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INFINITE PATHS WITH NO SMALL ANGLES

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Abstract. It is shown here that given a discrete (and infinite) set of points in the plane, it is possible to arrange them on a polygonal path so that every angle on the polygonal path is at least 9°. This has been known to hold for finite sets (with 20°). The main result holds for discrete sets in higher dimensions as well, with a smaller bound on the angle.

§1. *Introduction and the main result.* A set $X \subset \mathbb{R}^2$ is *discrete* by definition if every disk contains only finitely many elements of X. Of course, X is finite or countable. An ordering of the points of X is either $x_1, x_2, \ldots$ (a one-way infinite sequence) or $\ldots, x_{-1}, x_0, x_1, x_2, \ldots$ (a two-way infinite sequence) or $x_1, x_2, \ldots, x_n$ (when X is finite). Such an ordering is identified with a polygonal path P on X: its edges are the segments connecting x_i to x_{i+1}. The angle of P at x_i is just $\angle x_{i-1}x_ix_{i+1}$. The path is called α-good if all of its angles are at least α where $\alpha > 0$. In answer to a question of Fekete [3] from 1992 (see also [4]) and of Dumitrescu [2] from 2005, we proved in [1] the following result.

THEOREM 1. *If X is a finite set in the plane, then there is an α-good path on X with $\alpha = \pi/9$.*

The aim of this paper is to extend the above result of [1] to infinite, discrete sets $X \subset \mathbb{R}^2$. The condition of discreteness is quite natural. For instance, when X is the set of rational points on the x-axis, the ordering is either increasing or decreasing but it is unclear how to define angles along this path. Even worse, it is equally unclear what the definition of a path or an angle could be when X is the image of the rational points on the Peano curve. The following is our main result.

THEOREM 2. *Assume that $0 < \alpha < \pi/18$ and that X is a discrete set in the plane. Then there exists an α-good path on the points of X.*

Here one cannot guarantee that the path is one- or two-way infinite. The example showing this is when X is the set of positive integer points, and integer points, respectively, on the x-axis. The next example is interesting as it highlights the difficulties of finding an α-good path. Let $q \in \mathbb{R}$ be large and define $x_n = (q^{3n+1}, 0) \in \mathbb{R}^2$, $y_n = (0, q^{3n+2}) \in \mathbb{R}^2$, $z_n = (-q^{3n+3}, 0) \in \mathbb{R}^2$, and $X = \bigcup_0^\infty \{x_n, y_n, z_n\}$. Every pair of points in X determines a segment that is either

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almost vertical or almost horizontal. In view of Theorem 1 there is a good path on every finite subset of X. But how to extend such a path to an infinite one? What is an α-good path on X? How many α-good paths are there on X?

We observe that Kynčl [5] has recently improved the bound in Theorem 1 from $\pi/9$ to $\pi/6$, which is actually the best possible value of α. The details are not yet available, but most likely his result combined with our proof would imply that Theorem 2 holds for every $\alpha < \pi/12$.

§2. *Auxiliary lemmas.* The proof of Theorem 2 consists of several steps. We now introduce some notation and terminology and state the two main lemmas needed for the proof. For a point $z \in \mathbb{R}^2$, $|z|$ denotes its distance from the origin and $\overline{z}$ denotes the unit vector $z/|z|$ (assuming that $z \neq 0$). So $\overline{z} \in S^1$ where S^1 is the unit circle, so it can be thought of as a direction or angle. It will be convenient to use the notation $\overline{z} \in I$, meaning that I is an arc on S^1. Such an arc is just $I = (\beta, \gamma)$ where β, γ are angles and (β, γ) means the anticlockwise arc from β to γ. Given distinct points $u, v \in \mathbb{R}^2$ we let $\overline{uv}$ denote the unit vector $(v-u)/|v-u|$.

From now on we assume that $X \subset \mathbb{R}^2$ is infinite and discrete and $\alpha \in (0, \pi/18)$. We assume, without loss of generality, that the origin, to be denoted by 0, is not contained in X and also that $|x|$ and $|y|$ are different for each pair $x, y \in X$, $x \neq y$.

