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CENTRAL LIMIT THEOREMS FOR GAUSSIAN POLYTOPES

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Choose n random, independent points in $\mathbf{R}^d$ according to the standard normal distribution. Their convex hull K_n is the *Gaussian random polytope*. We prove that the volume and the number of faces of K_n satisfy the central limit theorem, settling a well-known conjecture in the field.

1. The main result. Let $\Psi_d = \Psi$ denote the standard normal distribution on $\mathbf{R}^d$, its density function is

$$\psi_d = \psi = \frac{1}{(2\pi)^{d/2}} \exp\left\{-\frac{x^2}{2}\right\},$$

where $x^2 = |x|^2$ is the square of the Euclidean norm of $x \in \mathbf{R}^d$. We will use this notation only for $d \geq 2$, for $d = 1$ the standard normal has density function

$$\phi = \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{x^2}{2}\right\}$$

with distribution Φ.

Fix $d \geq 2$ and choose a set $X_n = \{x_1, \dots, x_n\}$ of random independent points from $\mathbf{R}^d$ according to the normal distribution Ψ. The convex hull of these points, $K_n = \operatorname{Conv}(x_1, \dots, x_n)$, is the *Gaussian random polytope* or *Gaussian polytope* for short. This is one of the central models in the theory of random polytopes, initiated by Rényi and Sulanke in the 1960s. The main goal of this theory is to investigate the distributions of the key functionals (such as the volume) of random polytopes.

A cornerstone in probability theory is the central limit theorem. A sequence X_n of random variables satisfies the central limit theorem if for every t

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_n - \mathbf{E}X_n}{\sqrt{\operatorname{Var} X_n}} \leq t\right) - \Phi(t) = 0.$$

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It is a natural and important conjecture in the theory of random polytopes that the key functionals of K_n satisfy the central limit theorem, as n tends to infinity. This conjecture has been open for several decades, and very few partial results have been proved (see the next section).

In this paper, we develop a general frame work which enables us to confirm this conjecture for many functionals. Due to the length of the proofs, we will focus on the volume and the number of faces, perhaps the two most interesting parameters. Some other functionals (such as the intrinsic volumes of the probability content) will be discussed in Section 14.

For a convex polytope K, we use $\mathrm{Vol}(K)$ and $f_s(K)$ to denote its volume and number of faces of dimension s, respectively. Here are our main results.

THEOREM 1.1. *Let d be a fixed integer at least 2. There is a function $\varepsilon(n)$ tending to 0 as n tends to infinity such that the following holds. For any value of t,*

$$\left|\mathbf{P}\left(\frac{\mathrm{Vol}(K_n) - \mathbf{E}\,\mathrm{Vol}(K_n)}{\sqrt{\mathrm{Var}\,\mathrm{Vol}(K_n)}} \leq t\right) - \Phi(t)\right| \leq \varepsilon(n). \tag{1}$$

THEOREM 1.2. *Let d be a fixed integer at least 2 and s be a nonnegative integer at most $d-1$. There is a function $\varepsilon(n)$ tending to 0 as n tends to infinity such that the following holds. For any value of t,*

$$\left|\mathbf{P}\left(\frac{f_s(K_n) - \mathbf{E} f_s(K_n)}{\sqrt{\mathrm{Var}\, f_s(K_n)}} \leq t\right) - \Phi(t)\right| \leq \varepsilon(n). \tag{2}$$

REMARK 1.3. In both theorems, we can take $\varepsilon(n) = (\log n)^{-(d-1)/4+o(1)}$. (See Remarks 4.2, 3.3 and 8.3.)

In the next section, we give a brief survey about the study of Gaussian polytopes and random polytopes in general.

NOTATION. In the whole paper, we assume that n is large, whenever needed. The asymptotic notation are used under the assumption that $n \to \infty$. Given non-negative functions $f(n)$ and $g(n)$, we write $f(n) = O(g(n))$ ($f(n) = \Omega(g(n))$) if there is a positive constant C, independent of n, such that $f(n) \leq Cg(n)$ ($f(n) \geq Cg(n)$) for all sufficiently large value of n. We write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. In this case, we say that $f(n)$ and $g(n)$ have the same order of magnitude. Finally $f(n) = o(g(n))$ if $f(n)/g(n)$ tends to zero as n tends to infinity.

Consider a (measurable) subset S of $\mathbf{R}^d$. The probability content of S is

$$\Psi(S) = \int_S \psi(x)\,dx.$$

P, **E**, Var denote probability, expectation, variance, respectively. Let t_i, $i = 1, \dots, n$, be independent random variables and $Y = Y(t_1, \dots, t_n)$ be a random variable depending on $t_1, \dots, t_n$. $\mathbf{E}(Y|t_1, \dots, t_i)$ is the conditional expectation of Y conditioned on the first i variables. $\mathbf{I}_E$ is the indicator of the event E: $\mathbf{I}_E = 1$ if E holds and 0 otherwise.

2. History. Gaussian random polytopes were first considered by Rényi and Sulanke in their classical paper [16]. Naturally, the existence of central limit theorems should be one of the very first questions to ask. However, early results are very far from a possible answer of this question, due to the lack of tools. These results mostly focused on expectations. In particular, Rényi and Sulanke determined the expectation of $f_1(K_n)$ for a Gaussian polytope in $\mathbf{R}^2$. (Here and later f_i denotes the number of faces of dimension i.) In 1970, Raynaud [14] computed $\mathbf{E} f_{d-1}(K_n)$ in all dimensions. The general formula is

$$\mathbf{E} f_s(K_n) = \frac{2^d}{\sqrt{d}} \binom{d}{s+1} \beta_{s,d-1} (\pi \log n)^{(d-1)/2} (1 + o(1)), \tag{3}$$

where $s \in \{0, 1, \dots, d-1\}$ and $d \geq 1$, as $n \to \infty$. Here $\beta_{s,d-1}$ is the internal angle of the regular $(d-1)$-simplex at one of its s-dimensional faces. The formula was proved by Affentranger and Schneider [2] and by Baryshnikov and Vitale [6]; simpler proofs can be found in [12]. Recently Hug and Reitzner [13] obtained an estimate for the variance

$$\operatorname{Var} f_s(K_n) = O\big((\log n)^{(d-1)/2}\big). \tag{4}$$

In [10, 11], Hueter stated a central limit theorem for $f_0(K_n)$, but the proof had a gap, namely, the claimed estimate on the variance was not correct.

As far as the volume is concerned, Affentranger [1] determined the expectation of $\operatorname{Vol}(K_n)$:

$$\mathbf{E} \operatorname{Vol}(K_n) = \kappa_d (2 \log n)^{d/2} (1 + o(1)). \tag{5}$$

Here κ_d denotes the volume of B^d, the d-dimensional unit ball. An upper bound for the variance of $\operatorname{Vol}(K_n)$ is given by Hug and Reitzner [13]:

$$\operatorname{Var} \operatorname{Vol}(K_n) = O\big((\log n)^{(d-3)/2}\big). \tag{6}$$

We are not aware of a central limit theorem for the volume, prior to this paper.

Another popular model of random polytopes is the so-called uniform model, defined as follows. Let K be a convex set in $\mathbf{R}^d$ of volume one. Select n random points in K with respect to the uniform distribution and define the random polytope as the convex hull of these points. Similar to the situation with the Gaussian model, there is a vast amount of literature focusing on the expectations of the key functions (see [21] for a survey). As far as central limit theorems are concerned, the case $d = 2$ has been studied by Groeneboom [8], Groeneboom and Cabo [7], and Hsing [9].

They proved central limit theorems for random polyogon in the square and the unit disk. But their methods do not extend to higher dimensions.

In 2004 and 2005 there were several notable developments on the uniform model, especially in the case when the mother body K has smooth boundary: Vu [19] proved that several key functionals have distributions with exponential tails. Next, Reitzner [15] established a central limit theorem for a Poisson variant of the model. Further, Vu [20], using the results of the above two papers and a coupling argument, proved several central limit theorems for the uniform model. The central limit theorem when K is a polytope was established by Bárány and Reitzner [5].

The framework we develop in this paper makes use of ideas from [15, 19, 20] and also from [5]. Moreover, due to the obvious differences between the uniform measure and the Gaussian one, we also need to introduce several new ideas to handle technical obstacles.

Let us conclude this section with a few basic facts about the normal distribution. Let r be a positive number at least one. Let $B(r)$ denote the ball of radius r centered at the origin and $\overline{B(r)}$ be its complement. The probability content of $\overline{B(r)}$ is

$$\Psi(\overline{B(r)}) = \Theta(e^{-r^2/2}r^{d-2}). \tag{7}$$

Let $H(r)$ be a half space at distance r from the origin [$H(r)$ is not unique, but it does not matter]. The probability content of $H(r)$ is

$$\Psi(H(r)) = \Theta(e^{-r^2/2}r^{-1}). \tag{8}$$

3. Two more models.

It is hard to prove the CLT for K_n directly. We are going to take a detour and prove the CLT for some more convenient models, namely K_n' and Π_n, and next prove that the distributions of $\mathrm{Vol}(K_n)$ and $\mathrm{Vol}(K_n')$ and $\mathrm{Vol}(\Pi_n)$ are approximately the same.

