Discrete Comput Geom 38:177–187 (2007)
DOI: 10.1007/s00454-007-1332-9

Discrete & Computational
Geometry

Packing Cones and Their Negatives in Space*

Imre Bárány[1] and Jiří Matoušek[2]

[1]Rényi Institute of Mathematics, Hungarian Academy of Sciences,
PO Box 127, 1364 Budapest, Hungary
barany@math-inst.hu
and
Department of Mathematics, University College London,
Gower Street, London WC1E 6BT, England

[2]Department of Applied Mathematics and
Institute of Theoretical Computer Science (ITI), Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. Let C be a cone in $\mathbf{R}^3$ whose base B is a planar convex body in a horizontal plane π and whose tip is a point $v \notin \pi$. Let $\mathcal{C}$ be a packing formed by translates of C and $-C$ in $\mathbf{R}^3$. We exhibit an explicit constant $c > 0$ such that the density of any such $\mathcal{C}$ is smaller than $1 - c$, answering a question of Wlodek Kuperberg.

1. Introduction and Main Result

Let C be a cone, over a planar convex set B, in $\mathbf{R}^3$ and let $\mathcal{C}$ be a packing consisting of translates of C and $-C$ (no rotations allowed). Kuperberg [5] proved several years ago that the density $\delta(\mathcal{C})$ of $\mathcal{C}$ is less than 1 (for the reader's convenience, we outline a short proof at the end of this section). This immediately implies the existence of a constant $c > 0$ such that $\delta(\mathcal{C}) \leq 1 - c$ for every C and every packing $\mathcal{C}$ of translates of C and $-C$. Indeed, if $\sup_{C,\mathcal{C}} \delta(\mathcal{C}) = 1$, then one can choose a convergent subsequence of the cones such that the limiting cone tiles the space. However, then the density of the corresponding packing is 1.

This argument cannot give any explicit value for c. That is why Kuperberg [5] raised the following problem: Find an explicit constant $c > 0$ such that for every cone C, every packing by translates of C and $-C$ has density less than $1 - c$. The aim of this paper is to give such an explicit constant.

* Imre Bárány was supported by Hungarian National Foundation Grants T 046246 and T 037846.

Here a cone C is simply the convex hull of the base B and the tip v, where B is a convex compact set of nonzero area lying in a two-dimensional plane π, and $v \notin \pi$ is a point in $\mathbf{R}^3$.

Theorem 1.1. *There is an explicit constant $c > 0$ such that for every cone $C \subset \mathbf{R}^3$, every packing by translates of C and $-C$ has density smaller than $1 - c$.*

Remarks. Our proof actually works for larger class of packings, with the same constant c. Namely, let $\mathcal{F}$ be the family of all cones in $\mathbf{R}^3$ with tips at $(0, 0, 1)$ or at $(0, 0, -1)$ and with bases B in the plane $z = 0$ such that B contains the horizontal unit disk centered at the origin and is contained in the concentric disk of radius 2. Let $\mathcal{C}$ be a packing of translates of cones in $\mathcal{F}$. Then the density of $\mathcal{C}$ is at most $1 - c$. For this remark we are indebted to Wlodek Kuperberg.

Our method gives an extremely small value for c. (We have not tried to optimize the constants in the proofs.) It is very easy to see that if the base B tiles the plane, then there exists a packing $\mathcal{C}$ by translates of C and $-C$ whose density is $\frac{2}{3}$. The best construction we know of is more than 100 years old and is due to Minkowski [6]. It is a lattice packing by translates of an octahedron with density $\frac{18}{19}$, showing that the constant in Theorem 1.1 satisfies $c \le \frac{1}{19}$. Betke and Henk [1] proved that no lattice packing of octahedra can have a larger density.

Sketch of a Proof of $\delta(\mathcal{C}) < 1$. We assume that $\mathcal{C}$ is a packing of translates of C and $-C$, and we show that $\delta(\mathcal{C}) < 1$, the result of Kuperberg. This, of course, is weaker than Theorem 1.1, but the proof is simple.