Fix $\beta \in (0, \pi/18)$ and define K to be the cone consisting of vectors z with $\overline{z} \in [-\beta, \beta]$. As usual, let $-K$ be the reflection of K with respect to 0 and $K^* = K \cup (-K)$ be the corresponding double cone. Here is the cone lemma, an auxiliary result needed for Theorem 2.

LEMMA 1. *If $X \backslash K^*$ is finite, then there is an α-good path on X.*

The same conclusion holds, of course, if $X \backslash K_0^*$ is finite where K_0^* is a rotated copy (around the origin) of K^*. We now slightly reformulate the cone lemma. Let $\triangle = \triangle(X)$ denote the set of limit directions in X, that is, $z \in \triangle(X)$ if and only if there is a sequence of distinct elements $z_1, z_2, \ldots$ of X with $\lim \overline{z_n} = z$. Clearly $\triangle(X) \subset S^1$ is closed. When I is an arc on S^1 we define $I^* = I \cup (-I)$. Here is the cone lemma in a slightly different form, more suitable for our purposes.

LEMMA 2. *Assume that $\triangle(X) \subset I^*$ for some open arc $I \subset S^1$ of length $\pi/9$. Then there is an α-good path on X.*

It will suffice to prove Lemma 1 because of the following.

CLAIM 1. *Lemma 1 implies Lemma 2.*

Proof. Assume that the conditions of Lemma 2 hold. Since I is open and $\triangle(X)$ is closed, there is a closed arc $J \subset I$ with $\triangle(X) \subset J^*$. Let K_0 be the cone hull of J; then K_0 is a cone with half angle $\beta \in (0, \pi/18)$ and $X \backslash K_0^*$ is finite, so Lemma 1 applies. □

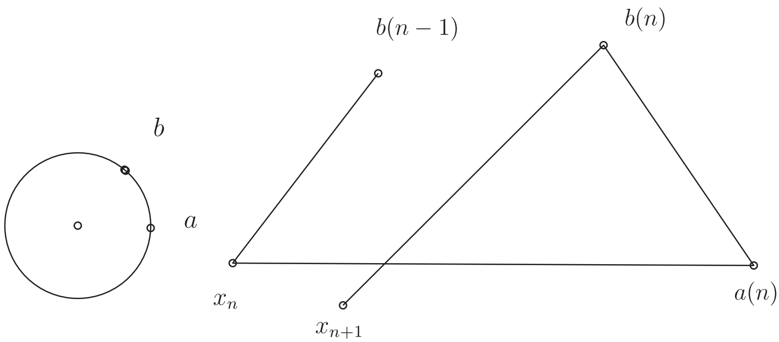

Figure 1: The construction of $a(n)$, $b(n)$, x_{n+1}.

Now we come to the second auxiliary lemma. A point $a \in X$ is called *sharp* if $\angle 0ab < \pi/18$ for every $b \in X$ with $|b| < |a|$. Set $\gamma = \pi/9$.

LEMMA 3. *Assume that all but finitely many elements of X are sharp. Then there is a γ-good path on X.*

For our purposes, an α-good path on X would do as well. But, as we will see later, the proof gives a γ-good path on X.

§3. *Proof of Theorem 2.* A pair $a, b \in X$ is said to be *fat* if all angles of the triangle $0ab$ are at least $\pi/18$. The proof of the following result is simple.

PROPOSITION 1. *If X contains infinitely many fat pairs, then there is an α-good path on X.*

Proof. We choose a sequence of (distinct) fat pairs, a_k, b_k, from X with $\lim \overline{a_k} = a \in S^1$ and $\lim \overline{b_k} = b \in S^1$. This is clearly possible, and $\angle a0b \geq \pi/18$. Also, $\angle a0b \leq \pi - 2\pi/18$ since the angles at a and b of the triangle $a0b$ are at least $\pi/18$.