We define K_n' first. Let c_0 be a large constant compared to the dimension d ($c_0 = 100d$ will satisfy all purposes). Define $R > 0$ via

$$R^2 = 2\log n + \log(\log n)^{c_0}. \tag{9}$$

We will use this definition later as well, for the time being we only need the following consequence.

$$e^{-R^2/2}R^{d-2} = \Theta\left(\frac{(\log n)^{(d-2)/2}}{n(\log n)^{c_0/2}}\right) = \Theta\left(\frac{1}{n(\log n)^{C_0}}\right), \tag{10}$$

where $C_0 = \frac{c_0}{2} - \frac{d-2}{2}$. Notice that the left-hand side is (up to a constant factor) the probability content of the complement of $B(R)$, the ball of radius R centered at the origin, see (7). The probability that one of n random points falls outside $B(R)$ is at most

$$O\left(n \times \frac{1}{n(\log n)^{C_0}}\right) = O\left(\frac{1}{(\log n)^{C_0}}\right).$$

By setting c_0 (and so C_0) sufficiently large, this probability will be negligible. This allows us to replace the normal distribution Ψ by the truncated distribution Ψ', restricted to $B(R)$. Ψ' is defined so that for any region S in $B(R)$, the measure of S is $\Psi'(S) = \frac{\Psi(S)}{\Psi(B(R))}$. To be precise, the density function ψ' of Ψ' is defined as

$$\psi'(x) = \psi(x)\frac{\mathbf{I}_{x\in B(R)}}{\Psi(B(R))},$$

where $\mathbf{I}$ is the indicator variable.

Let K'_n be the convex hull of a set of n random points chosen independently in $B(R)$ with respect to Ψ'. The central limit theorem for the K'_n model says the following.

THEOREM 3.1. *Let d be a fixed integer at least 2. There is a function $\varepsilon(n)$ tending to zero as n tends to infinity such that*

$$\left|\mathbf{P}\left(\frac{\mathrm{Vol}(K'_n) - \mathbf{E}(\mathrm{Vol}(K'_n))}{\sqrt{\mathrm{Var}\,\mathrm{Vol}(K'_n)}} \le t\right) - \Phi(t)\right| \le \varepsilon(n)$$

holds for all t.

Again, it is hard to prove this theorem directly. That is why we need the second model, the Poisson polytope.

We consider a Poisson point process, $X(n)$, of intensity n and underlying distribution Ψ' where Ψ' is the truncated Gaussian, that is, the Gaussian restricted to $B(R)$. Let S be a measurable subset of $\mathbf{R}^d$. The intersection of $X(n)$ with S consists of random points $\{x_1, \dots, x_k\} = X(n) \cap S$ where the number, k, of random points is Poisson distributed with expectation $n\Psi'(S)$ and for fixed k, the points are distributed independently. The property that we need most is that if S_1 and S_2 are disjoint measurable sets, then the two point sets $\{x_1, \dots, x_{k_1}\} = X(n) \cap S_1$ and $\{y_1, \dots, y_{k_2}\} = X(n) \cap S_2$ are independent, k_1 and k_2 are independently Poisson distributed. The *Poisson polytope* is, by definition, the convex hull of $X(n)$.

Another, equivalent and useful, way to look at Π_n is the following. First choose a random number n' with respect to the Poisson distribution with mean n. Next, generate n' random, independent points $x_1, \dots, x_{n'}$ with respect to Ψ', the truncated normal distribution on $\mathbf{R}^d$. Then Π_n is the convex hull, $\mathrm{Conv}\{x_1, \dots, x_{n'}\}$, of the chosen points. It is well known that n' is very close to n with high probability:

$$\mathbf{P}(|n' - n| \ge A\sqrt{n \log n}) \le n^{-A/4},$$

for every constant $A \ge 10$ (the constants 4 and 10 are just convenient choices and play no important role). So a good approximation of the Poisson polytope Π_n is $K_{n'}$ with n' Poisson distributed. Clearly, n' is concentrated on the interval $I = [n - A\sqrt{n \log n}, n + A\sqrt{n \log n}]$ and negligible outside this interval. The central limit theorem for the Poisson model is as follows.

THEOREM 3.2. *Let d be a fixed integer at least 2. There is a function $\varepsilon(n)$ tending to 0 as n tends to infinity such that the following holds. For any value of t,*

$$\left|\mathbf{P}\left(\frac{|\operatorname{Vol}(\Pi_n) - \mathbf{E}\operatorname{Vol}(\Pi_n)|}{\sqrt{\operatorname{Var}\operatorname{Vol}(\Pi_n)}} \le t\right) - \Phi(t)\right| \le \varepsilon(n).$$

REMARK 3.3. In both theorems above one can take $\varepsilon(n) = (\log n)^{-(d-1)/4+o(1)}$. This error term will be the dominating one when we apply Lemma 4.1 from the next section.

4. The plan of the proof. From now on we focus on the volume, the proof for the number of faces is basically the same and will be discussed in Section 13.

The proof is long and consists of many steps. To help the reader grasp the main ideas quickly, we first lay out the plan of the proof. The leading idea is coupling. In fact, our proof will involve two different couplings. Both of them are based on a simple lemma.

LEMMA 4.1. *Let Y_n and Y_n' be two sequences of random variables with means μ_n and μ_n', variances σ_n^2 and ${\sigma'}_n^2$, respectively. Assume that there are functions $\varepsilon_1(n), \varepsilon_2(n), \varepsilon_3(n), \varepsilon_4(n)$, all tending to zero as n tends to infinity such that:*

- $|\mu_n' - \mu_n| \le \varepsilon_1(n)\sigma_n'$,
- $|\sigma'_n - \sigma_n| \le \varepsilon_2(n)\sigma'_n$,
- *for any t*, $|\mathbf{P}(Y_n' \ge t) - \mathbf{P}(Y_n \ge t)| \le \varepsilon_3(n)$,
- *for any t*,

$$\left|\mathbf{P}\left(\frac{Y_n' - \mu_n'}{\sigma_n'} \le t\right) - \Phi(t)\right| \le \varepsilon_4(n).$$

Then there is a positive constant C such that for any t,

$$\left|\mathbf{P}\left(\frac{Y_n - \mu_n}{\sigma_n} \le t\right) - \Phi(t)\right| \le C\sum_{i=1}^{4}\varepsilon_i(n).$$

Basically, this lemma asserts that if Y_n' satisfies the CLT (the fourth condition) and Y_n is sufficiently close to Y_n' in distribution (the first three conditions), then Y_n also satisfies the CLT. We defer the routine proof to the end of this section. The lemma has been used in an implicit form in [20] and in [15].

REMARK 4.2. We can rewrite the error term $C\sum_{i=1}^4 \varepsilon_i(n)$ as $C\max_{i=1}^4 \varepsilon_i(n)$ (the two C's can have different values). In applications of Lemma 4.1, $\varepsilon_4(n)$ will be the dominating term.

We now present the plan for the proof of Theorem 1.1, which consists of the following steps.

- *Step* 1 (Variance). In this step, we show that the exact order of magnitude of $\operatorname{Var}\operatorname{Vol}(K_n)$ is $(\log n)^{(d-3)/2}$. The upper bound was obtained in [13]. We will prove the matching lower bound. Section 6 is devoted to this step. The necessary geometric tools are developed in Section 5. The variance plays a significant role and we will use the estimate obtained in this step several times later on.
- *Step* 2 (The first coupling). In this step, we couple K_n and K_n' in order to show that they satisfy the first three conditions of Lemma 4.1. This will be done in Section 7. Thus, it remains to verify the fourth, and critical, condition that $\operatorname{Vol}(K_n')$ satisfies the CLT. This task will take time and effort. We mention that the second condition of Lemma 4.1, together with Step 1, imply that the order of magnitude of $\operatorname{Var}\operatorname{Vol}(K_n')$ is $(\log n)^{(d-3)/2}$.
- *Step* 3 (The second coupling). In this step which is in Section 8, we couple Π_n with K_n'. Technically speaking, we are going to verify the first three conditions of Lemma 4.1 with respect to $\operatorname{Vol}(\Pi_n)$ and $\operatorname{Vol}(K_n')$. After this, both Theorem 1.1 and Theorem 3.1 follow from Theorem 3.2, the CLT for the Poisson model. This step is close to the coupling argument used for the uniform model [20]. However, the analysis for the current case is simpler, as strong concentration results are not needed. Again, the results imply that the order of magnitude of $\operatorname{Var}\operatorname{Vol}(\Pi_n)$ is $(\log n)^{(d-3)/2)}$.
- *Step* 4 (Sandwiching). In this step, we define a radius $r < R$ but very close to R, and prove that K_n' contains the ball $B(r)$ with high probability, namely, with probability $1-(\log n)^{-C}$. (For this end r has to be chosen carefully, see Remark 9.4) By definition, K_n' is contained in $B(R)$. So with high probability, K_n' is sandwiched between two very close balls. We will also prove that the Poisson polytope has the same property, that is, $B(r) \subset \Pi_n \subset B(R)$ with high probability. This is the content of Section 9.

The main idea behind the proof of Theorem 3.2, following Reitzner [15], is as follows. It is well known that if $\xi_1, \dots, \xi_n$ are independent variables with bounded means and variances, then the distribution of the normalized version of the sum $\sum_{i=1}^n \xi_i$ is approximately Gaussian. We are going to use a strengthening of this result, originally due to Stein [18], which asserts that it suffices to assume that the ξ_i are weakly dependent. The quantitative and technical statement below is from Rinott [17], which is slightly stronger than an earlier one due to Baldi and Rinott [3].

THEOREM 4.3. *Assume G is a graph with vertex set $V(G)$ and edge set $E(G)$, $|V(G)| = m$, and maximal degree D. Assume ξ_v is a random variable satisfying $|\xi_v| \le M$ almost surely for each $v \in V(G)$. Assume further that if there is no edge between a vertex in $V_1 \subset V(G)$ and a vertex of $V_2 \subset V(G)$ where V_1 and V_2 are disjoint, then the random variables $\{\xi_v : v \in V_1\}$ and $\{\xi_v : v \in V_2\}$ are independent. Then, writing $\xi = \sum_{v \in V(G)} \xi_v$, we have*

$$\left|\mathbf{P}\left(\frac{\xi - \mathbf{E}\xi}{\sqrt{\operatorname{Var}\xi}} - \Phi(t)\right)\right| \le \frac{DM}{\sqrt{\operatorname{Var}\xi}}\left(\frac{1}{\sqrt{2\pi}} + 16\frac{\sqrt{mD}M}{\sqrt{\operatorname{Var}\xi}} + 10\frac{mDM^2}{\operatorname{Var}\xi}\right).$$

In order to apply this result we have to make some geometric preparations and define the *dependency graph*.