For contradiction we assume $\delta(\mathcal{C}) = 1$ and let $\mathcal{C}_n$ be a packing by translates of C and $-C$ such that $\delta(\mathcal{C}_n)$ tends to 1. Let Q be a large cube. Then there are translated copies Q_n of Q such that, as n goes to infinity,

$$\sum_{C^* \in \mathcal{C}_n} \mathrm{Vol}(C^* \cap Q_n) \to \mathrm{Vol}\, Q.$$

Translate Q_n to Q together with the $C^* \in \mathcal{C}_n$ that intersect Q_n. We get finite packings by translates of C and $-C$ that cover Q almost perfectly. One can choose a convergent subsequence of these packings, and the limiting packing $\mathcal{C}^*$ tiles Q. Then C is a polytope. Let C^* be a cone in $\mathcal{C}^*$ which is close to the center of Q, and let T be a triangular facet of C^* adjacent to the tip. Every point p in the relative interior of T is covered (besides C^*) by another cone $C(p) \in \mathcal{C}^*$. Further, C^* and $C(p)$ are separated by the plane $\mathrm{aff}(T)$. Now if $C(p)$ is a translate of C, then $C(p) \cap \mathrm{aff}(T)$ is a vertex or an edge of $C(p)$. This implies that the translates of C in $\mathcal{C}^*$ can only cover a small portion (of measure zero) of T. For the rest of the points $p \in T$, $C(p)$ is a translate of $-C$. Consequently, p is covered by $-T$. However, that is impossible: a triangle T cannot be covered by internally disjoint translates of $-T$. □

Remark. There are several beautiful open questions about the density of packings of cones in $\mathbf{R}^d$, $d \ge 3$, some of them are very natural and look hard. We refer the interested reader to the forthcoming paper by Bezdek and Kuperberg [3] with the hope that it will be written up and published soon. Some information on these problems can also be found in [2].

2. Preparations

In this section we introduce notation and terminology, and state auxiliary lemmas needed in the proof.

We assume that the base B of the cone C lies in the horizontal plane π containing the origin 0. For a real number x we let $\pi(x)$ be the plane parallel to π at distance $|x|$ from $\pi = \pi(0)$, where $\pi(x)$ lies *below* π for $x > 0$ and *above* π for $x < 0$. This is opposite(!) to the usual convention for the position of the coordinate system, but we find our "reverse" convention more convenient in this paper.

Let $D \subset \pi$ be the unit disk centered at the origin. Since our problem is invariant under nondegenerate linear transformations, we can assume that B is sandwiched between $\frac{1}{2}D$ and D, that is, $\frac{1}{2}D \subset B \subset D$ (by Löwner's theorem [4]). Similarly, we may assume that the tip v of C is above the origin and at distance 1 from it (so it lies in $\pi(-1)$). The sandwiching easily implies the following two facts, whose elementary proofs are omitted.

Fact 2.1. *For every point p on the boundary of $B \subset \pi$, the angle between π and the line connecting p and v is between 45° and 60°.*

Fact 2.2. *For every p on the boundary of $B \subset \pi$, there is a wedge K in the plane π with apex at p and of angle 60° such that $K \cap (p + \frac{1}{2}D)$ is contained in B; see Fig. 1.*

Let C^* be a translated copy of C. We write $v(C^*)$ for its tip, $B(C^*)$ for its base, and we let $a(C^*)$ be the vertical coordinate of the base; that is, $B(C^*)$ lies in the plane $\pi(a(C^*))$. So $a(C^*) \in [0, 1]$ if and only if C^* intersects π. For a translate C_i of C we simply write a_i, B_i, v_i instead of $a(C_i)$, $B(C_i)$, $v(C_i)$.

We write $\mathrm{dist}(S_1, S_2)$ for the Euclidean distance between sets $S_1, S_2 \subset \mathbf{R}^3$. Of course, the distance between S_1 and S_2 is the infimum of $\mathrm{dist}(x, y)$ with the infimum taken over all $x \in S_1$ and $y \in S_2$.

We need three simple lemmas for the proof of the main theorem.

Lemma 2.3 (Avoidance Lemma). *Let C_1 and C_2 be disjoint translates of C, both intersecting π, and let $0 \le a_2 \le a_1 \le 1$; see Fig. 2. Then*

$$\mathrm{dist}(\pi \cap C_1, \pi \cap C_2) \ge a_2.$$

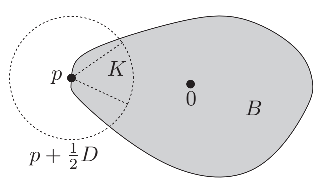

Fig. 1. Illustration to Fact 2.2.