We will construct an α-good path P on X of the form

$$x_1, a(1), b(1), x_2, a(2), b(2), x_3, \ldots$$

satisfying the condition

$$\text{for every } n, \text{ each } x \in X \text{ with } |x| < |x_n| \text{ appears before } x_n \text{ on } P. \qquad (1)$$

Here $a(n), b(n)$ are fat pairs from the sequence a_k, b_k. The construction is quite straightforward (see Figure 1). Evidently, x_1 is the shortest element of X. Assume that $P_n = x_1, a(1), b(1), x_2, \ldots, b(n-1), x_n$ has been constructed satisfying condition (1) and, further, that $\overline{x_n b(n)} \approx b$, meaning that $\overline{x_n b(n)}$ and b are less than $(\pi/18 - \alpha)/2$ apart on S^1. Clearly x_{n+1} has to be the shortest vector in X missing from P_n. Choose $a(n), b(n)$ from the sequence of fat pairs so far from x_n and x_{n+1} that $\overline{x_n a(n)} \approx a$ and $\overline{x_{n+1} b(n)} \approx b$ (with the same meaning of $\approx$ as before). It is not hard to see now that $P_{n+1} = P_n, a(n), b(n), x_{n+1}$ is an α-good path. □

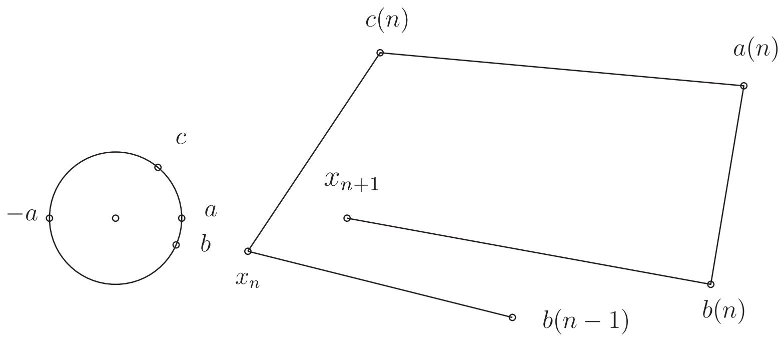

Figure 2: The construction of $a(n)$, $b(n)$, $c(n)$, x_{n+1}.

Next we call a pair $a, b \in X$ *balanced* if $\angle 0ab \geq \pi/18$, $\angle 0ba \geq \pi/18$ and $\angle a0b < \pi/18$.

PROPOSITION 2. *If X contains infinitely many balanced pairs, then there is an α-good path on X.*

Proof. We again choose a sequence of (distinct) balanced pairs a_k, b_k from X with $\lim \overline{a_k} = a \in S^1$ and $\lim \overline{b_k} = b \in S^1$. This is clearly possible, $\angle a0b \leq \pi/18$, and $a, b \in \triangle(X)$.

For $z \in S^1$, let I_z be the open arc of S^1 of length $\pi/9$, centered at z.

Assume that there exists $c \in \triangle(X)$ with $c \notin I_{-a} \cup I_b$. Let $c_k \in X$ be a sequence with $\overline{c_k} \to c$ and $|c_k| \to \infty$. We will construct an α-good path P on X of the form

$$x_1, c(1), a(1), b(1), x_2, c(2), a(2), b(2), x_3, \ldots$$

satisfying condition (1) where $a(n), b(n)$ are pairs from the sequence a_k, b_k and $c(n)$ is a subsequence of c_k (see Figure 2). The construction is similar to the previous one. We start with x_1, the shortest element in X. Assume that the path $P_n = x_1, c(1), a(1), b(1), \ldots, b(n-1), x_n$ has been constructed and satisfies condition (1), and, further, that $\overline{x(n)b(n-1)} \approx b$. Again, x_{n+1} has to be the shortest vector in X missing from P_n. Choose $c(n)$ so that $\overline{x_n c(n)} \approx c$ and then the pair $a(n), b(n)$ so far away from $c(n)$ and x_{n+1} that $\overline{a(n)c(n)} \approx -a$ and $\overline{x_{n+1}b(n)} \approx b$. It is clear that $P_{n+1} = P_n, c(n), a(n), b(n), x_{n+1}$ is an α-good path.