- *Step* 5 (The dependency graph). We subdivide the annulus $A(R,r) = B(R) \setminus B(r)$ into pairwise internally disjoint cells $W_1, \dots, W_m$. The cells are nice and wellbehaving, and they define the *dependency graph* G with vertex set $V(G) = \{1, \dots, m\}$ and the pair (i, j) forming an edge of G if W_i and W_j are far apart. (The actual definition is different, but this is the essence of it.) Note that the dependency graph is defined by geometric conditions. We will give an upper bound on the maximal degree of G, and on the volume of the cells. The details appear in Section 10. Note that randomness does not come up here but is present in the background.
- *Step* 6 (CLT for the Poisson model). In this step, we work with the Poisson model Π_n under condition B which says that $B(r) \subset \Pi_n$. The Baldi–Rinott theorem can be applied with $\xi_i = \mathrm{Vol}(\Pi_n \cap W_i)$ and dependency graph G. This is a technical step which is carried out in Section 11. It proves Theorem 3.2, the CLT for the Poisson model, but only under condition B. The role of the Poisson model is critical here, as it guarantees that ξ_i and ξ_j are independent whenever i and j are not adjacent in G.
- *Step* 7 (Removing condition B). This is a technical step which is another (this time simple), application of Lemma 4.1. It proves, finally, that $\mathrm{Vol}(\Pi_n)$ satisfies the CLT (Theorem 3.2) and so it finishes the proof of the main theorem.

The proof for Theorem 1.2 concerning the number of faces is similar and will be presented in Section 13. In the last Section 14, we discuss few other results which can be proved using the same method.

Let us now conclude this section with the proof of Lemma 4.1.

PROOF OF LEMMA 4.1. We have to show that for any x

$$\mathbf{P}\left(\frac{Y_n - \mu_n}{\sigma_n} \le x\right) = \Phi(x) + O\left(\sum_{i=1}^{4} \varepsilon_i(n)\right).$$

By the third condition of the lemma

$$\mathbf{P}\left(\frac{Y_n - \mu_n}{\sigma_n} \le x\right) = \mathbf{P}(Y_n \le \mu_n + x\sigma_n) = \mathbf{P}(Y'_n \le \mu_n + x\sigma_n) + O(\varepsilon_3(n)).$$

On the other hand,

$$\mathbf{P}(Y'_n \le \mu_n + x\sigma_n) = \mathbf{P}(Y'_n \le \mu'_n + x'\sigma'_n),$$

where $x' = \frac{\mu_n - \mu'_n}{\sigma'_n} + \frac{x\sigma_n}{\sigma'_n}$. The first two conditions of the lemma guarantee that x' is between the maximum and minimum of the four values $x(1 \pm \varepsilon_2(n)) \pm \varepsilon_1(n)$. Moreover, the fourth condition of the lemma yields

$$\mathbf{P}(Y'_n \le \mu'_n + x'\sigma'_n) = \Phi(x') + O(\varepsilon_4(n)).$$

Further,

$$\Phi(x') = \Phi(x) + (x' - x)\Phi'(x_0)$$

for some x_0 between x and x'. The difference $|x - x'|$ is at most $|x|\varepsilon_2(n) + \varepsilon_1(n)$. As $\Phi'(x)$ decays exponentially, it is easy to see that $|x|\Phi'(x_0) = O(1)$ and thus

$$\Phi(x') = \Phi(x) + O(\varepsilon_1(n) + \varepsilon_2(n)).$$

Putting everything together completes the proof:

$$\mathbf{P}\left(\frac{Y_n - \mu_n}{\sigma_n} \le x\right) = \Phi(x) + O(\varepsilon_1(n) + \varepsilon_2(n) + \varepsilon_3(n) + \varepsilon_4(n)). \qquad \square$$

5. A geometric construction. Here we give a geometric construction, á la Reitzner [15] and Bárány and Reitzner [5]. We use it in the next section for estimating $\operatorname{Var}\operatorname{Vol}(K_n)$ and $\operatorname{Var} f_s(K_n)$. A similar, if more subtle, construction will be needed for the dependency graph as well.

In the construction $b_1, b_2, \ldots$ are positive constants that depend on dimension only. Let $S(r)$ denote the sphere of radius r centered at the origin. We define

$$r^2 = 2\log n - \log\log n.$$

The choice of r is not arbitrary here: it ensures that $\Psi(\triangle_i) = \Theta(1/n)$ (see later). Next we choose a system of points $y_1, \ldots, y_m$ from the sphere $S(r)$ which is maximal with respect to the property that for distinct i and j

$$|y_i - y_j| \ge 2b_1.$$

Such a system can be found by an obvious greedy algorithm. The spherical caps on $S(r)$ with center at y_i and radius b_1 are pairwise disjoint, and the same spherical caps with radius $2b_1$ cover $S(r)$. This implies by volume comparison

CLAIM 5.1. *We have*

$$m = \Theta((\log n)^{(d-1)/2}).$$

Next, for each $i = 1, \ldots, m$ set

$$y_i^0 = \left(1 + \frac{1}{r^2}\right)y_i.$$

Thus $|y_i^0| = r + \frac{1}{r}$ and we have, for all $x \in \mathbf{R}^d$ with $r \le |x| \le r + \frac{1}{r}$ that

$$\psi(x) = \Theta\left(\frac{\sqrt{\log n}}{n}\right) \tag{11}$$

Next we let H_i denote the hyperplane with equation $z \cdot y_i = r^2$. For each $i = 1, \ldots, m$ we fix a regular $(d-1)$-dimensional simplex in H_i whose vertices $y_i^1, \ldots, y_i^d$ lie in the $(d-2)$-dimensional sphere

$$H_i \cap S(y_i, \sqrt{2}).$$

The center of this simplex is clearly y_i. The simplex $\triangle_i$ is now defined as the convex hull of the y_i^j, $j = 0, 1, \dots, d$.

CLAIM 5.2. *For all i*

$$\Psi(\triangle_i) = \Theta\left(\frac{1}{n}\right).$$

PROOF. It is clear that for $j = 1, \dots, d$

$$|y_i^j| = \sqrt{r^2 + 2} < r + \frac{1}{r} = |y_i^0|.$$

Then every $x \in \triangle_i$ satisfies $r \le |x| \le r + \frac{1}{r}$, and the claim follows from (11) as $\mathrm{Vol}\, \triangle_i = \Theta(\frac{1}{\sqrt{\log n}})$. □

As the final step of the construction, for $i = 1, \dots, m$, $j = 0, 1, \dots, d$, let $\triangle_i^j$ be a homothetic copy of $\triangle_i$ where the center of homothety is y_i^j and the factor of homothety is a small number $b_2 > 0$.

This is our geometric construction. Now we establish several properties of this construction.

CLAIM 5.3. *We have*

$$\Psi(\triangle_i^j) = \Theta\left(\frac{1}{n}\right).$$

PROOF. The density $\psi(x)$ satisfies (11) for all $x \in \triangle_i^j$. The claim follows as the volume of $\triangle_i^j$ is just b_2^d times that of $\triangle_i$. □

Assume now that z_j is an arbitrary point in $\triangle_i^j$, $j = 0, 1, \dots, d$. We define the cone C_i via

$$C_i = z_0 + \mathrm{pos}\{z_j - z_0 : j = 1, \dots, d\}.$$

The following lemma is crucial since it implies the independence structure of K_n needed when estimating the variance.

LEMMA 5.4. *For b_1 large enough and b_2 small enough the cone C_i contains all simplices $\triangle_k$ with $k \neq i$.*

PROOF. We have to check that the segment $[z_0, y_j^k]$ intersects $\mathrm{Conv}\{z_1, \dots, z_d\}$ whenever $j \neq i$ and $k \in \{0, 1, \dots, d\}$. This is the same as checking that the segment $[z_0, y_j^k]$ intersects $\mathrm{Conv}\{z_1', \dots, z_d'\}$ where $z_j' = \mathrm{aff}\{z_0, z_j\} \cap H_i$. If b_2 is small enough then the $(d-1)$-dimensional ball $B_i = H_i \cap B(y_i, \frac{\sqrt{2}}{2d})$ is contained in $\mathrm{Conv}\{z'_1, \dots, z'_d\}$. It is not hard to see that, for large enough b_1, the segment $[y_i^0, y_j^k]$ intersects $H_i \cap B(y_i, \frac{\sqrt{2}}{3d})$ which is a smaller shrunken copy of B_i. (Here again $j \neq i$ and $k \in \{0, 1, \dots, d\}$.) But z_0 is very close to y_i^0 if the factor of homothety, b_2 is very small, and then the segment $[z_0, y_j^k]$ intersects B_i. □

We need one more lemma for estimating the variance. Let H_i^j be the half space containing $\triangle_i^k$ for all $k = 1, \dots, d$ except $k = j$, not containing $\triangle_i^0$ and $\triangle_i^j$, and whose bounding hyperplane touches all $\triangle_i^k$ except $k = j$.

CLAIM 5.5. *If b_2 is small enough, then*

$$\Psi(H_i^j) = O(n^{-1}).$$

PROOF. Let H denote the hyperplane through the points y_i^k ($k = 0, 1, \dots, d$, $k \neq j$) for this proof. It is not hard to check that the distance of H from the origin is at least $r - \frac{d^2}{r}$. The bounding hyperplane of H_i^j tends to H as b_2 tends to zero. So for small enough b_2, the distance of H_i^j from the origin is at least $r - \frac{2d^2}{r}$. An application of (8) finishes the proof. □

6. The variance.

THEOREM 6.1. *We have* $\mathrm{Var}\,\mathrm{Vol}(K_n) = \Theta((\log n)^{(d-3)/2})$.

