proved respectively in Sects. 5 and 6.

Theorem 2.3 *There is a convex set $P \subset \mathbf{R}^2$ such that the following holds. Let P_n be an arbitrary sequence of minimizers, of $L_n(D)$, translated so that* $\min\{x : (x, y) \in P_n\}$ *is reached at the origin. Then* $\lim \operatorname{dist}(n^{-3/2} P_n, P) = 0$.

We explain in Sect. 9.1 how and why P is determined uniquely by C. Moreover, it is shown in Sect. 10 that P is a circular disk if and only if the unit ball D is an ellipse having a focus point at the origin, which is the case with Jarník's polygon.

To avoid some trivial complications in the proofs we assume that D is strictly convex. We emphasize however that the above results are valid without this extra condition.

3 Centering the norm

We introduce some notation. We write B for the Euclidean unit ball in $\mathbf{R}^2$ and $|x|$ for the Euclidean norm of $x \in \mathbf{R}^2$. Since D is compact convex and $0 \in \operatorname{int} D$, there are positive constants d_1, d_2 such that $d_1 B \subset D \subset d_2 B$, or, equivalently,

$$d_1|x| \le \|x\| \le d_2|x|, \quad \text{for every } x \in D.$$

Let $\langle , \rangle$ denote the Euclidean scalar product in $\mathbf{R}^2$. Write $e(t)$ for the Euclidean unit vector $(\cos t, \sin t)$, $t \in [0, 2\pi]$. Define the following open set:

$$D^\star = \{u \in \mathbf{R}^2 \quad ||x||_D + \langle u, x\rangle > 0, \forall x \in \mathbf{R}^2, x \ne 0\}$$

Each $u \in D^\star$ defines a norm $||.||_u$ on $\mathbf{R}^2$, with unit ball D_u via

$$||x||_u = ||x||_D + \langle x, u\rangle.$$

Fact 3.1 *For each $u \in D^*||.||_u$ is a norm, and* $\operatorname{Per}_D(P) = \operatorname{Per}_{D_u}(P)$ *for every $P \in \mathcal{P}_n$. Consequently $P \in \mathcal{P}_n$ is a minimizer for $L_n(D)$ if and only if it is one for $L_n(D_u)$.*

We need some further notation. As usual, the radial function of D_u, $r_u(t)$ is defined as $||e(t)||_u^{-1}$. Let $g(K)$ denote the center of gravity of $K \in \mathcal{K}$, that is, $g(K) = (\text{Area } K)^{-1} \int_K x dx$. When $K = D_u$ this can be written as

$$g(D) = (3\text{Area } D_u)^{-1} \int_0^{2\pi} r_u^3(t) e(t) dt.$$

The following simple result is important.

Theorem 3.2 *There exists a unique vector $u \in D^*$ such that $g(D_u)$ lies at the origin.*

Remark The uniqueness of this u is actually not needed and could be seen as a consequence of the sequel.

Proof We define a map $\Psi : D^* \to (0, \infty)$ by

$$\Psi(u) = \int_0^{2\pi} r_u^2(t) dt.$$

This map is differentiable and $\nabla\Psi(u) = -2 \int_0^{2\pi} r_u^3(t) e(t) dt$.

We **claim** now that

(i) Ψ is strictly convex on D^*,
(ii) if $v \in \partial D^*$ and $u \in D^*$ tends to v, then $\Psi(u)$ tends to infinity.

Before the proof we show how the claim implies the theorem. By (i) and (ii) Ψ takes its minimum at a unique $u_0 \in D^*$, and $\nabla\Psi(u_0) = 0$. Consequently $g(D_{u_0}) = 0$, indeed. Uniqueness follows since $\nabla\Psi(u) = 0$ implies that Ψ takes its minimum at u.

Proof of the claim For (i) we have to show that

$$\Psi\left(\frac{1}{2}(u+v)\right) \le \frac{1}{2}\left(\Psi(u) + \Psi(v)\right). \tag{3.1}$$

Observe that for all t

$$||e(t)|| + \left\langle \frac{1}{2}(u+v), e(t) \right\rangle = \frac{1}{2}[||e(t)|| + \langle u, e(t)\rangle + ||e(t)|| + \langle v, e(t)\rangle],$$

which is the same as

$$r^{-1}_{\frac{1}{2}(u+v)}(t) = \frac{1}{2}[r_u^{-1}(t) + r_v^{-1}(t)].$$

This implies, via elementary methods, that

$$r^{2}_{\frac{1}{2}(u+v)}(t) \le \frac{1}{2}[r_u^{2}(t) + r_v^{2}(t)],$$

with equality iff $u = v$. Integrating this inequality gives (3.1), and the case of equality is clear.

For the proof of (ii) we show that, for large enough N there is a small $\delta > 0$ so that $\Psi(u) > N$ for every $u \in D^*$ with $|u - v| < \delta$ (Euclidean distance). Note first that $v \in \partial D^*$ implies the existence of $\tau \in [0, 2\pi]$ with $||e(\tau)|| + \langle v, e(\tau)\rangle = 0$. Since D is convex, the function $r(t) = ||e(t)||^{-1}$ has left and right derivatives everywhere, and then so does $||e(t)||$. This implies that $||e(t) - e(\tau)||$ is bounded in absolute value by a constant times $t - \tau$ for every $t \in [0, 2\pi]$. Then on the same interval $r_u^{-1}(t) - r_u^{-1}(\tau) = ||e(t)|| - ||e(\tau)|| + \langle e(t) - e(\tau), u\rangle \le (const)|t - \tau|$ (because D^* is bounded). Further $r_u^{-1}(\tau) = r_u^{-1}(\tau) - ||e(\tau)|| - \langle v, e(\tau)\rangle = \langle e(\tau), u - v\rangle$ which is smaller than δ in absolute value. Then for $t \in [0, 2\pi]$

$$r(t) > (\text{const}|t - \tau| + \delta)^{-1}.$$

The integral of $r^2(t)$ on $[0, 2\pi]$ is then larger than const$/\delta$. □

From now on we work with the new norm $||.||_u$, and for simpler notation we drop the subscript u. As we have seen this does not affect the value of $L_n = L_n(D)$, or the set of minimizers for L_n. It is also clear that the new norm is strictly convex, again. We make another simplifying assumption, namely, that

$$\text{Area } D = 1 \tag{3.2}$$

This is just a convenient scaling of the unit ball which leaves the set of minimizers, and the corresponding E_n, C_n and consequently C, P (from Theorems 2.2 and 2.3) unchanged.

4 Auxiliary lemmas

We write $\mathbf{P}$ for the set of primitive vectors in $\mathbf{Z^2}$, i.e., $z = (x, y) \in \mathbf{Z^2}$ $(z \neq 0)$ is in $\mathbf{P}$ if x and y are relatively prime. The following two claims are very simple. The second one explains the crucial role played by $\mathbf{P}$ in all the problem.