The same argument works, exchanging the roles of a and b, when there is $c \in \triangle(X)$ with $c \notin I_{-b} \cup I_a$. Thus we can assume that there is no $c \in \triangle(X)$ with $c \notin I_{-a} \cup I_b$ or $c \notin I_{-b} \cup I_a$. This means that, with $I = I_a \cap I_b$, $\triangle(X) \subset I^*$. Now the cone lemma (Lemma 2) can be applied since I is an open interval of length at most $\pi/9$. $\square$

Thus we are left with the case when there are only finitely many fat pairs and finitely many balanced pairs in X. Choose r so large that all fat and balanced pairs in X are inside D_r, the disk of radius r centered at 0. We claim then that every $x \in X \setminus D_r$ is sharp.

Indeed, consider $x \in X \backslash D_r$ and assume that $z \in X$ with $|z| < |x|$. Then, of course, $\angle 0zx > \angle 0xz$. If $\angle 0xz \geq \pi/18$, then the pair x, z is either fat (since $\angle x0z \geq \pi/18$) or balanced (if $\angle x0z < \pi/18$). But both cases are excluded as x is outside D_r. Thus $\angle 0xz < \pi/18$ and x is sharp.

A direct application of Lemma 3 completes the proof of the theorem. □

§4. *Proof of the cone lemma.* We need a stronger version of Theorem 1 which is proved in [**1**]. To state it we require two additional definitions.

Given a path $z_1, z_2, \ldots, z_n$ the directions $\overline{z_2 z_1}$ and $\overline{z_{n-1} z_n}$ are called the *end directions* of the path. We call a subset R of S^1 a *restriction* if it is the disjoint union of two closed arcs $R_1, R_2 \subset S^1$ such that both have length 4γ and their distance from each other (along the unit circle) is larger than 2γ. (Recall that $\gamma = \pi/9$.) We call the path $z_1, \ldots, z_n$ *R-avoiding* if the two end directions are not in the same R_i ($i = 1, 2$) and the path is γ-good.

THEOREM 3. *Let X be a finite set of points in the plane. For every restriction R there is an R-avoiding path on all the points of X.*

We now begin the proof of the cone lemma. Call a pair $a, b \in X$ *steep* if the angle between the x-axis and the line through a and b is at least 2γ.

If there is no steep pair in X, then ordering the points of X by increasing first component gives an α-good path on X, even with $\alpha = 5\pi/9$.

We let C be the cone consisting of all $z \in \mathbb{R}^2$ with $\overline{z} \in [-\pi/18, \pi/18]$, and set $C^* = C \cup (-C)$. Since $\beta < \pi/18$, the cone K lies in the interior of the cone C. One more piece of notation: z^1 denotes the first coordinate of $z \in \mathbb{R}^2$.

Assume next that there are only finitely many steep pairs in X. For $t_i > 0$ define the strip $T_i = \{x \in \mathbb{R}^2 : |x^1| \leq t_i\}$. Choose t_1 so large that T_1 contains all steep pairs and the set $X \backslash K^*$ as well. Next choose $t_2 \in \mathbb{R}$ so large that $X \backslash T_2 \subset x + C^*$ for every $x \in X \cap T_1$. Such a t_2 exists because $K \subset C$.

Set $R_1 = [-2\gamma, 2\gamma]$, $R_2 = [\pi - 2\gamma, \pi + 2\gamma]$. Then $R = R_1 \cup R_2 \subset S^1$ is a restriction, so by Theorem 3, there is an R-avoiding path, $P = x_1, x_2, \ldots, x_n$ on $X \cap T_2$ (even with $\alpha = \pi/9$). One end direction of P is not in R_1 and the other one is not in R_2. For the sake of simplicity assume that $\overline{x_2 x_1} \notin R_1$ and $\overline{x_{n-1} x_n} \notin R_2$.