PROOF. The upper bound (6) has been proved by Hug and Reitzner [13]. So we need to give a lower bound on $\mathrm{Var}\,\mathrm{Vol}(K_n)$.

Let $X_n = \{x_1, \dots, x_n\}$ denote our random sample of n points. Denote by A_i the event that exactly one random point (out of the sample X_n) is contained in each simplex $\triangle_i^j$, $j = 0, 1, \dots, d$, and no further point of X_n is contained in $H_i^+ \cup \bigcup_{j=1}^d H_i^j$. Here H_i^+ is the half space not containing the origin whose bounding hyperplane is H_i. Since H_i^+ is farther from the origin than H_i^j ($j > 0$), Claim 5.5 implies $\Psi(H_i^+) = O(1/n)$.

LEMMA 6.2. *There is a positive constant b_3 such that, for every $i = 1, \dots, m$*

$$\mathbf{P}(A_i) \geq b_3.$$

PROOF. Assuming that A_i has occurred, let $x_j \in X_n$ denote the unique point of X_n in $\triangle_i^j$, $j = 0, 1, \dots, d$, and set $X = X_n \setminus \{x_0, \dots, x_d\}$. As $\Psi(\triangle_i^j) = \Omega(1/n)$

and $\Psi(H_i^j) = O(1/n)$ we have

$$\begin{aligned}\mathbf{P}(A_i) &= \binom{n}{d+1}\mathbf{P}(x_j \in \triangle_i^j, j = 0, \dots, d)\mathbf{P}\left(X \cap \left(H_i^+ \cup \bigcup_{k=1}^d H_i^k\right) = \varnothing\right)\\ &= \binom{n}{d+1}\prod_0^d \Psi(\triangle_i^j)\left(1 - \Psi\left(H_i^+ \cup \bigcup_{k=1}^d H_i^k\right)\right)^{n-d-1}\\ &\geq c_1 n^{d+1} \cdot \frac{1}{n^{d+1}}\left(1 - \frac{c}{n}\right)^{n-d-1} \geq b_3 > 0.\end{aligned}$$

Here c is $(d+1)$ times the implicit constant in Claim 5.5, and c_1 is another constant that depends on d only. $\square$

So we can bound the expected number of A_i from below:

$$\mathbf{E}\left(\sum_1^m \mathbf{I}_{A_i}\right) = \sum_1^m \mathbf{P}(A_i) = \Omega(m).$$

We start bounding $\operatorname{Var}\operatorname{Vol}(K_n)$ from below. Let $\mathcal{F}$ denote the position of all random points from X_n except those in $\triangle_i^0$ with $\mathbf{I}_{A_i} = 1, i = 1, \dots, m$. We decompose the variance under condition $\mathcal{F}$:

$$\begin{aligned}\operatorname{Var}\operatorname{Vol}(K_n) &= \mathbf{E}\operatorname{Var}(\operatorname{Vol}(K_n)|\mathcal{F}) + \operatorname{Var}\mathbf{E}(\operatorname{Vol}(K_n)|\mathcal{F})\\ &\geq \mathbf{E}\operatorname{Var}(\operatorname{Vol}(K_n)|\mathcal{F}).\end{aligned} \tag{12}$$

Suppose condition $\mathcal{F}$ holds and $\mathbf{I}_{A_i} = \mathbf{I}_{A_j} = 1$. Clearly, the unique $x_i \in \triangle_i^0$ and $x_j \in \triangle_j^0$ ($x_i, x_j \in X_n$) are vertices of K_n, and, because of Lemma 5.4, there is no edge between x_i and x_j. Then the change in K_n when x_i is moved is independent of the change when x_j is moved. This implies that the change in $\operatorname{Vol}(K_n)$ when x_i is moved is independent of the change when x_j is moved, showing that

$$\operatorname{Var}(\operatorname{Vol}(K_n)|\mathcal{F}) = \sum_{i:\mathbf{I}_{A_i}=1} \operatorname{Var}_{x_i}\operatorname{Vol}(K_n)$$

where the variance in the sum is taken when x_i is changing within $\triangle_i^0$.

We now evaluate this variance. Let $z_j \in X_n$ be the unique random point in $\triangle_i^j$ ($j = 1, \dots, d$). Denote the simplex $\operatorname{Conv}\{x_i, z_1, \dots, z_d\}$ by $\triangle$. The change in $\operatorname{Vol}(K_n)$ when x_i changes within $\triangle_i^0$ equals the change in $\operatorname{Vol}(\triangle)$ and

$$\operatorname{Var}_{x_i}\operatorname{Vol}(\triangle) = \mathbf{E}\big(\operatorname{Vol}(\triangle) - \mathbf{E}_{x_i}\operatorname{Vol}(\triangle)\big)^2.$$

The base of $\triangle$, $\operatorname{Conv}\{z_1, \dots, z_d\}$, is a fixed $(d-1)$-dimensional simplex, of constant $(d-1)$-dimensional volume. Its height varies nearly between $\frac{1}{r}(1 - b_2)$ and $\frac{1}{r}$, so the expectation $\mathbf{E}_{x_i}\operatorname{Vol}(\triangle)$ is about $\Theta(1/r)$. Moreover, the height of $\triangle$

changes on a small interval of length about b_2/r, so the volume is a linear (but not constant) function on a positive fraction of this interval. Consequently,

$$(\mathrm{Vol}(\triangle) - \mathbf{E}_{x_i} \mathrm{Vol}(\triangle))^2 = \Omega\left(\frac{1}{(\sqrt{\log n})^2}\right) = \Omega\left(\frac{1}{\log n}\right)$$

holds on a positive fraction of $\triangle_i^0$. This implies that

$$\mathrm{Var}_{x_i} \mathrm{Vol}(\triangle) = \Omega\left(\frac{1}{\log n}\right).$$

Putting this into formula (12) and using (6) completes the proof. $\square$

The same method, with the same notation, works for $\mathrm{Var}\, f_s(K_n)$, so we present it here.

THEOREM 6.3. *We have* $\mathrm{Var}\, f_s(K_n) = \Theta((\log n)^{(d-3)/2})$.

PROOF. The upper bound is again due to Hug and Reitzner [13].

The method for the lower bound is similar to the one in [15]. We assume $s \in \{0, 1, \dots, d-1\}$. Condition A_i is the same as in Lemma 6.2 except that we require exactly two points from X_n to be in $\triangle_i^0$. Also, we let $\mathcal{F}$ denote the position of all random points from X_n except those two in $\triangle_i^0$ with $\mathbf{I}_{A_i} = 1$, $i = 1, \dots, m$. Then Lemma 6.2 remains valid for the new A_i. We can decompose the variance under condition $\mathcal{F}$ the same way and we still get (12). An identical analysis applies and gives

$$\mathrm{Var}(f_s(K_n)|\mathcal{F}) \geq \sum_{i\ \mathbf{I}_{A_i}=1} \mathrm{Var}_{x_i,y_i} f_s(K_n)$$

where the variance in the sum is taken when x_i, y_i are changing within $\triangle_i^0$. Here x_i and y_i are the two points from X_n contained in $\triangle_i^0$. The proof of the following claim is simple and left as an exercise.

CLAIM 6.4. *We have*

$$\mathrm{Var}_{x_i,y_i} f_s(K_n) = \Theta(1).$$

This finishes the proof of Theorem 6.3. $\square$

7. The first coupling.

Here we show that the random variables $\mathrm{Vol}(K_n)$ and $\mathrm{Vol}(K_n')$ satisfy the first three conditions of Lemma 4.1.

LEMMA 7.1. *We have*

$$|\mathbf{E}\,\mathrm{Vol}(K_n') - \mathbf{E}\,\mathrm{Vol}(K_n)| \leq \sqrt{\mathrm{Var}\,\mathrm{Vol}(K_n)}(\log n)^{-C_0/2},$$

$$|\mathrm{Var}\,\mathrm{Vol}(K_n') - \mathrm{Var}\,\mathrm{Vol}(K_n)| \leq \mathrm{Var}\,\mathrm{Vol}(K_n)(\log n)^{-C_0/2}.$$

Furthermore, for all t,

$$\left|\mathbf{P}(\mathrm{Vol}(K'_n) \geq t) - \mathbf{P}(\mathrm{Vol}(K_n) \geq t)\right| \leq (\log n)^{-C_0/2}.$$

PROOF. Choose n points $t_1, \dots, t_n$ in $\mathbf{R}^d$ with respect to the normal distribution Ψ. Let A denote the event that all n points fall inside $B(R)$. [Recall that R is defined in (9).] For every nonnegative integer i, let B_i be the event that all n points fall inside $B(4^{i+1}R)$ but there is at least one point outside $B(4^i R)$. Trivially

$$\overline{A} = \bigcup_{i=0}^{\infty} B_i.$$

Let $Y = Y(t_1, \dots, t_n)$ be a nonnegative random variable depending on $t_1, \dots, t_n$. Now choose n points $t'_1, \dots, t'_n$ in $\mathbf{R}^d$ with respect to the truncated distribution Ψ' and define Y' accordingly. It is clear that

$$\mathbf{E}(Y|A) = \mathbf{E}(Y').$$

Let c be a nonnegative constant. We say that Y is *c-bounded* if $\mathbf{E}(Y|A) \leq \mathrm{Vol}(B(R))^c$ and $\mathbf{E}(Y|B_i) \leq \mathrm{Vol}(B(4^{i+1}R))^c$ for all $i \geq 0$.