Claim 4.1 *For all* $n \geq 3$, $L_n < L_{n+1}$.

Proof Let $P_{n+1} = \operatorname{conv}\{z_0, z_1, \ldots, z_n\}$ be a minimizer for L_{n+1} and set $P_n^* = \operatorname{conv}\{z_1, \ldots, z_n\}$. Then $L_n \leq \operatorname{Per} P_n^* < L_{n+1}$. □

Claim 4.2 $E_n \subset \mathbf{P}$.

Proof Assume P_n is a minimizer and the edge $z_2 - z_1 \notin \mathbf{P}$, say. Then the segment $[z_1, z_2]$ contains an integer $z \in \mathbf{Z^2}$ distinct from z_1, z_2. The convex lattice n-gon $\operatorname{conv}\{z_1, z, z_3, \ldots, z_n\}$ has shorter D-perimeter than P_n because the triangle $\operatorname{conv}\{z_1, z_2, z_3\}$ contains the triangle $\operatorname{conv}\{z_1, z, z_3\}$ so the latter has shorter D-perimeter. □

The following lemma will be useful when proving that most points in $C_n \cap \mathbf{P}$ belong to E_n.

Lemma 4.3 *Assume* $a, b \in E_n$ *and* $a \neq \pm b$. *Let* T *be the parallelogram with vertices* $0, a, b, a + b$. *If* $x, y \in (T \cap \mathbf{P}) \setminus E_n$ *and* $x \neq y$, *then* $x + y \notin T$.

Proof If $x+y \in T$ were the case, then set $E^* = E_n \cup \{x, y, z\} \setminus \{a, b\}$ where $z = a+b-x-y$. The increasing slope construction works now because $\sum_{z \in E^*} z = 0$ and gives rise to a convex lattice $(n + 1)$-gon P if there is no $u \in E_n$ with $u = \lambda z$ with $\lambda > 0$. If there is such a u, we replace u and z by $u + z$ in E^*, and the increasing slope construction gives a convex lattice n-gon P. We claim that P has shorter D-perimeter than P_n. This clearly finishes the proof.

To prove $\operatorname{Per} P < \operatorname{Per} P_n$ we have to show that $\|x\| + \|y\| + \|z\| < \|a\| + \|b\|$. According to our assumptions, $x = \alpha_1 a + \beta_1 b$, $y = \alpha_2 a + \beta_2 b$, and $z = \alpha_3 a + \beta_3 b$ with $\alpha_i, \beta_i \geq 0$ for all i and $\sum \alpha_i = \sum \beta_i = 1$. Thus (Fig. 1)

$$\begin{aligned} \|x\| + \|y\| + \|z\| &\leq \alpha_1\|a\| + \beta_1\|b\| + \alpha_2\|a\| + \beta_2\|b\| + \alpha_3\|a\| + \beta_3\|b\| \\ &= \|a\| + \|b\|. \end{aligned}$$

□

5 The density of P

In what follows $c, c_1, c_2, ..$ denote positive constants independent of n. We will also use Vinogradov's convenient $\ll$ notation: $f(n) \ll g(n)$ means that there are positive constants c and n_0 such that $cf(n) \leq g(n)$ for all $n \geq n_0$. Of course, the constants do not depend on n. But they depend on D, more precisely, they depend on the constants d_1, d_2. $f(n) \gg g(n)$ has the same meaning but with $f(n) \geq cg(n)$. We will also use the big Oh and little oh notation.

We need some standard estimates on the distribution of lattice points and primitive points in a convex body $K \in \mathcal{K}$. We assume that $0 \in K$. Let L denote the Euclidean perimeter of K. We suppose that $L > 100$, say, but we think of K as "large". In fact, in most applications L tends to infinity. The following estimate is simple and well-known.

$$\Big| |K \cap \mathbf{Z^2}| - \operatorname{Area} K \Big| \leq 2L. \tag{5.1}$$

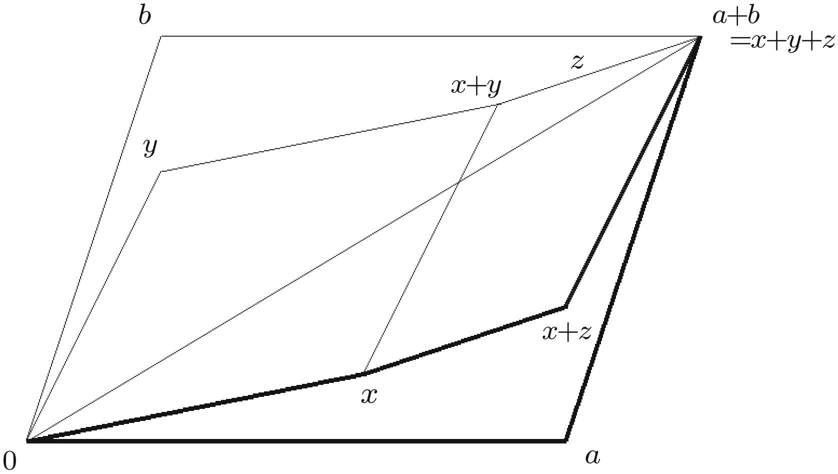

Fig. 1 The proof of Lemma 4.3

This implies, with the standard method using the Möbius function, that

$$\left| |K \cap \mathbf{P}| - \frac{6}{\pi^2} \text{Area } K \right| \le 3L \log L. \tag{5.2}$$

Assume next that $f : \mathbf{R^2} \to \mathbf{R}$ is a 1-homogeneous function, that is, $f(\lambda x) = \lambda f(x)$ for every $x \in \mathbf{R^2}$ and $\lambda \ge 0$. Write $M = \max\{|f(z)| : z \in K\}$. Let $Q(z)$ denote the aligned unit square centered at $z \in \mathbf{Z^2}$. Define the variance of f on K as

$$V = \max\{|f(x) - f(z)| : x \in Q(z) \text{ and } Q(z) \cap K \ne \emptyset \text{ and } z \in \mathbf{Z^2}\}$$

Under these conditions the following estimates hold.

$$\left| \sum_{z \in K \cap \mathbf{Z^2}} f(z) - \int_K f(z)dz \right| \le V \text{Area } K + 4ML, \tag{5.3}$$

$$\left| \sum_{z \in K \cap \mathbf{P}} f(z) - \frac{6}{\pi^2} \int_K f(z)dz \right| \le (2V \text{Area } K + 5ML) \log L. \tag{5.4}$$

The proof of the four estimates (5.1), (5.2), (5.3), (5.4) is postponed to Appendix 1. Proofs of analogous results can also be found in [3] or [1].