Let $x_{n+1}, x_{n+2}, \ldots$ and $x_0, x_{-1}, x_{-2}, \ldots$ respectively be the points of $(X \backslash T_2) \cap K$ in increasing order and the points of $(X \backslash T_2) \cap (-K)$ in decreasing order.

CLAIM 2. *The path $\ldots, x_{-1}, x_0, x_1, \ldots, x_n, x_{n+1}, \ldots$ is α-good on X.*

Proof. We only have to check $\angle x_{n-1} x_n x_{n+1} \geq \alpha$ and $\angle x_0 x_1 x_2 \geq \alpha$. By symmetry it suffices to check the latter. Either x_1, x_2 is a steep pair (see Figure 3), in which case $x_1, x_2 \in T_1$ and $\angle x_0 x_1 x_2 \geq 2\gamma - \pi/18 = \pi/6 > \alpha$ because $x_0^1 < 0$, $x_1 \in X \cap T_1$, and $x_0 \in X \backslash T_2 \subset x_1 - C$ imply $\overline{x_0 x_1} \in -C$; or x_1, x_2 is not a steep pair, in which case $\overline{x_2 x_1} \in R_2$ because $\overline{x_2 x_1} \notin R_1$. But then $\angle x_0 x_1 x_2 \geq \pi - \gamma - \pi/18 = 13\pi/18 > \alpha$. □

So we are left with the case when there are infinitely many steep pairs. We first construct an α-good path on X under the extra condition that $X \subset K$, and explain how to extend the argument for the general case later.

Let Π be the set of steep pairs in X. We will use them to create U-turns on the α-good path to be constructed.

We recursively define numbers $t_0 = 0 < t_1 < t_2 < \cdots$ and pairs $\{a_i, b_i\}$ and $\{c_i, d_i\}$ in Π (all of them distinct points of X) satisfying conditions (Ai) and (Ci) below. We set $T_i = \{z \in \mathbb{R}^2 \mid 0 \leq z^1 \leq t_i\}$. The conditions are as follows.

(Ai) $\{a_i, b_i\} \in \Pi,\ a_i, b_i \in T_i \setminus T_{i-1}$ and $X \setminus T_i \subset (a_i + C) \cap (b_i + C)$.
(Ci) $\{c_i, d_i\} \in \Pi,\ c_i, d_i \in T_{i+1} \setminus T_i$ and $X \cap T_i \subset (c_i - C) \cap (d_i - C)$.

The recursive definition starts with choosing a steep pair a_1, b_1 and then t_1 so large that (A1) is satisfied. This is possible since the angle of K is smaller than that of C. Assume that $t_i, a_i, b_i, c_{i-1}, d_{i-1}$ have been defined for all $i = 1, 2, \ldots, k$ (except c_{-1}, d_{-1}, which are not needed) and satisfy all conditions. Then we choose a steep pair, c_k, d_k, outside T_k satisfying (Ck). Next we choose another steep pair a_{k+1}, b_{k+1} outside T_k (both distinct from c_k, d_k). Finally, we fix t_{k+1} so large that condition (A$k+1$) holds. This is clearly possible.

The construction of the α-good path is now easy (see Figure 4). We add a dummy point $a_0 = (-1, 0)$. Theorem 1 guarantees the existence of a γ-good path Q_i on the finite set

$$((X \cap (T_i \setminus T_{i-1})) \setminus \{a_i, b_i, c_{i-1}, d_{i-1}\}) \cup \{a_{i-1}, c_i\}.$$

Neither a_{i-1} nor c_i is an interior point of Q_i because of condition (A$i-1$) and (Ci). In the case of a_0 this follows from $X \subset K$. Thus a_{i-1} and c_i are the endpoints of Q_i with the end direction at a_{i-1} in C and at c_i in $-C$. It follows now that the path $Q_1, d_1, b_1, Q_2, d_2, b_2, Q_3, d_3, \ldots$ is α-good on $X \cup \{a_0\}$. Deleting the dummy vertex from it gives an α-good path on X. This completes the proof when $X \subset K$.