LEMMA 7.2. *If Y is c-bounded then*

$$|\mathbf{E}(Y) - \mathbf{E}(Y')| = O(\mathbf{E}(Y)(\log n)^{-C_0+cd/2}).$$

PROOF. We start with the identity

$$\mathbf{E}(Y) = \mathbf{E}(Y|A)\mathbf{P}(A) + \mathbf{E}(Y|\overline{A})\mathbf{P}(\overline{A}).$$

Since $\mathbf{E}(Y|A) = \mathbf{E}(Y')$, the triangle inequality implies that

$$|\mathbf{E}(Y) - \mathbf{E}(Y')| \leq \mathbf{E}(Y')\mathbf{P}(\overline{A}) + \mathbf{E}(Y|\overline{A})\mathbf{P}(\overline{A}). \tag{13}$$

To estimate $\mathbf{E}(Y|\overline{A})$, observe that

$$\mathbf{E}(Y|\overline{A}) = \sum_{i=0}^{\infty} \mathbf{E}(Y|B_i\overline{A})\mathbf{P}(B_i|\overline{A}). \tag{14}$$

The (c-boundedness) assumption of the lemma implies

$$\begin{aligned}\mathbf{E}(Y|B_i\overline{A}) &= \mathbf{E}(Y|B_i) \leq \mathrm{Vol}(B(4^{i+1}R))^c \\ &= O\big(4^{cd(i+1)}R^{cd}\big) = O\big(4^{cd(i+1)}(\log n)^{cd/2}\big).\end{aligned}$$

Furthermore, as B_i implies $\overline{A}$,

$$\mathbf{P}(B_i|\overline{A}) = \frac{\mathbf{P}(B_i)}{\mathbf{P}(\overline{A})} = O((\log n)^{C_0}\mathbf{P}(B_i)).$$

On the other hand, $\mathbf{P}(B_i)$ is at most the probability that there is a point outside $B(4^i R)$. By the union bound and (7), this probability is

$$O(n\Psi(\overline{B(4^i R)})) = O(n\exp(-4^{2i}R^2/2)(4^i R)^{d-2}). \tag{15}$$

For $i = 0$, the right-hand side of (15) is $\Theta((\log n)^{C_0})$ by the definition of R. For $i \geq 1$, the right-hand side of (15) is at most n^{-2i}, as

$$\exp(-4^{2i}R^2/2) = n^{(-1+o(1))4^{2i}} \leq n^{-2i-1}.$$

This shows that

$$\sum_{i=0}^{\infty} \mathbf{E}(Y|B_i\overline{A})\mathbf{P}(B_i|\overline{A}) = O\left(\sum_{i=0}^{\infty} 4^{cd(i+1)}(\log n)^{cd/2}n^{-2i}\right) = O((\log n)^{cd/2}).$$

Therefore the right-hand side of (13) is at most

$$O((\log n)^{cd/2})\mathbf{P}(\overline{A}) = O((\log n)^{-C_0+cd/2}),$$

proving the lemma. $\square$

Let Y be the volume. It is clear that Y is 1-bounded. Applying Lemma 7.2, we have

$$|\mathbf{E}\,\mathrm{Vol}(K_n') - \mathbf{E}\,\mathrm{Vol}(K_n)| = O(\mathbf{E}\,\mathrm{Vol}(K_n)(\log n)^{-C_0+d/2}) = O((\log n)^{-C_0+d}),$$

since $\mathbf{E}\,\mathrm{Vol}(K_n) = \Theta((\log n)^{d/2})$. Moreover $\mathrm{Var}\,\mathrm{Vol}(K_n) = \Theta((\log n)^{(d-3)/2})$. By setting c_0 sufficiently large, it thus follows that

$$|\mathbf{E}\,\mathrm{Vol}(K_n') - \mathbf{E}\,\mathrm{Vol}(K_n)| = O\big(\sqrt{\mathrm{Var}\,\mathrm{Vol}(K_n)}(\log n)^{-C_0/2}\big).$$

We will use this estimate for proving the statement about the difference between the two variances. But first, let Y be the square of the volume. It is clear that Y is 2-bounded. Thus, Lemma 7.2 yields

$$\begin{aligned}&|\mathbf{E}\,\mathrm{Vol}(K_n')^2 - \mathbf{E}\,\mathrm{Vol}(K_n)^2| \\ &\quad = O(\mathbf{E}(\mathrm{Vol}(K_n))^2(\log n)^{-C_0+d}) = O((\log n)^{-C_0+3d}),\end{aligned}$$

since $\mathrm{Vol}(K_n)^2 = O((\log n)^{2d})$, which (by the definition of variance) implies,

$$\begin{aligned}&|\,\mathrm{Var}\,\mathrm{Vol}(K_n') - \mathrm{Var}\,\mathrm{Vol}(K_n)| \\ &\quad = O((\log n)^{-C_0+3d}) + |(\mathbf{E}\,\mathrm{Vol}(K_n'))^2 - (\mathbf{E}\,\mathrm{Vol}(K_n))^2|.\end{aligned}$$

On the other hand,

$$\begin{aligned}&|(\mathbf{E}\,\mathrm{Vol}(K_n'))^2 - (\mathbf{E}\,\mathrm{Vol}(K_n))^2| \\ &\quad = |\mathbf{E}\,\mathrm{Vol}(K_n') + \mathbf{E}\,\mathrm{Vol}(K_n)||\mathbf{E}\,\mathrm{Vol}(K_n') - \mathbf{E}\,\mathrm{Vol}(K_n)|,\end{aligned}$$

where $|\mathbf{E}\operatorname{Vol}(K_n') - \mathbf{E}\operatorname{Vol}(K_n)|$ is $O((\log n)^{-C_0+d})$ by the previous argument. Furthermore

$$|\mathbf{E}\operatorname{Vol}(K_n') + \mathbf{E}\operatorname{Vol}(K_n)| = O(\mathbf{E}\operatorname{Vol}(K_n)) = O((\log n)^{d/2}).$$

Putting everything together, we obtain

$$\begin{aligned}|\operatorname{Var}\operatorname{Vol}(K_n') - \operatorname{Var}\operatorname{Vol}(K_n)| &= O((\log n)^{-C_0+3d}) + O((\log n)^{-C_0+d}(\log n)^{d/2})\\ &= O((\log n)^{-C_0+3d}).\end{aligned}$$

Again, by setting c_0 large, we have

$$|\operatorname{Var}\operatorname{Vol}(K_n') - \operatorname{Var}\operatorname{Vol}(K_n)| = O(\operatorname{Var}\operatorname{Vol}(K_n)(\log n)^{-C_0/2}),$$

as claimed.

To bound the difference between the two probabilities, define

$$Y = \mathbf{I}_{\operatorname{Vol}(K_n)\ge t}.$$

In this case, Y is bounded from above by 1, thus it is 0-bounded. Since $\mathbf{E}(Y) = \mathbf{P}(\operatorname{Vol}(K_n) \ge t)$, the claim follows instantly. □

We have the following:

COROLLARY 7.3. *We have* $\operatorname{Var}\operatorname{Vol}(K_n') = \Theta((\log n)^{(d-3)/2})$.

8. The second coupling. In this section we will show that the first three conditions of Lemma 4.1 are satisfied for the random variables $\operatorname{Vol}(\Pi_n)$ and $\operatorname{Vol}(K_n')$. The fourth condition is just Theorem 3.2, whose proof will come later. The first three conditions of Lemma 4.1 are stated next.

LEMMA 8.1. *For all sufficiently large n we have*

$$|\mathbf{E}\operatorname{Vol}(\Pi_n) - \mathbf{E}\operatorname{Vol}(K_n')| \le n^{-1/2+o(1)}\sqrt{\operatorname{Var}\operatorname{Vol}(K_n')},$$

$$|\operatorname{Var}\operatorname{Vol}(\Pi_n) - \operatorname{Var}\operatorname{Vol}(K_n')| \le n^{-1/2+o(1)}\operatorname{Var}\operatorname{Vol}(K_n');$$

moreover, the following holds for all t:

$$\big|\mathbf{P}\big(\operatorname{Vol}(\Pi_n) \le t\big) - \mathbf{P}\big(\operatorname{Vol}(K_n') \le t\big)\big| \le n^{-1/2+o(1)}.$$

This lemma plus Theorem 3.2 imply Theorem 3.1, that is, the central limit theorem for $\operatorname{Vol}(K_n')$, which, in turn, implies Theorem 1.1. So we will still have to prove Theorem 3.2, a major task which is the content of the next four sections. We mention further that Lemma 8.1 implies the following.

COROLLARY 8.2. *We have* $\operatorname{Var}\operatorname{Vol}(\Pi_n) = \Theta((\log n)^{(d-3)/2})$.

REMARK 8.3. Let us notice that when applying Lemma 4.1, the dominating error term comes from Theorem 3.2. Indeed, the error terms come from the first coupling are at most $(\log n)^{-C}$, where C can be arbitrarily large. The error terms from Lemma 8.1 is even smaller, $n^{-1/2+o(1)}$. This implies the estimate on the error term in Remark 1.3.

Lemma 8.1 is a consequence of the following lemma.