These estimates will be used quite often in the case when $K = \lambda K_0$, and $\lambda \to \infty$ with K_0 fixed. Then formulae (5.1), (5.2), (5.3), (5.4) have the following simpler form:

$$|K \cap \mathbf{Z^2}| = \lambda^2 \text{Area } K_0(1 + O(\lambda^{-1})), \tag{5.5}$$

$$|K \cap \mathbf{P}| = \frac{6}{\pi^2} \lambda^2 \text{Area } K_0(1 + O(\lambda^{-1} \log \lambda)). \tag{5.6}$$

$$\sum_{z \in K \cap \mathbf{Z^2}} f(z) = \lambda^3 \int_{K_0} f(z)dz + O(\lambda^2), \tag{5.7}$$

$$\sum_{z \in K \cap \mathbf{P}} f(z) = \frac{6}{\pi^2} \lambda^3 \int_{K_0} f(z)dz + O(\lambda^2 \log \lambda). \tag{5.8}$$

The constant in the big Oh notation depends only on K_0. Here K_0 is either a convex set or a starshaped set with boundary consisting of finitely many line segments.

6 Asymptotics of L_n

We show first that $\lim n^{-3/2}L_n(D)$ exists and equals $\frac{\pi}{\sqrt{6}}\int_D ||x||dx$. So the formula for $\alpha(D)$ in Theorem 2.2 holds with $C = D$ (but, of course, only after the centering procedure).

We start with the lower bound.

Claim 6.1 $\liminf n^{-3/2}L_n \geq \frac{\pi}{\sqrt{6}}\int_D ||x||dx$.

Proof Here we use the following *density principle*. The sum of the lengths of n distinct primitive vectors is at least as large as the sum of the lengths of the n shortest (distinct) primitive vectors. We will see the same principle in action again.

Let $v_1, \dots, v_n$ be the n shortest, in $||.||$-norm, vectors in $\mathbf{P}$ (ties broken arbitrarily). Set $\lambda = \max\{\|v_i\| : i = 1, \dots, n\}$. Then $(\operatorname{int} \lambda D) \cap \mathbf{P} \subset \{v_1, \dots, v_n\} \subset \lambda D$. The boundary of λD contains at most $\operatorname{Per}_B \lambda D \leq 2\pi d_2 \lambda$ lattice points. So $|\lambda D \cap \mathbf{P}| - 2\pi d_2\lambda \leq n \leq |\lambda D \cap \mathbf{P}|$. Using (5.6) with λD gives, together with Area $D = 1$, that

$$|\lambda D \cap \mathbf{P}| = \frac{6}{\pi^2}\lambda^2(1 + O(\lambda^{-1}\log \lambda)).$$

This shows that $n = \frac{6}{\pi^2}\lambda^2(1 + O(\lambda^{-1}\log\lambda))$ implying that $\lambda = \left(\frac{\pi}{\sqrt{6}} + o(1)\right)n^{1/2}$. Using this in (5.8) with λC gives

$$L_n \geq \sum_1^n \|v_i\| \geq \sum_{z \in \operatorname{int}(\lambda D)\cap\mathbf{P}} \|z\| \geq \left(\frac{6}{\pi^2} - O(\lambda^{-1}\log\lambda)\right)\lambda^3 \int_D \|z\|dz.$$

□

Claim 6.2 $\limsup n^{-3/2}L_n \leq \frac{\pi}{\sqrt{6}}\int_D ||x||dx$.

Proof Consider now the convex polygon whose edges are $v_1, \dots, v_n$ plus an extra edge $v_0 = -\sum_{i=1}^n v_i$ which may have the same direction as one of the other n vectors. This polygon is either an n-gon or an $n+1$-gon. Since, by Claim 4.1, $L_n < L_{n+1}$, we obtain

$$\begin{aligned} L_n < L_{n+1} &\leq \sum_{i=0}^n ||v_i|| = \sum_{i=1}^n ||v_i|| + \left\|-\sum_{i=1}^n v_i\right\| \\ &\leq \sum_{z\in\lambda D\cap\mathbf{P}} ||z|| + \left\|-\sum_{z\in\lambda D\cap\mathbf{P}} z\right\| + 2|\mathbf{P}\cap\partial\lambda D| \end{aligned}$$

with the notations of Claim 6.1. Claim 5.8 bounds the two first terms respectively by $\left(\frac{6}{\pi^2} + O(\lambda^{-1}\log\lambda)\right)\lambda^3\int_C \|z\|dz$ and $O(\lambda^2\log\lambda)$. The third term is bounded by λ times the number of lattice points of the boundary of λC and so it is less than $2\pi d_2\lambda^2$. □

7 Bounding C_n

Our next target is to give bounds on the width and diameter of $C_n = \operatorname{conv} E_n$.

Claim 7.1 *The width of E_n, $w(E_n)$, satisfies $w(E_n) \gg n^{1/2}$.*

Proof Set $w = w(E_n)$. Clearly,

$$L_n = \sum_{v \in E_n} \|v\| \gg \sum_{v \in E_n} |v| \geq M_n(w),$$

where $M_n(w)$ is the sum of the lengths of the n shortest (in Euclidean norm) distinct vectors in $\mathbf{Z}^2$ lying in a strip of width w.

A simple yet technical computation, delayed to Appendix 1, shows that $w \leq \gamma n^{1/2}$ (where $\gamma \in (0, 1/2]$) implies $M_n(w) \gg n^{3/2}/\gamma$. This finishes the proof of Claim 7.1, because then $n^{3/2} \gg L_n \gg M_n(w) \gg n^{3/2}/\gamma$ would lead to contradiction if γ were too small. □

Claim 7.2 *Assume the smallest Euclidean ball centred at* 0 *and containing E_n is RB. Then $R \ll n^{1/2}$.*

Proof Assume a is the farthest point (in Euclidean distance) from the origin in E_n. Then $|a| = R$. Claim 6.2 implies that $|a| \leq L_n \ll n^{3/2}$. Since $w(E_n) \gg n^{1/2}$ by the previous claim, there is a point $b \in E_n$ whose distance from the line $\{x = ta : t \in \mathbf{R}\}$ is $\geq \frac{1}{2} w(E_n) \gg n^{1/2}$.

The perimeter of the triangle $\triangle = \operatorname{conv}\{0, a, b\}$ is $|a| + |b| + |a - b| \leq 4|a|$ because $|b| \leq |a|$ and $|a - b| \leq |a| + |b| \leq 2|a|$. Here Area $\triangle = \frac{1}{2}|a|h$ where h is the corresponding height of $\triangle$. Since $w(E_n) \geq n^{1/2}$, $h \gg n^{1/2}$.