In the general case we proceed as follows. If there are infinitely many steep pairs both in K and $-K$, then we choose a steep pair $a_1, b_1 \in K$ and another $a_{-1}, b_{-1} \in -K$ and then fix t_1 so large that $T_1 = \{z \in \mathbb{R}^2 : |z^1| \leq t_1\}$ contains all $Z \setminus K^*$ and, further, the conditions

$$(X \setminus T_1) \cap K \subset (a_1 + C) \cap (b_1 + C)$$
$$(X \setminus T_1) \cap (-K) \subset (a_{-1} - C) \cap (b_{-1} - C)$$

are satisfied. This is clearly possible. We then proceed the same way as before, but moving in two directions.

If, finally, there are infinitely many steep pairs in K yet only finitely many in $-K$, then an obvious combination of the previous methods produces an α-good path on X. The details are straightforward and therefore omitted. □

Remark. The bound $\alpha < \pi/18$ comes from this part of the proof. Namely, Theorem 1 gives the γ-good path Q_i; its endpoints are forced to be a_{i-1} and c_i only when the angle of K is less than $\pi/9$.

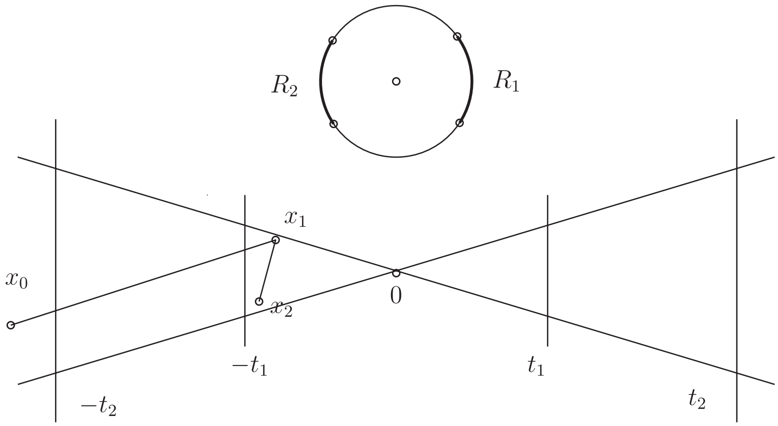

Figure 3: The angle $\angle x_0 x_1 x_2$.

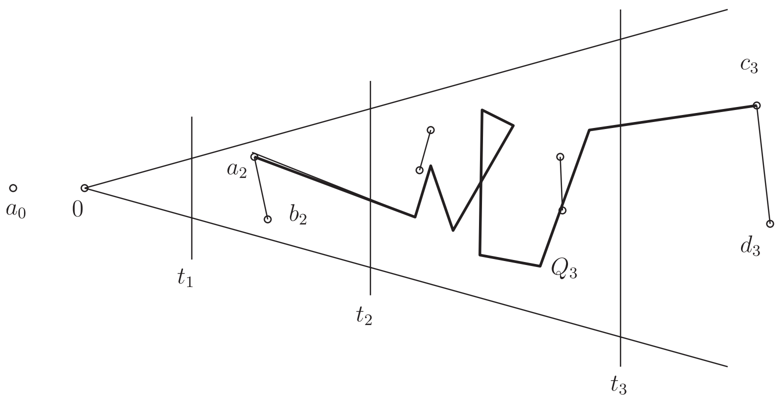

Figure 4: The construction of Q_3.

§5. *Proof of Lemma 3.* Let $r > 0$ be so large that $X_0 = X \cap D_r$ contains all non-sharp points of X. Order the elements of $X \setminus X_0$ by increasing distance from the origin, so if $x_1, x_2, \ldots$ is this order, then $|x_n| < |x_{n+1}|$ for all $n \in \mathbb{N}$. Set, further, $X(n) = X_0 \cup \{x_1, \ldots, x_n\}$ and fix a γ-good path, P_n, on $X(n)$.