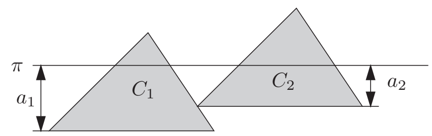

Fig. 2. The avoidance lemma.

For the next lemma and for the rest of the paper we set $r_0 = \frac{1}{12}$.

Lemma 2.4 (Local Boundedness Lemma). *Let rD be the disk in π of radius r centered at 0, where $r \in (0, r_0]$, and let $\mathcal{C}^+$ be a packing of translates of C (no $-C$ allowed here). Then rD intersects at most one cone from $\mathcal{C}^+$ with $a(C^*) \geq 2r$ and at most six cones with $a(C^*) < 2r$.*

For $0 \leq h_2 < h_1$ we let $\mathrm{Cyl}(r, h_1, h_2)$ denote the vertical cylinder of radius r with axis passing through 0 and bounded from above by the plane $\pi(h_2)$ and from below by $\pi(h_1)$. Let C_0 denote the translate of C whose tip is at the origin.

Lemma 2.5 (Special Cylinder Lemma). *Let α be a sufficiently small positive real number, and let $\beta \in (0, \alpha)$. For every $R \in (0, r_0]$, and for every packing $\mathcal{C}^+$ of translates of C with $C_0 \in \mathcal{C}^+$ there is an r with*

$$\left(\frac{2\beta}{2+\alpha}\right)^6 R \leq r \leq R$$

such that C_0 is the only cone of $\mathcal{C}^+$ intersecting the interior of $\mathrm{Cyl}(r, \alpha r, \beta r)$; *see Fig.* 3.

We use these lemmas in the proof of the main theorem. Their proofs are given in Section 5.

3. One More Lemma and Proof of the Main Theorem

We assume that $r \in (0, r_0]$, $\alpha > 0$, and $\beta \in (0, \alpha/2]$ have been fixed. Let Z be the cylinder $\mathrm{Cyl}(r, \alpha r, \beta r)$ and let T be its axis, that is, the segment of the vertical line through 0 between the planes $\pi(\alpha r)$ and $\pi(\beta r)$. We also set $\gamma = \alpha - \beta$ and $\eta = \alpha^2$.

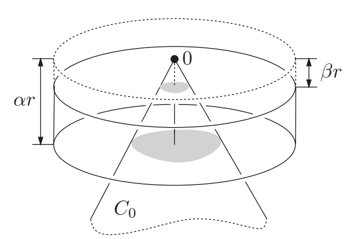

Fig. 3. Illustration to the special cylinder lemma.

Lemma 3.1 (Main Lemma). *Let $\mathcal{C}^+$ be a packing of translates of C in which each element is disjoint from T. Then*

$$\operatorname{Vol}\left(Z \setminus \bigcup \mathcal{C}^+\right) \geq \eta \operatorname{Vol} Z.$$

The proof is given in the next section. We are actually going to use the lemma for the translates of $-C$ in the given packing of C and $-C$.

Proof of Theorem 1.1. We specify the parameters now, but we work with their numerical values only later. So let $\alpha = 1/(384\pi)$, and $\beta = \gamma = \alpha/2$. We choose $R = r_0 = \frac{1}{12}$.

Lemma 2.5 applies to every positive cone $C_i \in \mathcal{C}$ and to the packing $\mathcal{C}^+$ consisting of the positive cones in $\mathcal{C}$. This gives, for every C_i, a cylinder $Z_i = \operatorname{Cyl}(r_i, \alpha r_i, \beta r_i) + v_i$ that is intersected only by C_i and possibly by some translates of $-C$ in $\mathcal{C}$, but by no cone in $\mathcal{C}^+ \setminus \{C_i\}$.