Then by (5.2) for large enough n,

$$\left| |\mathbf{P} \cap \frac{1}{2}\triangle| - \frac{6}{\pi^2} \text{Area}\, \frac{1}{2}\triangle \right| \leq 3 \cdot 2|a| \log 2|a| \ll h|a| \frac{\log |a|}{\sqrt{n}} \ll \text{Area}\, \triangle \frac{\log n}{\sqrt{n}}$$

implying that $|\mathbf{P} \cap \frac{1}{2}\triangle| \geq \frac{1}{8}$Area $\triangle$, again when n is large enough.

Assume now that Area $\triangle > 16n$. Then $|\mathbf{P} \cap \frac{1}{2}\triangle| \geq 2n$. Since $|E_n| \leq n$, $\frac{1}{2}\triangle$ contains two distinct points $x, y \in \mathbf{P} \setminus E_n$ and, evidently, $x + y \in \triangle$. Then $x, y, x + y \in \operatorname{conv}\{0, a, b, a + b\}$ contradicting Lemma 4.3.

Thus Area $\triangle = \frac{1}{2}|a|h \leq 16n$, and so $R = |a| \ll n^{1/2}$. □

We need one more fact about C_n:

Claim 7.3 *Assume rB is the largest Euclidean ball centered at* 0 *and contained in C_n. Then $r \gg n^{1/2}$.*

Proof Let a be the nearest point to 0 on the boundary of C_n. Thus $r = |a|$. Define $E^+ = E_n \cap \{x \in \mathbf{R}^2 : ax > 0\}$ and $E^- = E_n \cap \{x \in \mathbf{R}^2 : ax < 0\}$, and set $f(x) = ax/|a|$ which is just the component of $x \in \mathbf{R}^2$ in direction a. To have simpler notation we write $f(X) = \sum_{x \in X} f(x)$ when $X \in \mathbf{R}^2$ is a finite set. Since $\sum_{z \in E_n} z = 0$, $f(E^+) + f(E^-) = 0$ (because $f(z) = 0$ when $az = 0$). We will show, however, that $|a| \leq \gamma n^{1/2}$, for a suitably small $\gamma > 0$, implies that

$$f(E^+) + f(E^-) < 0. \tag{7.1}$$

Define $F^+ = \{x \in \mathbf{R}^2 : 0 < f(x) \leq \gamma n^{1/2}\} \cap RB$ with $R \ll n^{1/2}$ from Claim 7.2. The density principle tells now that $f(E^+) \leq f(\mathbf{P} \cap F^+) \leq f(\mathbf{Z}^2 \cap F^+)$ and the last sum can be estimated as follows. Let $Q(z)$ be the unit cube centred at z. Again, Area $Q(z) \cap F^+ \geq 1/4$ for all $z \in \mathbf{Z}^2 \cap F^+$. This implies that, for large enough n,

$$m := |\mathbf{Z}^2 \cap F^+| \leq \text{Area}\, F^+/4 \ll R|a| \ll \gamma n.$$

We use now (5.3):

$$\left| f(\mathbf{Z^2} \cap F^+) - \int_{F^+} f(z)dz \right| \ll R|a|.$$

It is easy to see that $\int_{F^+} f(z)dz \ll |a|^2 R$ implying that $f(\mathbf{Z^2} \cap F^+) \ll |a|^2 R \ll \gamma^2 n^{3/2}$.

Define $F^- = \{x \in \mathbf{R^2} : 0 > f(x) \geq -\lambda\gamma n^{1/2}\} \cap RB$ where $\lambda > 0$ is chosen so that F^- contains exactly $n - m - k$ lattice points. Here k is the number of lattice points on the line $ax = 0$ so $k \leq 2R + 1 \ll n^{1/2}$. Note that $\lambda\gamma n^{1/2} \ll R$ since $E_n \subset RB$ consists of exactly n vectors. Choosing γ small enough guarantees that $m < 0.1n$ which, in turn, guarantees that $\lambda > 1$ and further, that $|F^- \cap \mathbf{Z^2}| \geq 0.8n$. The Euclidean perimeter of F^- is at most $4R + \lambda\gamma n^{1/2} \ll R$ and (5.1) shows that $\big||F^- \cap \mathbf{Z^2}| - \text{Area } F^-\big| \ll R$. Clearly Area $F^- \ll R\lambda\gamma n^{1/2}$, implying that

$$0.8n < |F^- \cap \mathbf{Z^2}| \leq \left(1 + O\left(\frac{1}{\lambda\gamma n^{1/2}}\right)\right) \text{Area } F^- \ll R\lambda\gamma n^{1/2} \ll \lambda\gamma n,$$

which implies $\lambda\gamma \gg 1$.

The density principle says now that $f(E^-) \leq f(F^-)$ (note that f is negative on F^- and E^-), and $f(F^-)$ can be estimated using (5.3):

$$\left| f(F^-) - \int_{F^-} f(z)dz \right| \ll R^2 \ll n,$$

because $\max\{|f(x)| : x \in F^-\} \leq R$. Now $f(z)$ is negative on F^-. It is easy to check that $\lambda^2\gamma^2 nR \ll -\int_{F^-} f(z)dz \ll \lambda^2\gamma^2 nR$. So we have

$$-f(F^-) \geq \int_{F^-} -f(z)dz + O(n) \gg \int_{F^-} -f(z)dz \gg \lambda^2\gamma^2 nR \gg n^{3/2}$$

This shows that (7.1) indeed holds if $\gamma > 0$ is chosen small enough because $0 < f(\mathbf{Z^2} \cap F^+) \ll \gamma^2 n^{3/2}$ and $-f(\mathbf{Z^2} \cap F^-) \gg n^{3/2}$. □

Corollary 7.4 *There are positive numbers r and R (depending only on D) such that for all* $n \geq 3$

$$rB \subset (\text{Area } C_n)^{-1/2} C_n \subset RB.$$

8 Almost all primitive points of C_n are in E_n

We state now a geometric lemma which is about a special kind of approximation. The technical proof is postponed to Appendix 3.

Lemma 8.1 *Assume* $K \in \mathcal{K}$ *is a convex polygon with* $rB \subset K \subset RB$. *Then for every* $\delta \in (0, 0.02(r/R)^2]$ *there are vertices* $v_1, \ldots, v_m$ *of* K *such that with* $Q = \text{conv}\{v_1, \ldots, v_m\}$ *the following holds:*

- $Q \subset K \subset (1 + 4R^2 r^{-2}\delta)Q,$
- *for all* i, *the angle* $\angle v_i 0 v_{i+1}$ *is at least* δ.