LEMMA 8.4. *Let A be a constant at least* 10. *For any integer n' between n and, $n + A\sqrt{n \log n}$*

$$|\mathbf{E}\,\mathrm{Vol}(K'_{n'}) - \mathbf{E}\,\mathrm{Vol}(K'_n)| \le n^{-1/2+o(1)},$$

$$|\,\mathrm{Var}\,\mathrm{Vol}(K'_{n'}) - \mathrm{Var}\,\mathrm{Vol}(K'_n)| \le n^{-1/2+o(1)}.$$

Moreover, for all t,

$$\left|\mathbf{P}\big(\mathrm{Vol}(K'_{n'}) \le t\big) - \mathbf{P}\big(\mathrm{Vol}(K'_n) \le t\big)\right| \le n^{-1/2+o(1)}.$$

PROOF OF LEMMA 8.1 VIA LEMMA 8.4. Let A be a constant at least 10. We will use the fact that the probability that a Poisson variable with mean n falls outside the interval $I = [n - A\sqrt{n \log n}, n + A\sqrt{n \log n}]$ is less than $n^{-A/4}$. As $\mathrm{Vol}(\Pi_n)$ is bounded from above by $\mathrm{Vol}(B(R))$, we have

$$\mathbf{E}\,\mathrm{Vol}(\Pi_n) = \sum_{n' \in I} \mathbf{E}(\mathrm{Vol}(K'_{n'}))\mathbf{P}(n = n') + O(n^{-A/4}\,\mathrm{Vol}(B(R))).$$

As $\mathrm{Vol}(B(R)) = O((\log n^{d/2}))$, the last term on the right-hand side is $O(n^{-A/4+o(1)}) = O(n^{-1})$ as $A \ge 10$. So the first statement of Lemma 8.4 implies

$$\begin{aligned} |\mathbf{E}\,\mathrm{Vol}(\Pi_n) - \mathbf{E}\,\mathrm{Vol}(K'_n)| &\le \sum_{n' \in I} |\mathbf{E}(\mathrm{Vol}(K'_{n'})) - \mathbf{E}(\mathrm{Vol}(K'_n))|\mathbf{P}(n = n') + O(n^{-1}) \\ &\le n^{-1/2+o(1)}. \end{aligned}$$

Taking into account the fact that $\mathbf{E}(\mathrm{Vol}(K'_n)) = \Theta((\log n)^{d/2})$ and $\mathrm{Var}(\mathrm{Vol}(K_n)) = \Theta((\log n)^{(d-3)/2})$, one can deduce the first statement of Lemma 8.1. The third statement of the same lemma can be proved the same way.

Now we turn to the second statement. For every number n' in the interval I, let $E_{n'}$ denote the event that n' is sampled (according to the Poisson distribution with mean n) and E_0 denote the event that the sampled number does not belong to the interval. The events $E_{n'}$ (with $n' \in I$ or $n' = 0$) form a partition of the space. Thus,

$$\mathrm{Var}\,\mathrm{Vol}(\Pi_n) = \mathbf{E}_{n'}(\mathrm{Var}(\mathrm{Vol}(\Pi_n)|E_{n'})) + \mathrm{Var}\,\mathbf{E}(\mathrm{Vol}(\Pi_n|E_{n'})),$$

where $n' \in I$ or $n' = 0$. Notice that $\mathrm{Vol}(\Pi_n)|E_{n'} = \mathrm{Vol}(K'_{n'})$. The rest of the proof is a calculation similar to the one above and is left as an exercise. $\square$

Let $H(r)$ be a half space at distance $r > 0$ from the origin. Define r so that the probability content of $H(r) \cap B(R)$ is $\gamma \log n/n$ for some large constant γ. As $\Psi'(H(r)) = \Theta(e^{-r^2/2}r^{-1})$, $r = \Theta(\sqrt{\log n})$. For the proof of Lemma 8.4 we need the following claim.

CLAIM 8.5. *The constant γ can be chosen so that K_n' contains $B(r)$ with probability at least* $1 - \frac{1}{n}$.

We explain the proof of this claim after the proof of Lemma 9.1 in the next section.

PROOF OF LEMMA 8.4. Let us consider a number n' as in the lemma. Let Ω denote the product space $B(R)^n$, equipped with the n-fold product of Ψ'. A point P in Ω is an ordered set $(x_1, \dots, x_n)$ of n random points (we generate the points one by one). The x_i are the coordinates of P. We use $Y(P)$ to denote the volume of the convex hull of P and μ to denote the expectation of $Y(P)$.

REMARK 8.6. Of course $Y(P)$ is just another way to express $\mathrm{Vol}(K_n')$. It is however more convenient to use this notation in the proof below as it emphasizes the fact that Y is a function from Ω to $\mathbb{R}$.

Define Ω', P', μ' similarly (with respect to n'). Let us first consider the expectations. Consider a point $P' = (x_1, \dots, x_{n'})$ in Ω' and the canonical decomposition

$$P' = P \cup Q,$$

where $P = (x_1, \dots, x_n)$ and $Q = (x_{n+1}, \dots, x_{n'})$. In order to compare μ and μ', we rewrite μ as

$$\mu = \int_{\Omega'} Y(P)\, dP'.$$

We have

$$\mu' - \mu = \int_{\Omega'} (Y(P') - Y(P))\, dP'.$$

Now we are going to decompose Ω' into three parts $\Omega_1', \Omega_2', \Omega_3'$ as follows:

- $\Omega_1' = \{P' \mid \mathrm{Conv}(P) \text{ does not contain the ball } B(r)\}$;
- $\Omega_2' = \{P' \mid \mathrm{Conv}(P) \text{ contains the ball } B(r) \text{ and } B(r) \text{ does not contain } Q\}$;
- $\Omega_3' = \Omega' \backslash (\Omega_1' \cup \Omega_2')$.

The measure of Ω_1' is the probability that the convex hull of a set of n random points does not contain $B(r)$, which is $O(1/n)$, according to Claim 8.5. The measure of Ω_2' is bounded from above by the probability that $B(r)$ does not contain Q. This probability, by the union bound, is at most

$$|Q| \times \Psi'(\overline{B(r)}) = O\big(\sqrt{n \log n}\big) \times \frac{(\log n)^{O(1)}}{n} = n^{-1/2+o(1)}.$$

Since $Y(P')$ and $Y(P)$ are at most the volume of $B(R)$, which is $O((\log n)^{d/2})$, $Y(P') - Y(P)$ is $O((\log n)^d)$. Thus

$$(16)\quad \int_{\Omega_1' \cup \Omega_2'} (Y(P') - Y(P))\, dP' = O\big((\log n)^d n^{-1/2+o(1)}\big) = n^{-1/2+o(1)}.$$

To estimate the integral over Ω_3', recall that in this region, $\mathrm{Conv}(P) = \mathrm{Conv}(P')$ since

$$P' \backslash P = Q \subset B(r) \subset \mathrm{Conv}(P).$$

It follows that

$$(17)\quad \int_{\Omega_3'} \big(Y(P') - Y(P)\big)\, dP' = 0.$$

Equations (16) and (17) together imply that

$$\mu' - \mu = n^{-1/2+o(1)},$$

proving the first part of the lemma.

The third part of the lemma follows now directly: the measure of $\Omega_1' \cup \Omega_2'$ is at most $n^{-1/2+o(1)}$, and on the rest of Ω' the polytopes $\mathrm{Conv}\, P = K_n'$ and $\mathrm{Conv}\, P' = K_{n'}'$ coincide.

The proof for the variance is similar. Notice that the variance of $\mathrm{Vol}(K_n)$ is

$$s = \int_{\Omega'} |Y(P) - \mu|^2\, dP'$$

and the variance of $\mathrm{Vol}(K_{n'})$ is

$$s' = \int_{\Omega'} |Y(P') - \mu'|^2\, dP'.$$

We have

$$(18)\quad |s' - s| = \left| \int_{\Omega'} \big((Y(P') - \mu')^2 - \big(Y(P) - \mu\big)^2\big)\, dP' \right| \le \int_{\Omega'} |\mathcal{D}(P')|\, dP',$$

where

$$\mathcal{D}(P') = \big(Y(P') - \mu'\big)^2 - \big(Y(P) - \mu\big)^2.$$

It is obvious that

$$\mathcal{D}(P') = \big(\big(Y(P') - \mu'\big) + \big(Y(P) - \mu\big)\big)\big(\big(Y(P') - \mu'\big) - \big(Y(P) - \mu\big)\big).$$

By the triangle inequality,

$$|\mathcal{D}(P')| \le \big(Y(P) + Y(P') + \mu + \mu'\big)\big(|Y(P') - Y(P)| + |\mu' - \mu|\big).$$

Since $Y(P')$ and $Y(P)$ are at most the volume of $B(R)$, which is $O((\log n)^{d/2})$, $|\mathcal{D}|$ is $O((\log n)^d)$. Thus, by arguing as before,

$$(19)\quad \int_{\Omega_1' \cup \Omega_2'} |\mathcal{D}(P')|\, dP' = O\big((\log n)^d n^{-1/2+o(1)}\big) = n^{-1/2+o(1)}.$$

To estimate the integral over Ω_3', notice that in this region, $\mathrm{Conv}(P) = \mathrm{Conv}(P')$. Therefore,

$$\int_{\Omega_3'} |\mathcal{D}(P')|\,dP' \le \int_{\Omega_3'} (Y(P) + Y(P') + \mu + \mu')|\mu' - \mu|\,dP'.$$

But we just proved that $|\mu' - \mu| \le n^{-1/2+o(1)}$. Furthermore, all $Y(P')$, $Y(P)$, μ', μ are bounded from above by the volume of $B(R)$, which is $O((\log n)^{d/2})$. So

$$\int_{\Omega_3'} ((Y(P') - \mu') + (Y(P) - \mu))|\mu - \mu'|\,dP' \le n^{-1/2+o(1)}. \tag{20}$$

Equations (19) and (20) together imply that

$$|s' - s| \le n^{-1/2+o(1)}, \tag{21}$$

concluding the proof. $\square$

9. Sandwiching K_n'. By definition, K_n' is contained in $B(R)$. In this section we will show that K_n' contains the ball $B(r)$ with high probability where the radius r is very close to R. Recall that R is defined in (9) via

$$R^2 = 2\log n + \log(\log n)^{c_0}.$$

The definition of r comes a little later, we set first $\rho > 0$ via

$$\rho^2 = 2\log n - \log\log n + \log(c\log\log n)^{-2} \tag{22}$$

where c is a constant to be specified soon. Choose a system of points $y_1, \dots, y_m$ from the sphere $S(\rho)$ maximal with respect to the property that, for $i \ne j$,

$$|y_i - y_j| \ge 2c_1.$$

As $\rho = \sqrt{2\log n}(1 + o(1))$ as n goes to infinity, we have, just as in Claim 5.1

$$m = \Theta\big((\log n)^{(d-1)/2}\big).$$

Define the half space $H_i^+ = \{x \in \mathbf{R}^d : y_i \cdot x \ge \rho^2\}$ and the cap C_i as

$$C_i = H_i^+ \cap B\big(\sqrt{\rho^2 + c_1^2}\big).$$

These caps are pairwise disjoint, and for $x \in C_i$

$$\psi(x) = \Theta\left(\frac{c\sqrt{\log n}\log\log n}{n}\right).$$

As $\mathrm{Vol}\, C_i = \Theta((\log n)^{-1/2})$, we have

$$\Psi(C_i) = \Theta\left(\frac{c\log\log n}{n}\right) \quad \text{and} \quad \Psi'(C_i) = \Theta\left(\frac{c\log\log n}{n}\right), \tag{23}$$

since $C_i \subset B(R)$. Set now $r = \rho - 5c_1^2/\rho$; it is clear then that this r satisfies

$$5c_1^2 < \rho^2 - r^2 < 10c_1^2. \tag{24}$$

LEMMA 9.1. *For every $C > 0$ the constants c, c_1 can be chosen so that the following holds. K_n' contains $B(r)$ with probability at least* $1 - (\log n)^{-C}$.