We also note that all r_i are larger than the fixed positive number

$$\left(\frac{2\beta}{2+\alpha}\right)^6 R = \left(\frac{\alpha}{2+\alpha}\right)^6 R,$$

and so

$$\operatorname{Vol} Z_i = \pi \gamma {r_i}^3 \geq \pi \gamma \left(\frac{\alpha}{2+\alpha}\right)^{18} R^3 =: c_0.$$

The negative cones in $\mathcal{C}$ are disjoint from the axis of Z_i because this axis is contained in C_i. Then the main lemma obviously can be applied to Z_i and to the packing $\mathcal{C}^-$ formed by the negative cones in $\mathcal{C}$. So the negative cones in $\mathcal{C}$ occupy at most a $1-\eta$ fraction of Z_i. The only positive cone intersecting Z_i is C_i, and $\operatorname{Vol}(Z_i \cap C_i) \leq (\alpha r_i)^3 \pi/3$. Thus $\mathcal{C}$ altogether misses at least

$$\eta \operatorname{Vol} Z_i - \frac{\pi}{3}(\alpha r_i)^3 = \left(\eta - \frac{\alpha^3}{3\gamma}\right) \operatorname{Vol} Z_i = \left(\eta - \frac{2\alpha^2}{3}\right) \operatorname{Vol} Z_i = \frac{\eta}{3} \operatorname{Vol} Z_i$$

of the volume of Z_i, since we have chosen $\eta = \alpha^2$.

Using the avoidance lemma (Lemma 2.3) it is easy to check that the cylinders Z_i are disjoint. Consequently, for each positive cone $C_i \in \mathcal{C}^+$, an $\eta/3$ fraction of the volume of the cylinder Z_i is left uncovered by $\mathcal{C}$.

The same applies to the negative cones in $\mathcal{C}$ as well. Now when computing the density of $\mathcal{C}$, we consider a large cube Q containing n cones from the packing, with at least half of them positive, say. If $n \operatorname{Vol} C < \frac{1}{2} \operatorname{Vol} Q$, then the density in Q is small, smaller than $\frac{2}{3}$ for $\operatorname{Vol} Q$ sufficiently large, since the cones from $\mathcal{C}$ that intersect Q but are not contained in Q can cover only a small portion of Q. So we now suppose that $n \operatorname{Vol} C \geq \frac{1}{2} \operatorname{Vol} Q$. Then in the cylinders Z_i corresponding to the positive cones from $\mathcal{C}$ contained in Q, a volume of at least

$$\frac{n}{2} \frac{\eta}{3} c_0$$

is uncovered by $\mathcal{C}$, while the volume of Q is at most $2n \operatorname{Vol} C \leq (2\pi/3)n$. This implies that $\mathcal{C}$ leaves an ε fraction of Q uncovered, where

$$\varepsilon = \frac{\eta c_0}{4\pi} = \frac{1}{8} \frac{\alpha^{21} R^3}{(2+2\alpha)^{18}} \approx 5.327 \cdot 10^{-75}. \qquad \square$$

Remark. By fine-tuning the parameters in this argument and in the proof of the main lemma it is possible to get $\varepsilon \approx 10^{-42}$. This is much larger than the ε above but still extremely small.

4. Proof of the Main Lemma

For simpler notation we translate the upper face of the considered cylinder to the plane $\pi(0)$. So here we assume that $Z = \operatorname{Cyl}(r, \gamma r, 0)$, $\gamma = \beta - \alpha$.

We argue by contradiction; so we assume that $\mathcal{C}^+$ is a packing of translates of C with $T \cap \bigcup \mathcal{C}^+ = \emptyset$ that misses less than an η fraction (of the volume) of Z. We suppose that all cones in $\mathcal{C}^+$ intersect Z.

We set $\rho = 2\sqrt{\eta} r$ and we let $V = \operatorname{Cyl}(\rho, \gamma r, \gamma r/2)$ be a smaller cylinder in the lower half of Z.

Claim 4.1. *There is a $C_1 \in \mathcal{C}^+$ intersecting V such that $a(C_1) < \gamma r + 2\rho$.*

Proof. By the choice of ρ, the cylinder V has volume $2\eta \operatorname{Vol} Z$, and so it is met by some element of $\mathcal{C}$, say by C_1. Since C_1 is disjoint from the axis T of the cylinder Z, there exists a halfspace H with T on its boundary and disjoint from C_1. Since $\operatorname{Vol}(H \cap V) \geq \eta \operatorname{Vol} Z$, there exists another $C_2 \in \mathcal{C}^+$ intersecting V.