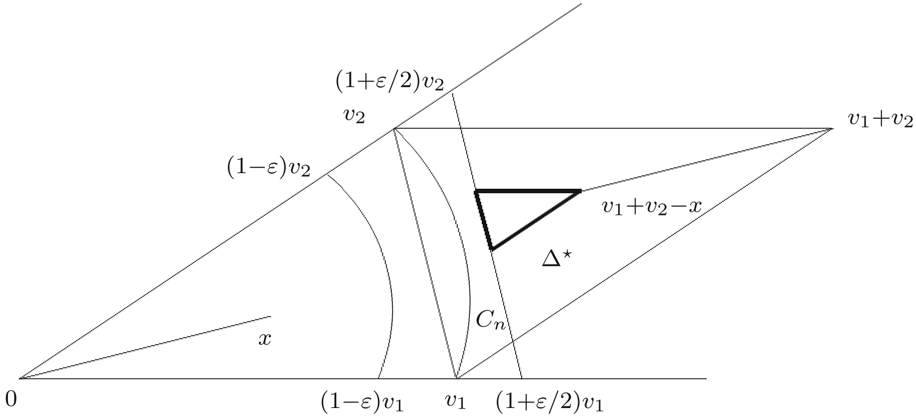

Fig. 2 The proof of Lemma 8.2

Lemma 8.2 *For every $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon, D)$ such that for all $n \geq n_0$, $(1-\varepsilon)C_n \cap \mathbf{P} \subset E_n$.*

Proof Let r_n, resp. R_n be the maximal, minimal radius such that $r_n B \subset C_n \subset R_n B$. It follows from Corollary 7.4 that $R_n/r_n \leq c$ with a suitable positive constant depending only on D. Thus Lemma 8.1 can be applied with $K = C_n$ and $\delta = \varepsilon/(8c^2)$ (if $\varepsilon \leq 0.02/8$ which we can clearly assume). We get a polygon $Q = \operatorname{conv}\{v_1, \ldots, v_m\}$ satisfying $C_n \subset (1+\varepsilon/2)Q$.

Assume, contrary to the statement of the lemma, that there is an $x \in (1-\varepsilon)C_n \cap \mathbf{P} \setminus E_n$. One of the cones $\operatorname{pos}\{v_i, v_{i+1}\}$ contains x, the cone $W := \operatorname{pos}\{v_1, v_2\}$, say. Define $\triangle = \operatorname{conv}\{0, v_1, v_2\}$. Thus $\triangle \subset C_n \cap W \subset (1+\varepsilon/2)\triangle$, see Fig. 2. As $x \in (1-\varepsilon)C_n \cap W$, $v_1+v_2-x \in W \setminus (1+\varepsilon)\triangle$. The triangle $\triangle^* = ((v_1+v_2-x) - W) \setminus (1+\varepsilon/2)\triangle$ is disjoint from C_n. We claim that it contains a primitive point y. This will finish the proof since then x, y, $x+y$ all lie in the parallelogram with vertices $0, v_1, v_2, v_1+v_2$ contradicting Lemma 4.3.

We prove the claim by using (5.2): Area $\triangle^* \gg \varepsilon^3 n$ because its angle at v_1+v_2-x is at least δ, and the neighbouring sides are of length at least $\varepsilon|v_1|/2$ and $\varepsilon|v_2|/2$ and $|v_1|, |v_2| \gg n^{1/2}$. Further, its perimeter is at most $|v_1|+|v_2|+|v_1-v_2| \ll n^{1/2}$. Thus

$$\left| |\triangle^* \cap \mathbf{P}| - \frac{6}{\pi^2} \text{Area } \triangle^* \right| \ll (\log n) n^{1/2}.$$

Here $\frac{6}{\pi^2}$ Area $\triangle^*$ is of order $\varepsilon^3 n$ and the error term is of order $(\log n)n^{1/2}$. Since ε fixed, $\triangle^*$ contains a primitive vector if n is large enough. □

9 The limit shape of C_n

In this section we prove Theorem 2.2.

The Blaschke selection theorem and Corollary 7.4 imply that $(\text{Area } C_n)^{-1/2} C_n$ contains a convergent (in Hausdorff metric) subsequence. Assume next that $(\text{Area } C_{n_k})^{-1/2} C_{n_k}$ is a convergent subsequence whose limit is a convex body $C \in \mathcal{K}$.

Our target is to show that $C = D$. This will prove Theorem 2.2: if every convergent subsequence of $(\text{Area } C_n)^{-1/2} C_n$ converges to D then the sequence itself converges to D.

Define $\lambda_k = \sqrt{\text{Area } C_{n_k}}$ and set, for simpler writing, $C^k = \lambda_k^{-1} C_{n_k}$. Since E_{n_k} contains n_k primitive points, we get, using Lemma 8.2, that $|C_{n_k} \cap \mathbf{P}|$ is $n_k(1+o(1))$. Now $||C_{n_k} \cap \mathbf{P}|$

$-\frac{6}{\pi^2}\text{Area}\, C_{n_k}|$ is controlled by the perimeter of C_{n_k} which is of the order of $\sqrt{\text{Area} C_{n_k}}$ by convexity of C_n and Corollary 7.4. Inequality 5.2 then yields $n_k = \frac{6}{\pi^2}\text{Area}\, C_{n_k}(1+o(1))$ and $\lambda_k = \frac{\pi}{\sqrt{6}}\sqrt{n_k}(1+o(1))$.

For every $\delta > 0$, $(1-\delta)C \subset C^k \subset (1+\delta)C$ for all large enough k. It follows immediately that Area $C = 1$. We show next that $\int_C z dz = 0$ (which implies $g(C) = 0$). For this it suffices to prove that $\int_C f(z)dz = 0$ in the case when f is the linear function $f(z) = x$ and $f(z) = y$ where $z = (x, y)$. Choose $\varepsilon > 0$ and then, using Lemma 8.1, k_0 so large that, for $k > k_0$,

$$(1-\varepsilon/2)C_{n_k} \cap \mathbf{P} \subset E_{n_k} \subset C_{n_k} \cap \mathbf{P}.$$

It follows now that there is a k_1 so that for all $k > k_1$

$$(1-\varepsilon)\lambda_k C \cap \mathbf{P} \subset E_{n_k} \subset (1+\varepsilon)\lambda_k C \cap \mathbf{P}. \tag{9.1}$$

Using the notation $f(X) = \sum_{z\in X} f(z)$ when $X \subset \mathbf{R^2}$ is finite, we have $f(E_{n_k}) = 0$. Next,

$$\begin{aligned} |f(\mathbf{P} \cap \lambda_k C)| &= |f(\mathbf{P} \cap \lambda_k C) - f(E_{n_k})| \\ &\le |f\,(\mathbf{P} \cap [(1+\varepsilon)\lambda_k C \setminus (1-\varepsilon)\lambda_k C])\,| \\ &\ll \varepsilon\lambda_k \max\{f(z) : z \in \lambda_k C\} \ll \varepsilon n_k. \end{aligned}$$

On the other hand, by (5.8),

$$|f(\mathbf{P} \cap \lambda_k C)| = \frac{6}{\pi^2}\lambda_k^3 \int_C f(z)dz \left(1 + O(\lambda_k^{-1} \log \lambda_k)\right)$$

as one can check easily. So if $\int_C f(z)dz \neq 0$, then $f(\mathbf{P} \cap \lambda_k C)$ is of order $n_k^{3/2}$. But as we have just shown, $|f(\mathbf{P} \cap \lambda_k C)| \ll \varepsilon n_k$. So indeed, $\int_C f(z)dz = 0$.