CLAIM 3. *For every $n \in \mathbb{N}$, x_n is an endpoint of every γ-good path on $X(n)$.*

Proof. Assume to the contrary that x_n is an interior point of such a path. Then the two neighbors of x_n (a, b, say) are in $X(n)$ and $\angle 0x_n a$ and $\angle 0x_n b$ are both smaller than $\pi/18$ and therefore $\angle a x_n b < \pi/9$, which is a contradiction. □

For every $1 \leq n \leq k$ we define, by backward induction on n, a γ-good path $P_k[n]$ on $X(n)$ as follows. Set $P_k[k] = P_k$. If $P_k[n]$ has been defined and $n > 1$, then, by the previous claim, x_n is an end vertex of $P_k[n]$. Delete this end vertex from $P_k[n]$ to get $P_k[n-1]$.

Let L be an infinite subset of the natural numbers and $n \in \mathbb{N}$. There exists an infinite subset L' of L such that for every $j, k \in L'$ the two paths $P_j[n]$ and $P_k[n]$ are the same. Indeed, partition L by paths on $X(n)$, that is, for every $k \in L$, $k \geq n$, the element k is put into the class $P_k[n]$. Since there are finitely many paths on $X(n)$, one of the classes L' is infinite.

Next we define, by induction, a chain of infinite sets $L_1 \supset L_2 \supset L_3 \supset \cdots$ with the property that, for every $j, k \in L_n$, the two paths $P_j[n]$ and $P_k[n]$ are the same. Start with $L_0 = \mathbb{N}$. Let $n \geq 1$ and assume that $L = L_{n-1}$ has been defined. The previous argument gives a suitable infinite $L' \subset L$, and we set $L_n = L'$. The sets form an infinite chain $L_0 = \mathbb{N}, L_1, L_2, \ldots$ with each L_n infinite and containing L_{n+1}, and, further, for $i, j \in L_n$ with $n \geq 1$, the condition $P_j[n] = P_k[n]$ is satisfied.

For $n \in \mathbb{N}$ let Q_n be the path $P_k[n]$ for some $k \in L_n$. For $n < m$, Q_n is a subpath of Q_m by construction. Define the infinite path Q as the union of the paths Q_n. The path Q is an infinite γ-good path on X. □

Remark. In the example of §1 there are neither fat nor balanced pairs, and the conditions of Lemma 2 do not hold. So in our proof, the α-good path on X is found via the above procedure. The argument in Claim 3 can be used to show that all α-good paths on X are of the following form. The order of the x_ns and z_ns is $\ldots, z_2, z_1, x_1, x_2, x_3, \ldots$ and y_1 is either between x_1 and x_2 or between x_1 and z_1 and, for $n \geq 2$, y_n is either between x_n and x_{n+1} or between z_{n-1} and z_n. It is easy to see that each such path is indeed α-good.

§6. *Higher dimensions.* In the paper [**1**] we proved the higher-dimension analogue of Theorem 1 in the following form.

THEOREM 4. *For every $d \geq 2$ there is a positive α_d such that for every finite set of points $X \subset \mathbb{R}^d$ there exists an α_d-good path on X.*

The actual value of α_d is $\pi/42$ (for $d > 2$); see [**1**]. The proof of Theorem 2 goes through in higher dimensions without any real difficulty, and gives the following result.

THEOREM 5. *For every $d \geq 2$ for every discrete set of points $X \subset \mathbb{R}^d$ and every $\alpha \in (0, \pi/84)$ there exists a α-good path on X.*

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References

1. I. Bárány, A. Pór and P. Valtr, Paths without small angles. *SIAM J. Discrete Math.* (2009), to appear.
2. A. Dumitrescu, 2005, personal communication.
3. S. Fekete, Geometry and the traveling salesman problem, *PhD Thesis*, Department of Combinatorics and Optimization, The University of Waterloo, 1992.
4. S. Fekete and G. J. Woeginger, Angle-restricted tours in the plane. *Comput. Geom.* **8** (1997), 195–218.
5. J. Kynčl, 2009, personal communication.

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