For contradiction let us suppose that both $a(C_1) \geq \gamma r + 2\rho$ and $a(C_2) \geq \gamma r + 2\rho$. Then both C_1 and C_2 intersect $\pi(\gamma r)$ and both are at a distance of at most ρ from T, implying that

$$\operatorname{dist}(\pi(\gamma r) \cap C_1, \pi(\gamma r) \cap C_2) \leq 2\rho.$$

However, by the avoidance lemma (Lemma 2.3)

$$\operatorname{dist}(\pi(\gamma r) \cap C_1, \pi(\gamma r) \cap C_2) > 2\rho,$$

a contradiction. Thus we have $a(C_1) < \gamma r + 2\rho$ or $a(C_2) < \gamma r + 2\rho$, and at least one of the cones C_1 and C_2 satisfies the requirements of the claim. $\square$

Now let $C_1 \in \mathcal{C}^+$ be as in the claim, and let us put

$$a_1 = \min(\gamma r, a(C_1)).$$

Since C_1 intersects V, we have $a_1 \geq \gamma r/2$. Let $C_2, \ldots, C_m$ be the cones in $\mathcal{C}^+$ with $a_i = a(C_i) \leq a_1$, where the notation is chosen so that $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$.

We denote by $\tilde{C}$ a general element of our packing $\mathcal{C}^+$. For every $\tilde{C} \in \mathcal{C}^+$ different from $C_1, \ldots, C_m$ we have $a_1 < a(\tilde{C})$. For $x \in [0, \alpha r)$ we define

$$\tilde{C}(x) = \text{Area}(\tilde{C} \cap \pi(x) \cap Z).$$

The function $C_i(\cdot)\colon [0, \gamma r] \to \mathbf{R}$ is positive, continuous, and increasing on $[0, a_i]$, and is equal to zero on (a_i, a_1) for $i = 1, \ldots, m$. For $\tilde{C} \in \mathcal{C}^+$ different from these C_i, the function $C(\cdot)$ is nonnegative, continuous, and nondecreasing on $[0, a_1]$.

We denote by M the volume missed by $\mathcal{C}^+$ from Z. Set $a_{m+1} = 0$. Then

$$\begin{aligned} M &= \int_0^{\gamma r} \left(r^2\pi - \sum_{\tilde{C} \in \mathcal{C}^+} \tilde{C}(x) \right) dx \\ &\geq \int_0^{a_1} \left(r^2\pi - \sum_{\tilde{C} \in \mathcal{C}^+} \tilde{C}(x) \right) dx \\ &= \sum_{i=1}^{m} \int_{a_{i+1}}^{a_i} \left(r^2\pi - \sum_{\tilde{C} \in \mathcal{C}^+} \tilde{C}(x) \right) dx \\ &\geq \sum_{i=1}^{m} \int_{a_{i+1}}^{a_i} \left(\sum_{\tilde{C} \in \mathcal{C}^+} \tilde{C}(a_i) - \sum_{\tilde{C} \in \mathcal{C}^+} \tilde{C}(x) \right) dx \\ &\geq \sum_{i=1}^{m} \int_{a_{i+1}}^{a_i} (C_1(a_i) - C_1(x))\, dx. \end{aligned}$$

Here the last inequality holds since $\tilde{C}(a_i) \geq \tilde{C}(x)$ for $x \in [a_i, a_{i+1}]$, and hence we can restrict the summation to the single cone C_1. The previous inequality follows from $\sum_{\tilde{C} \in \mathcal{C}^+} C(x) \leq r^2\pi$, which holds since $\mathcal{C}^+$ is a packing. We need a simple claim, whose proof is postponed to the end of this section.