An almost identical proof, this time with the 1-homogeneous function $f(z) = \|z\|$ gives

$$\frac{\pi}{\sqrt{6}} \int_C \|x\| dx = \alpha(D).$$

We only give a sketch: Eq. (9.1) shows that

$$\left| \sum_{z\in\mathbf{P}\cap\lambda_k C} \|z\| - \sum_{z\in\mathbf{P}\cap E_{n_k}} \|z\| \right| \ll \varepsilon n_k.$$

Here $\sum_{z\in\mathbf{P}\cap E_{n_k}} \|z\| = L_{n_k}$ and so $\lim n_k^{-3/2} \sum_{z\in\mathbf{P}\cap\lambda_k C} \|z\| = \alpha(D)$. The estimate (5.4) says now that

$$\left| \sum_{z\in\mathbf{P}\cap\lambda_k C} \|z\| - \frac{6}{\pi^2} \int_{\lambda_k C} \|x\| dx \right| \ll n_k \log n_k,$$

and $\frac{\pi}{\sqrt{6}} \int_C \|x\| dx = \alpha(D)$ follows.

So $C \in K$ satisfies $g(C) = 0$, Area $C = 1$, and $\frac{\pi}{\sqrt{6}} \int_C \|x\| dx = \alpha(D)$. We show that $C = D$ then. Letting $\rho(.)$ denote the radial function of C, we have

$$\int_C \|x\| dx = \frac{1}{3} \int_0^{2\pi} \rho(t)^3 / r(t) dt = \int_D \|x\| dx = \frac{1}{3} \int_0^{2\pi} r^2(t) dt.$$

We use Hölder's inequality :

$$\int_0^{2\pi} \rho^2 \le \left(\int_0^{2\pi} \frac{\rho^3}{r}\right)^{2/3} \left(\int_0^{2\pi} r^2\right)^{1/3}$$

which is an equality if and only if ρ and r are proportional. But this is an equality in our case. Further, $\int_0^{2\pi} \rho^2 = \int_0^{2\pi} r^2 = 2$ and so $\rho = r$. Thus C and D have the same radial functions. □

9.1 Proof of Theorem 2.3

This is fairly simple once we know that $(\text{Area}\, C_n)^{-1/2} C_n$ tends to D. Recall that $e(t) = (\cos t, \sin t)$. When a minimizer P_n is translated as Theorem 2.3 specifies, the sum of the edges of P_n having direction between $e(0)$ and $e(t)$ is very close to the sum of the primitive vectors in C_n whose direction is between $e(0)$ and $e(t)$ in D_n. The latter, divided by $n^{3/2}$ is very close to $P(t) = \int_{D(t)} z dz$ where $D(t)$ is the set of vectors in D with direction between $e(0)$ and $e(t)$. The curve $P(t)$ is closed (because $g(D) = 0$) and convex (this has been shown in [2]), so it is the boundary of a convex set P. The simple and straightforward checking of

$$\lim \text{dist}\,(n^{-3/2} P_n, P) = 0$$

is left to the reader. We remark that the convexity of $P(t)$ follows also from the fact that the boundary of P_n, after suitable rescaling, tends to $P(t)$. □

The same construction $D \to P$ with $P(t) = \int_{D(t)} z dz$ is used, with a similar purpose, in [2]. Further properties of the construction are also established there.

10 An example

We concentrate now on the cases when the solution is constant which correspond to the case when the limit shape of the polygon is a circle.

Lemma 10.1 *The limit shape is a round if and only if $1/r$ is of the form $a + b\cos\theta + c\sin\theta$, or, in other words, when r is the radial function of an ellipse having its focus point at the origin.*

Proof Suppose $1/r$ is of the form above. Then the unique solution u of the centering procedure is equal to $(-b, -c)$, r_u is constant, and D_u is a circle. Conversely, if D_u is a circle, it means that r_u is constant. In other words, $1/r = 1/r_u - b\cos t - c\sin t$, where (b, c) are the coordinates of u. □

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11 Appendix 1

Lemma 11.1 *Let $M_n(w)$ be the sum of the lengths of the n shortest (in Euclidean norm) distinct vectors in $\mathbf{Z^2}$ lying in a strip of width w, centred at the origin. Suppose $\gamma \in (0, 1/2]$, then $w \le \gamma n^{1/2}$ implies $M_n(w) \gg n^{3/2}/\gamma$.*

Proof It is clear that this set of vectors is just the set of lattice points contained in $A := dB \cap T$ where T is a strip of width w, centred at the origin, and d is a suitable radius making $A \cap \mathbf{Z^2}$ have exactly n elements (ties broken arbitrarily). Let φ denote the angle that the strip T makes with the x-axis of $\mathbf{R^2}$. We may assume by symmetry that $\varphi \in [0, \pi/4]$.

Observe first that $d \ge \sqrt{n}/2$ since otherwise the disk dB would contain fewer than n lattice points. Let $Q(z)$ denote the unit square centred at $z \in \mathbf{R^2}$ and let ℓ_k be the line with equation $x = k$ (k is an integer). Clearly, ℓ_k intersects S in a segment of length $w \cos\varphi$, and so $\ell_k \cap \mathbf{Z^2}$ contains at least $\lfloor w/\cos\varphi \rfloor$ and at most $\lfloor w/\cos\varphi \rfloor + 1$ lattice points from S.

Assume first that $w/\cos\varphi \ge 1$. As is easy to see, Area $A \cap Q(z)$ is at least $1/4$ for $z \in A \cap \mathbf{Z^2}$. Hence, Area $A \ge n/4$. Since Area $A < 2dw$, $d > n/(4w)$ follows.