REMARK 9.2. This lemma is an analog of a result from [4] for the uniform model (see Section 2 for the definition). It is also a similar to Lemma 4.2 from [19], which was proved using VC-dimension techniques. While in those results the probability that K_n does not contain $B(r)$ is at most n^{-C}, here we have the weaker bound $(\log n)^{-C}$. The same bound was required in the uniform model when K is a polytope; see [5].

PROOF OF LEMMA 9.1. We claim first that every half space $H(r)$ at distance r from the origin contains a C_i for some $i = 1, \dots, m$. Assume y is the nearest point of $H(r)$ to the origin. Then $|y| = r$ and $y^* = \rho y/r$ lies on $S(\rho)$. As the system $y_1, \dots, y_m$ is maximal, there is a y_i with $|y^* - y_i| < 2c_1$. Define $\alpha \in (0, \pi/2)$ by $\sin\alpha = c_1/\rho$; it follows that the angle between y and any vector from C_i is at most 3α. Consequently, C_i is contained in the half space with normal y and at distance $\rho\cos 3\alpha$ from the origin. A simple computation shows now that for large enough n

$$\rho\cos 3\alpha > \rho - \frac{5c_1^2}{\rho} = r.$$

CLAIM 9.3. *There is a constant $b > 0$ depending only on d such that for all large enough n*

$$\mathbf{P}\big(B(r) \setminus K_n' \neq \varnothing\big) = O\left(\frac{(\log n)^{(d-1)/2}}{(\log n)^{bc}}\right).$$

PROOF. If $B(r)$ is not part of K_n', then there is a half space $H(r)$ at distance r from the origin which is disjoint from the random sample X_n. Then there is a cap $C_i \subset H(r)$. Then $C_i \cap X_n = \varnothing$. Consequently

$$\begin{aligned}
\mathbf{P}(C_i \cap X_n = \varnothing \text{ for some } i) &\le \sum_{i=1}^{m} \mathbf{P}(C_i \cap X_n = \varnothing)\\
&\le \sum_{i=1}^{m} (1 - \Psi'(C_i))^n \le m\left(1 - b\frac{c\log\log n}{n}\right)^n\\
&\le m\exp\{-bc\log\log n\}\\
&= \frac{m}{(\log n)^{bc}} = O\left(\frac{(\log n)^{(d-1)/2}}{(\log n)^{bc}}\right).
\end{aligned}$$

Here b is the constant coming from (23). □

Choosing the constants c and c_1 suitably completes the proof. □

REMARK 9.4. It is the choice of r from (22) and (24) that produces the bound $(\log n)^{-C}$. Also this choice of r gives the estimates in the next section. For the CLT for the volume, we could have taken $\rho^2 = 2\log n - \log(c\log n)^3$ and $r = \rho - 5c_1^2/\rho$ as well. This would have given

$$\Psi'(C_i) = \Theta\left(\frac{c' \log n}{n}\right), \tag{25}$$

and $1/n^{-c'}$ for the probability that K_n' does not contain $B(r)$. But this choice does not work for $f_s(K_n)$ (see Remark 13.7). That is why we used (22) and (24) for the definition of r.

The proof of Claim 8.5 goes along very similar lines. One can take $\rho^2 = 2\log n - \log(\gamma' \log n)^3$, for instance, and use the same argument. We omit the details.

One can prove similarly that Π_n contains $B(r)$ with high probability. Here is the quantitative statement, the routine proof is left to the interested reader.

LEMMA 9.5. *For every $C > 0$ the constants c, c_1 can be chosen so that the following holds. Π_n contains $B(r)$ with probability at least $1 - (\log n)^{-C}$.*

REMARK 9.6. Note that K_n is sandwiched between $B(R)$ and $R(r)$ with high probability, and both r, $R = \sqrt{2\log n}(1 + o(1))$. This almost implies (5) for the expectation of $\mathrm{Vol}(K_n)$, the only trouble being that K_n can have arbitrarily large volume when it is not contained in $B(R)$.

10. The dependency graph. With the notation of the previous section we define the annulus $A(R, r) = B(R) \setminus B(r)$, and let V_i denote the Voronoi region of y_i $(i = 1, \dots, m)$. This means that $x \in V_i$ if and only if $|x - y_i| \le |x - y_j|$ for all j. The sets $W_i = V_i \cap A(R, r)$ will be called *cells* and will play an important role in the central limit theorems. The following estimate will be needed.

CLAIM 10.1. *For each i,*

$$\Psi'(W_i) = \Theta\left(\frac{\log\log n}{n}\right).$$

PROOF. This is quite simple and similar to (23) and is therefore omitted. $\square$

The *dependency graph* $G(V, E)$ has, by definition, vertex set $V(G) = \{1, \dots, m\}$ and edge set $E(G)$ with $(i, j) \in E(G)$ if and only if there are $a_i \in W_i$ and $a_j \in W_j$ and $b \in A(R, r)$ such that the segments $[a_i, b]$ and $[b, a_j]$ lie completely in $A(R, r)$. In other words, if and only if $[a_i, b] \cap B(r) = \varnothing$ and $[a_j, b] \cap B(r) = \varnothing$ for some $a_i \in W_i$, $a_j \in W_j$ and $b \in A(R, r)$. Let D denote the maximal degree in the dependency graph.

THEOREM 10.2. *We have* $D = O((\log\log n)^{(d-1)/2})$.

PROOF. This is a simple matter using elementary geometry. Observe first that if the segment $[a, b] \subset A(R, r)$ and 2γ is the angle between vectors a and b, then $\cos\gamma \geq r/R$. We can estimate $\sin\gamma$ using the definitions of R and r:

$$\sin\gamma \leq \sqrt{1 - \left(\frac{r}{R}\right)^2} = \frac{1}{R}\sqrt{R^2 - r^2} = \frac{1}{R}\sqrt{R^2 - \left(\rho - \frac{5c_1^2}{\rho}\right)^2}$$
$$\leq \frac{2}{R}\sqrt{\log(\log n)^{2c_0}} = O\left(\sqrt{\frac{\log\log n}{\log n}}\right).$$

Suppose next that $a_i \in W_i$ and let $2\alpha_i$ be the angle between a_i and y_i. Set $a_i^* = \rho a_i/|a_i| \in S(\rho)$. The maximality of the system $y_1, \dots, y_m$ implies that $|a_i^* - y_i| \leq 2c_1$, which, in turn, shows that $\sin\alpha_i \leq c_1/\rho$. Consequently $\alpha = O((\log n)^{-1/2})$.

Assume $(i, j) \in E(G)$ and let $a_i \in W_i$, $a_j \in W_j$ and $b \in A(R, r)$ be the vectors such that the segments $[a_i, b]$ and $[a_j, b]$ are disjoint from $B(r)$. Let 2β be the angle between vectors y_i, y_j. Then

$$\beta \leq \alpha_i + \gamma + \alpha_j = O\left(\sqrt{\frac{\log\log n}{\log n}}\right).$$

This, of course, implies that for $(i, j) \in E(G)$

$$|y_j - y_i| \leq 2R\sin\beta = O(\sqrt{\log\log n}).$$

This means that all y_j with $(i, j) \in E(G)$ are contained in a ball, centered at y_i and of radius $O(\sqrt{\log\log n})$. Since all $y_j \in S(\rho)$ and since they are at distance $2c_1$ apart, the usual volume estimate gives the statement of the theorem. □

We establish one more inequality here.

CLAIM 10.3. *For each* i,

$$\mathrm{Vol}(W_i) = \Theta\left(\frac{\log\log n}{\sqrt{\log n}}\right).$$

PROOF. For each $t \in [r, R]$, $W_i \cap S(t)$ has constant, that is, $\Theta(1)$ $(d-1)$-dimensional volume, so $\mathrm{Vol}(W_i) = O(R - r)$, and

$$R - r = \frac{1}{R + r}(R^2 - r^2) = \frac{1}{R}\Theta(\log\log n)$$

as we have seen in the previous proof. □

11. Central limit theorem for the Poisson model. We are going to apply the Baldi–Rinott theorem for Π_n conditioned on $B(r) \subset \Pi_n$. This condition will be denoted by B. Recall from Lemma 9.5 that

$$\mathbf{P}\big(B(r) \subset \Pi_n\big) \geq 1 - (\log n)^{-C}.$$

Assume condition B holds and define the random variable $\xi_i = \mathrm{Vol}(W_i \cap \Pi_n)$. Clearly, $\xi := \sum_1^m \xi_i = \mathrm{Vol}(\Pi_n) - \mathrm{Vol}(B(r))$. This shows that, under condition B, the CLT for ξ holds if and only if it holds for $\mathrm{Vol}(\Pi_n)$.