Claim 4.2. *For $0 \leq x \leq y \leq a_1$ we have*

$$C_1(y) - C_1(x) \geq \frac{r}{4}(y - x).$$

We continue the last inequality for the missed volume M:

$$\begin{aligned} M &\geq \sum_{i=1}^{m} \int_{a_{i+1}}^{a_i} (C_1(a_i) - C_1(x))\, dx \\ &\geq \sum_{i=1}^{m} \int_{a_{i+1}}^{a_i} \frac{r}{4}(a_i - x)\, dx = \frac{r}{8} \sum_{i=1}^{m} (a_i - a_{i+1})^2 \\ &\geq \frac{r}{8} \cdot \frac{\left(\sum_1^m (a_i - a_{i+1}) \right)^2}{m} = \frac{r}{8} \cdot \frac{(a_1)^2}{m} \geq \frac{\gamma^2 r^3}{32m}. \end{aligned}$$

By now we are almost finished with the proof. First, all C_i intersect the disk $rD \subset \pi$, and, for each $i = 1, \dots, m$,

$$a_i \le a_1 < \gamma r + 2\rho < \alpha r + 2\sqrt{\eta} r = 3\alpha r < 2r.$$

Thus by the last part of Lemma 2.4 we have $m \le 6$. Second, since $\operatorname{Vol} Z = \gamma r^3 \pi$, we have

$$M \ge \frac{\gamma^2 r^3}{32m} \ge \frac{\gamma}{192\pi} \operatorname{Vol} Z = \eta \operatorname{Vol} Z,$$

contrary to our assumption that $M < \eta \operatorname{Vol} Z$. □

Proof of Claim 4.2. We recall that C_1 is the cone in $\mathcal{C}^+$ intersecting the smaller cylinder V, avoiding the axis T (of V and Z), and satisfying $a(C_1) < \gamma r + 2\rho$. We write T_1 for the axis of C_1. For $0 \le x \le a(C_1)$, we let $p_0(x)$ denote the point in the slice $C_1 \cap \pi(x)$ nearest to T. Clearly, $p_0(x)$ is unique and lies on the boundary of $C_1 \cap \pi(x)$. We denote by $T(x)$ the point $T \cap \pi(x)$, and $T_1(x)$ is the point $T_1 \cap \pi(x)$.

It follows easily from $a(C_1) < \gamma r + 2\rho$ that T_1 is far from T: their distance is at least $\frac{1}{2} - \rho$.

Since C_1 intersects V, we have $|T(a_1) - p_0(a_1)| \le \rho$. The segment $[p_0(a_1), v_1]$, where v_1 is the tip of C_1, lies in C_1, and so, by Fact 2.1, the point $[p_0(a_1), v_1] \cap \pi(x) \in C_1$ is at distance at most $\rho + (a_1 - x)$ from $T(x)$. This implies that, for all $x \in [0, a_1]$,

$$|T(x) - p_0(x)| \le \rho + (a_1 - x) \le 2\sqrt{\eta} r + \gamma r < 3\alpha r.$$

Further, for all $x \in [0, a_1]$, $\pi(x) \cap \partial C_1$ is a closed convex curve in $\pi(x)$; see Fig. 4. The part of this closed convex curve that lies in Z consists of connected components; let $L(x)$ denote the component containing $p_0(x)$. Since T_1 is far from T, $C_1 \cap \pi(x)$ cannot lie completely in Z. Thus $L(x)$ is a convex curve with two distinct endpoints. Consequently, the length $\ell(x)$ of $L(x)$ satisfies

$$\ell(x) \ge 2(r - |T(x) - p(x)|) \ge 2(r - 3\alpha r) \ge \tfrac{3}{2} r.$$

Let p be an arbitrary point of the curve $L(x)$, and let q be the intersection point of $\pi(y)$ and the line through p and v_1. Further, let C^*, L^*, and p^* denote the orthogonal projection of $C_1 \cap \pi(x)$, $L(x)$, and p, respectively, onto $\pi(y)$; see Fig. 5. We have

$$C_1(y) - C_1(x) = \operatorname{Area}(((C_1 \cap \pi(y)) \setminus C^*) \cap Z).$$

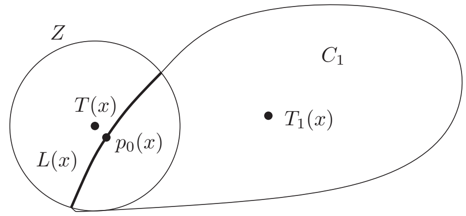

Fig. 4. Proof of Claim 4.2—the situation in the plane $\pi(x)$.

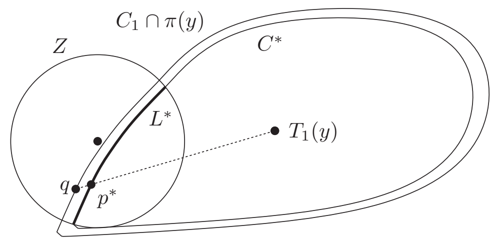

Fig. 5. Proof of Claim 4.2 continued.