For simpler notation write $u = (d\cos\varphi)/2$. For the lines ℓ_k with $k \in [u, 2u - w/2]$, $\ell_k \cap A$ contains at least $\lfloor w/\cos\varphi \rfloor$ lattice points. Since $w < u$, there are at least $\lfloor 2u - w/2 \rfloor - \lfloor u \rfloor \gg u$ such lines. All of them have distance at least $(d - w)/2 \gg d$ from the origin. Consequently, using the bounds $w \le \gamma n^{1/2}$ and $d \ge n^{1/2}/2$ generously,

$$M_n(w) \gg d \left\lfloor \frac{w}{\cos\varphi} \right\rfloor u \gg d^2 w \ge \left(\frac{n}{4w}\right)^2 w \gg \frac{1}{\gamma} n^{3/2}.$$

Assume next that $w/\cos\varphi < 1$. There are at most six $z \in A \cap \mathbf{Z^2}$ such that $Q(z)$ intersects the boundary of dB. For the other $z \in A \cap \mathbf{Z^2}$, $Q(z)$ intersects the boundary of A in one or two line segments, whose total length is between $1/\cos\varphi$ and $2/\cos\varphi$. For distinct lattice points in $A \cap \mathbf{Z^2}$ the corresponding segments do not overlap. This implies that

$$\frac{n-6}{\cos\varphi} \le 4d \le \frac{2(n-6)}{\cos\varphi}.$$

Each line ℓ_k with $|k| \le 2u/3$ contains at most one lattice point from A. The remaining points from $A \cap \mathbf{Z^2}$, and there are at least $n - 2\lfloor 2u/3 \rfloor - 1$ of them, are at distance $\frac{d}{3} - 1$ from the origin. Hence, we see

$$M_n(w) \ge \left(\frac{d}{3} - 1\right)\left(n - \lfloor 2\frac{d\cos\varphi}{3} \rfloor - 1\right) \gg n^2.$$

□

12 Appendix 2

We prove here estimates (5.1), (5.2), (5.3), (5.4).

Call $Q(z)$, $z \in \mathbf{Z^2}$, *inside square* if $Q(z) \subset K$, and *outside square* if $z \notin K$ but $Q(z) \cap K \ne \emptyset$ and a *boundary square* if $z \in K$ and $Q(z) \setminus K \ne \emptyset$, and write I, B respectively O for the set of $z \in \mathbf{Z^2}$ for which $Q(z)$ is an inside, boundary, and outside square.

It is easy to check that $|B| + |O| \le 2L$. The estimate (5.1) follows from the fact that $|K \cap \mathbf{Z^2}| = |I| + |B|$ and Area $K = |I| + \sum_{z\in B\cup O}$ Area $K \cap Q(z)$. Then $|K \cap \mathbf{Z^2}| -$ Area $K = \sum_{z\in B}(1 -$ Area $K \cap Q(z)) - \sum_{z\in O}$ Area $K \cap Q(z))$. This implies (5.1) as each term in the sum is at most one in absolute value.

Let x and y denote the components of $z \in \mathbf{R^2}$. For the proof of (5.2) we use the Möbius function:

$$|K \cap \mathbf{P}| = \sum_{z \in K \cap \mathbf{Z^2}, z \neq 0} \sum_{d|x, d|y} \mu(d) = \sum_{d=1}^{\infty} \mu(d) \left(|\frac{1}{d} K \cap \mathbf{Z^2}| - 1 \right).$$

The terms in the sum are all zero for $d \geq L$ because the perimeter of K is L and $0 \in K$. Next, $|\frac{1}{d} K \cap \mathbf{Z^2}|$ can be estimated easily by (5.1):

$$\left| |\frac{1}{d} K \cap \mathbf{Z^2}| - \text{Area } \frac{1}{d} K \right| \leq 2 \frac{L}{d}.$$

As is well-known, $\sum_1^{\infty} \frac{\mu(d)}{d^2} = \frac{\pi^2}{6}$ and $|\sum_{d>L} \frac{\mu(d)}{d^2}| \leq \frac{1}{L}$. Putting these estimates together gives that the left hand side of (5.2) is at most

$$\frac{\text{Area } K}{L} + \sum_1^L \left| \frac{2L}{d} - 1 \right| \leq \frac{L}{4\pi} + 2L(1 + \log L) + L < 3L \log L,$$

if L is large enough. Here we also used the isoperimetric inequality in the form Area $A < \frac{L^2}{4\pi}$.

The proof of (5.3) starts the same way as that of (5.1):

$$\sum_{z \in K \cap \mathbf{Z^2}} f(z) = \sum_{z \in I \cup B} f(z) \text{ and}$$

$$\int_K f(z) dz = \sum_{z \in I} \int_{Q(z)} f(u) du + \sum_{z \in B \cup O} \int_{Q(z) \cap K} f(u) du.$$

So the difference on the left hand side of (5.3) consists of two parts: the first is $\sum_{z \in I} \int_{Q(z)} |f(z) - f(u)| du \leq V|I| \leq V \text{Area } K$. The second is just

$$\sum_{z \in B} \left(f(z) - \int_{Q(z) \cap K} f(u) du \right) - \sum_{z \in O} \int_{Q(z) \cap K} f(u) du,$$

which in absolute value is at most $2M|B| + M|O| \leq 4ML$.

Finally we turn to (5.4). The same way as above one shows that

$$\sum_{z \in K \cap \mathbf{P}} f(z) = \sum_{d=1}^{L} d\mu(d) \sum_{z \in \frac{1}{d} K \cap \mathbf{Z^2}} f(z)$$

where we can include $z = 0$ in the summation since $f(0) = 0$. Next

$$\sum_{z \in \frac{1}{d} K \cap \mathbf{Z^2}} f(z) = \int_{\frac{1}{d} K} f(u) du + \text{ error term} = \frac{1}{d^3} \int_K f(u) du + \text{ error term}$$

where the error term is at most $V\frac{\text{Area}\,K}{d^2}+4\frac{M}{d}\frac{L}{d}$ from (5.3). So the left hand side of (5.4) is at most

$$\frac{1}{L}\int_K |f(u)|du+\sum_1^L|\frac{\mu(d)}{d}|(V\text{Area}\,K+4ML)$$
$$\le\frac{1}{L}M\text{Area}\,K+(1+\log L)(V\text{Area}\,K+4ML)\le(2V\text{Area}\,K+5ML)\log L,$$

using the isoperimetric inequality again.

13 Appendix 3

We start the proof of Lemma 8.1 with the following Claim.

Claim 13.1 *Suppose a,b,c,d are vertices of K (in anticlockwise order), $[a,b]$ and $[c,d]$ are edges of K, and $\angle b0c<3\delta$. Let x be the intersection point of the lines through a,b and c,d, and let y be the intersection point of the lines through $0,x$ and a,c. Then $|x-y|\le 4\delta(R/r)^2|y|$.*

Proof The condition $rB\subset K\subset RB$ implies that $\beta=\angle 0xb=\angle 0ba-\angle xba>\arcsin r/R-3\delta$ since

$$\sin\angle 0ba=\frac{d(0,\ell_{a,b})}{|b|}$$

($\ell_{a,b}$ being the line through a and b) $|b|<R$, $d(0,\ell_{a,b})>r$ by assumption, so that $\sin\angle 0ba>r/R$, see Fig. 3.

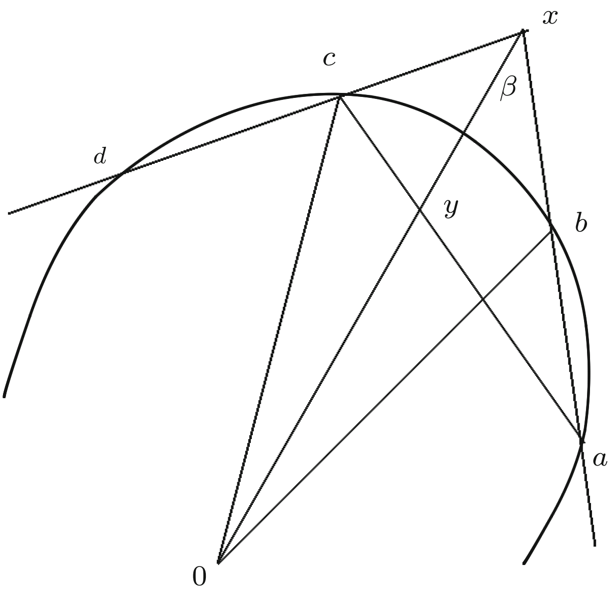

Fig. 3 The proof of Claim 13.1

Further $\angle xyc = \angle 0xa - \angle x0b > \beta$. The sine theorem in the triangle x, y, c shows that

$$\frac{|x-y|}{|x-c|} = \frac{\sin \angle cxy}{\sin \angle cyx},$$

and similarly, the sine theorem in the triangle $x, 0, c$ shows that

$$\frac{|x-c|}{|x|} = \frac{\sin \angle c0x}{\sin \angle 0cx}.$$

Multiplying them gives

$$\frac{|x-y|}{|x|} = \frac{\sin \angle cxy \sin \angle c0x}{\sin \angle cyx \sin \angle 0cx} < \frac{\sin 3\delta}{(r/R)\sin\beta}.$$

Next, since $|y| = |x| - |x-y|$, we have

$$\frac{|x|}{|x| - |x-y|} = \frac{1}{1 - \frac{|x-y|}{|x|}} < \frac{1}{1 - \frac{\sin 3\delta}{(r/R)\sin\beta}}$$

We use this inequality next in the form

$$\frac{|x-y|}{|y|} < \frac{\sin 3\delta}{(r/R)\sin\beta} \cdot \frac{|x|}{|x| - |x-y|} < \frac{\sin 3\delta}{(r/R)\sin\beta - \sin 3\delta} < 4\delta\left(\frac{R}{r}\right)^2,$$

where we only have to check the validity of the last inequality. This is a matter of direct computation using that $\sin\beta > \sin(\arcsin(r/R) - 3\delta) > (r/R)\cos 3\delta - \sin 3\delta$ and the assumption that $\delta < 0.02(r/R)^2$ implying, in particular, that $\delta < 0.02$. What is to be checked now is that

$$\tan 3\delta \left[1 + 4\delta\left(\frac{R}{r}\right)^2\left(\frac{r}{R} + 1\right)\right] \le 4\delta.$$

Here $\delta(R/r)^2 < 0.02$ and so the expression in the square bracket is at most 1.16 and the inequality follows. We omit the details. □

The **Proof** of Lemma 8.1 is an algorithm that constructs the vertex set V of Q. We start with $V = \emptyset$. We call the edge $[a, b]$ of K *special* if $\angle a0b \ge \delta$. Let W be a cone with apex at 0 and angle δ. It follows that if W is disjoint from all special edges, then it contains a vertex of K.

Case 1 Let $[a_1, b_1], [a_2, b_2], \dots, [a_k, b_k]$ be consecutive special edges in anticlockwise order so that $\angle b_i 0 a_{i+1} < 3\delta$ for all $i = 1, \dots, k-1$ (or up to k if $\angle b_k 0 a_1 < 3\delta$). We call this a maximal chain of consecutive special edges if there is no special edge $[a, b]$ with $\angle b0a_1 < 3\delta$ or $\angle b_k 0 a < 3\delta$.

For such a maximal chain we put the vertices $a_1, \dots, a_k, b_k$ (or $a_1, \dots, a_k$ if $\angle b_k 0 a_1 < 3\delta$) into V, and we do so for all such maximal chains.

Case 2 Let $[a_1, b_1]$ and $[a_2, b_2]$ be consecutive special edges with vertices a_1, b_1, a_2, b_2 in anticlockwise order so that $\gamma := \angle b_1 0 a_2 \ge 3\delta$. Then we choose $\delta' \in [\delta, 3\delta]$ so that γ/δ' is an odd integer, say $2h + 1$. This is always possible since there is an odd integer between $\gamma/(3\delta)$ and γ/δ because their difference is $\gamma/\delta - \gamma/(3\delta) = 2\gamma/(3\delta) \ge 2$.

Subdivide now the cone pos $\{b_1, a_2\}$ into $2h + 1$ subcones, each of angle δ' and pick a vertex $u_1, \dots, u_h$ from every second subcone. Finally, put $b_1, u_1, \dots, u_h, a_2$ into V.

If there are only two special edges $[a_1, b_1]$ and $[a_2, b_2]$, then one has to do the same construction between edges $[a_2, b_2]$ and $[a_1, b_1]$ as well. If there is only one special edge, then the construction is carried out from b_1 to a_1 as if one had two special edges $[a_1, b_1]$ and $[b_1, a_1]$.

Finally, if there are no special edges, then we chose a $\delta' \in [\delta, 2\delta]$ so that $2\pi/\delta'$ is an even integer, $2h$, say. This is evidently possible. Subdivide the plane into cones of angle δ' (with apex at 0) and choose a vertex $u_1, \dots, u_h$ from every second cone, and set $V = \{u_1, \dots, u_h\}$.

The algorithm is finished. By construction $\angle v_i 0 v_{i+1} \geq \delta$. We now check the condition $K \subset (1 + 4\delta(R/r)^2)Q$. Let $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ be four consecutive vertices of Q in anticlockwise order. Rename these points as a, b, c, d as in the Claim. Then $K \cap \text{pos}\,(b, c) \setminus Q$ is contained in the triangle b, c, x from the Claim. Now $y \in Q$ because y lies on the segment $[a, c]$, and so $x \in (1 + 4\delta(R/r)^2)Q$ according to the Claim. So the triangle b, c, x is contained $(1 + 4\delta(R/r)^2)Q$. □

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