Since C^* is a homothetic copy of $C_1 \cap \pi(y)$ with center of homothety $T_1(y)$, the points q, p^*, $T_1(y)$ are collinear. It follows from Fact 2.1 that $|q - p^*| \geq \frac{1}{2}(y - x)$.

Further, Fact 2.2 shows that the angle between the segment $[q, p^*]$ and the tangent line to L^* at p^* is at least 30°. Define

$$F = \bigcup_{p \in L(x)} [q, p^*].$$

It is now clear that

$$\text{Area}\, F \geq \tfrac{1}{2}(y - x)\ell(x) \sin 30^\circ = \tfrac{1}{4}(y - x)\ell(x).$$

It is not hard to see that F almost coincides with $((C_1 \cap \pi(x)) \backslash C^*) \cap Z$. More precisely, let L' be the set of those $p \in L(x)$ for which the segment $[q, p^*]$ is contained in Z. One can show readily that the length of L' is at least $\frac{2}{3}\ell(x)$; we omit the elementary details. Finally we have

$$\begin{aligned} C_1(y) - C_1(x) &= \text{Area}\big(((C_1 \cap \pi(x)) \backslash C^*) \cap Z\big) \\ &\geq \tfrac{2}{3}\text{Area}\, F \geq \tfrac{1}{6}(y - x)\ell(x) \\ &\geq \tfrac{1}{4}(y - x)r. \end{aligned}$$

□

5. Proof of the Auxiliary Lemmas

Proof of Lemma 2.3. The cone C_1 intersects the plane $\pi(a_2)$ and $C_1 \cap \pi(a_2)$ is a homothetic copy of the base B. This homothetic copy and $B(C_2)$ are disjoint and so they can be separated in $\pi(a_2)$ by a line ℓ. For $i = 1, 2$ let ℓ_i be the line that is the intersection of π with the affine hull of $v(C_i)$ and ℓ. The strip between ℓ_1 and ℓ_2 separates $\pi \cap C_1$ and $\pi \cap C_2$. Its width is at least a_2, as one can easily see using Fact 2.1. □

Proof of Lemma 2.4. We show first that there is at most one cone $C^* \in \mathcal{C}^+$ with $a(C^*) > 2r$. If there were two, C_1 and C_2, then

$$\text{dist}(C_1 \cap \pi, C_2 \cap \pi) \geq 2r$$

by the avoidance lemma (Lemma 2.3). However, since rD lies in the $2r$-neighborhood (in π) of $C_1 \cap \pi$, C_2 cannot intersect rD, a contradiction.

Next, let $C_1, \dots, C_m \in \mathcal{C}^+$ be the cones intersecting rD. We are done if $m \le 1$. For $m \ge 2$ we may assume $a(C_i) \le 2r$ for all $i \ge 2$. For each $i = 2, \dots, m$ there is a point $p_i \in rD \cap \partial C_i$. Since for $i \ge 2$, $\pi \cap C_i$ is a copy of the base B scaled by a factor between $1 - 2r$ and 1, Fact 2.2 implies the existence of a planar wedge $K_i \subset \pi$, with apex at p_i and angle 60°, such that $G_i = (p_i + (\frac{1}{2} - r)D) \cap K_i$ lies completely in C_i.

An elementary computation (using $r \le \frac{1}{12}$) shows that G_i intersects the boundary of the disk $\frac{1}{2}D$ in an arc longer than 0.15π. (We omit the details of this argument.) Since these arcs are disjoint, there are at most $\pi/0.15\pi = 6.66\dots$ of them. Thus $m \le 7$ follows. □

Proof of Lemma 2.5. Let $C_0, C_1, \dots, C_m \in \mathcal{C}$ be the cones intersecting the cylinder $\mathrm{Cyl}(R, \alpha R, 0)$ with $a_1 \ge a_2 \ge \cdots \ge a_m$.

We show first that $a_i < 2R$ for every i. This is satisfied if $a_i \le \alpha R$, so suppose $a_i > \alpha R$. In this case $C_i \cap \pi(\alpha R)$ intersects the cylinder $\mathrm{Cyl}(R, \alpha R, 0)$ so the distance between $C_i \cap \pi(\alpha R)$ and $C_0 \cap \pi(\alpha R)$ is at most $R - (\alpha/2)R$. The avoidance lemma applied to C_i and C_0 in the plane $\pi(\alpha R)$ shows that

$$\mathrm{dist}(C_i \cap \pi(\alpha R), C_0 \cap \pi(\alpha R)) \ge a_i - \alpha R.$$

So we have $a_i \le R + (\alpha/2)R < 2R$.

With $a_i < 2R$ proved, Lemma 2.4 applies and shows that $m \le 6$.

Next we want to define r whose existence is stated in the lemma. If $a_1 \le \beta R$, then $r = R$ will clearly do. So we suppose $a_1 > \beta R$.

We call an index $j \in \{1, \dots, m-1\}$ a *big drop* if

$$a_{j+1} \le \frac{2\beta}{2+\alpha}\, a_j.$$

First we assume that there is a big drop, and let j be the first big drop (that is, no $i < j$ is a big drop). Then, for all $i < j$,

$$a_{i+1} > \frac{2\beta}{2+\alpha}\, a_i, \qquad \text{implying} \quad a_j > \left(\frac{2\beta}{2+\alpha}\right)^{j-1} a_1 > \left(\frac{2\beta}{2+\alpha}\right)^{j-1} \beta R.$$

In this case $r = 2a_j/(2+\alpha)$ will do. Indeed, for $i > j$ we have $a_i \le a_{j+1} \le \beta r$, and thus C_i lies completely above the considered cylinder $\mathrm{Cyl}(r, \alpha r, \beta r)$. For $i \le j$, the avoidance lemma (applied in $\pi(\alpha r)$) and Fact 2.1 show that $C_i \cap \pi(\alpha r)$ is at least at a distance of

$$(a_i - \alpha r) + \frac{\alpha}{2} r \ge a_j - \frac{\alpha}{2} r = r$$

from the axis of C_0. This implies that C_i does not intersect the interior of $\mathrm{Cyl}(r, \alpha r, 0)$. Also,

$$r > \left(\frac{2\beta}{2+\alpha}\right)^{j} R \ge \left(\frac{2\beta}{2+\alpha}\right)^{5} R$$

since $j \le m - 1 \le 5$.

Next, we assume that there is no big drop. Then $r = 2a_m/(2+\alpha)$ will do. Indeed, in this case C_i is disjoint from the interior of $\mathrm{Cyl}(r, \alpha r, 0)$ for each $i \geq 1$. This can be checked the same way as in the previous paragraph. Finally,

$$r > \left(\frac{2\beta}{2+\alpha}\right)^m R \geq \left(\frac{2\beta}{2+\alpha}\right)^6 R. \qquad \square$$

Acknowledgments

For their hospitality and support we thank CNRS, and the universities of Jussieu and Marne-la-Vallée where most of the research reported here took place. We are also grateful to W. Kuperberg for useful and inspiring comments and for careful reading of the manuscript.

References

1. U. Betke, M. Henk, Densest lattice packings of 3-polytopes, *Comput. Geom.* **16** (2000), 157–186.
2. A. Bezdek, On the density of packings of congruent bodies, in: F. Glatz ed., *Lectures at the Hungarian Academy of Sciences*, MTA Press, Budapest, 1998, pp. 117–126 (in Hungarian).
3. A. Bezdek, W. Kuperberg, Packing space with convex cones, Manuscript, 2006.
4. L. Danzer, B. Grunbaum, V. Klee, Helly's theorem and its relatives, in: V. Klee ed., *Convexity*, Proc. Symp. Pure Math., Vol. VII, AMS, Providence, RI, 1963, pp. 101–108.
5. W. Kuperberg, private communication (2001).
6. H. Minkowski, Dichteste gitterformige Lagerung kongruenter Körper, *Nachr. K. Ges. Wiss. Göttingen, Math.-Phys. KL* (1904), 311–355, also in: *Gesammelte Abhandlungen*, vol. II, pp. 3–42, Leipzig, 1911.

Received February 27, 2006, *and in revised form May* 6, 2006. *Online publication July* 13, 2007.