CLAIM 11.1. *Assume condition B holds. Given disjoint subsets V_1, V_2 of the vertex set of the dependency graph with no edge between them, the random variables $\{\xi_i : i \in V_1\}$ are independent of the random variables $\{\xi_j : j \in V_2\}$.*

PROOF. The intersection $W_i \cap \Pi_n$ is determined by the facets of Π_n intersecting W_i. These facets are determined by their vertices. If there are no common vertices for the facets intersecting the W_i with $i \in V_1$ and the W_j with $j \in V_2$, then the corresponding ξ_i are independent. This is exactly how the dependency graph has been defined. $\square$

Write $\mathbf{P}^*$, $\mathbf{E}^*$, Var^* for $\mathbf{P}$, $\mathbf{E}$, Var under condition B. In the next section we will prove the following estimates.

LEMMA 11.2. *We have*

$$|\mathbf{E}^* \mathrm{Vol}(\Pi_n) - \mathbf{E}\,\mathrm{Vol}(\Pi_n)| \leq (\log n)^{-C_0/4}\sqrt{\mathrm{Var}\,\mathrm{Vol}(\Pi_n)},$$
$$|\mathrm{Var}^* \mathrm{Vol}(\Pi_n) - \mathrm{Var}\,\mathrm{Vol}(\Pi_n)| \leq (\log n)^{-C_0/4}\mathrm{Var}\,\mathrm{Vol}(\Pi_n),$$
$$\big|\mathbf{P}^*(\mathrm{Vol}(\Pi_n) \leq t) - \mathbf{P}\big(\mathrm{Vol}(\Pi_n) \leq t\big)\big| \leq (\log n)^{C_0/4}.$$

The inequality for the variances shows that

$$\mathrm{Var}^* \mathrm{Vol}(\Pi_n) = \Omega(\mathrm{Var}\,\mathrm{Vol}(\Pi_n)) = \Omega\big((\log n)^{(d-3)/2}\big).$$

We have seen that the maximal degree in G is $O((\log\log n)^{(d-1)/2})$ (Theorem 10.2), and $\xi_i = \mathrm{Vol}(W_i) = O(\log\log n/\sqrt{\log n})$. So the Baldi–Rinott theorem applies and gives the following CLT.

THEOREM 11.3. *Let d be a fixed integer at least 2. For any value of t,*

$$\left|\mathbf{P}^*\left(\frac{|\mathrm{Vol}(\Pi_n) - \mathbf{E}^* \mathrm{Vol}(\Pi_n)|}{\sqrt{\mathrm{Var}^* \mathrm{Vol}(\Pi_n)}} \leq t\right) - \Phi(t)\right| = O\left(\frac{(\log\log n)^{(d+4)/2}}{(\log n)^{(d-1)/4}}\right).$$

This theorem and Lemma 11.2 show that $\mathrm{Vol}(\Pi_n)$ and $\mathrm{Vol}(\Pi_n)|B$ satisfy conditions of Lemma 4.1. So our main central limit theorem, Theorem 1.1, follows as soon as we prove Lemma 11.2. This is our next (and final) task.

12. Proof of Lemma 11.2. This is similar to, and much simpler than, the proof in Section 7. The first step is a copy of Lemma 7.1.

LEMMA 12.1. *Let B denote the condition that $B(r) \subset K_n'$. Then we have, for large enough n,*

$$|\mathbf{E}(\mathrm{Vol}(K_n')|B) - \mathbf{E}\,\mathrm{Vol}(K_n')| \le (\log n)^{-C_0/2},$$

$$|\,\mathrm{Var}(\mathrm{Vol}(K_n')|B) - \mathrm{Var}\,\mathrm{Vol}(K_n')| \le (\log n)^{-C_0/2}.$$

Furthermore, for all t,

$$\left|\mathbf{P}\big(\mathrm{Vol}(K_n') \le t\,|B\big) - \mathbf{P}\big(\mathrm{Vol}(K_n') \le t\big)\right| \le (\log n)^{-C_0/2}.$$

PROOF. We use the first few lines of the proof of Lemma 7.2 with condition A replaced by B, events B_i do not appear yet. Then (13) says that

$$\left|\mathbf{E}(Y|B) - \mathbf{E}(Y)\right| \le \big(\mathbf{E}(Y|B) + \mathbf{E}(Y|\overline{B})\big)\mathbf{P}(\overline{B}), \tag{26}$$

where $Y = Y(t_1, \dots, t_n)$ is a c-bounded, nonnegative random variable.

When Y is just the volume, Y is bounded by $O((\log n)^{d/2})$ so its expectation, under any condition, is bounded the same way. Since $\mathbf{P}(\overline{B}) \le (\log n)^{-C_0}$ by Lemma 9.1, we are finished with the first inequality.

The third is proved by setting $Y = \mathbf{I}_{\mathrm{Vol}(K_n') \le t}$. The second inequality follows the same way as the corresponding inequality for variances in Lemma 7.1. □

We show finally how this lemma implies Lemma 11.2.

PROOF OF LEMMA 11.2. We give the proof for $\mathbf{E}$ first. As before, write E_n' for the event that $|X(n)| = n'$.

$$\begin{aligned}
|\mathbf{E}^*\,\mathrm{Vol}(\Pi_n) - \mathbf{E}\,\mathrm{Vol}(\Pi_n)| &= \left|\sum_0^\infty \big(\mathbf{E}(\mathrm{Vol}(K_{n'}')|B) - \mathbf{E}\,\mathrm{Vol}(K_{n'}')\big)\mathbf{P}(n = n')\right| \\
&\le \sum_{n' \in I} (\log n')^{-C_0/2}\mathbf{P}(n = n') + O((\log n)^{d/2} n^{A/4}) \\
&= O((\log n)^{-C_0/2}).
\end{aligned}$$

This suffices for the expectations as $\mathrm{Var}\,\mathrm{Vol}(\Pi_n) = \Theta((\log n)^{(d-3)/2})$ by Corollary 8.2. Of course, we chose C_0 large enough.

The proof for Var* and $\mathbf{P}^*$ is similar and is left to the reader. □

We want to emphasize here that the proofs of Theorems 3.2, 3.1 and 1.1 have finally been completed at this point.

13. Proof of Theorem 1.2. The proof of Theorem 1.2 follows the plan in Section 4 closely. In fact, most of the arguments are the same as in the proof of Theorem 1.1, except for a few technical modifications, and a single extra difficulty: finding the right bound M on the number of s-faces intersecting cell W_i. Thus, instead of working out all details, we only state the main steps and point out what modifications are needed, plus explain how the bound M can be found.

We have seen in Theorem 6.3 that the variance satisfies

$$\mathrm{Var}(f_s(K_n)) = \Theta\big((\log n)^{(d-1)/2}\big).$$

13.1. *The first coupling.* Lemma 7.1 still holds if one replaces Vol by f_s. Notice that the proof of this lemma only requires the c-bounded property. The number of faces has this property (for some sufficiently large constant c). Indeed, one can show that with very high probability (say $1 - n^{-100d}$) the number of vertices is at most $(\log n)^d$. This, together with a simple geometric argument shows that the number of faces is c-bounded for some constant c. The same proof goes for the square of the number of faces.

After the first coupling, it is left to prove the following variant of Theorem 3.1.

THEOREM 13.1. *Let s be an integer between* 0 *and* $d-1$. *There is a function $\varepsilon(n)$ tending to zero as n tends to infinity such that for all t*

$$\left|\mathbf{P}\left(\frac{f_s(K'_n) - \mathbf{E} f_s(K'_n)}{\sqrt{\mathrm{Var}\, f_s(K'_n)}} \le t\right) - \Phi(t)\right| \le \varepsilon(n).$$

13.2. *The second coupling.* The proof for the second coupling is almost the same as before. A small technical modification one needs to make here is to introduce a new part Ω'_0 in the partition which contains those P' where $\mathrm{Conv}(P')$ has more than (say) $(\log n)^d$ vertices. The probability of Ω'_0 will be less than $n^{-1/2}$. Now define $\Omega'_3 = \Omega \backslash (\Omega'_0 \cup \Omega'_1 \cup \Omega'_2)$. The rest of the proof is the same. In fact, since both the expectation and variance of $f_s(K'_n)$ are also polylogarithmic in n (similar to those of the volume), the error term $n^{-1/2+o(1)}$ remains unchanged in all these estimates.

After the second coupling one needs the f_s variant of Theorem 3.2.

THEOREM 13.2. *Let d be a fixed integer at least* 2 *and* $0 \le s \le d-1$. *There is a function $\varepsilon(n)$ tending to* 0 *as n tends to infinity such that the following holds. For any value of t,*

$$\left|\mathbf{P}\left(\frac{|f_s(\Pi_n) - \mathbf{E} f_s(\Pi_n)|}{\sqrt{\mathrm{Var}\, f_s(\Pi_n)}} \le t\right) - \Phi(t)\right| \le \varepsilon(n). \tag{27}$$

REMARK 13.3. One can take $\varepsilon(n) = (\log n)^{-(d-1)/4+o(1)}$. This error term will be the dominating one when we apply, twice, Lemma 4.1.

13.3. *The dependency graph.* The dependency graph is the same as before with

$$m = \Theta\big((\log n)^{(d-1)/2}\big), \qquad D = O\big((\log\log n)^{(d-1)/2}\big)$$

and

$$\Psi'(W_i) = \Theta\big((\log\log n)/n\big).$$

For proper accounting $f_s(\Pi_n)$ we have to define the random variable $\xi_i = f(W_i, s)$ suitably. For this purpose we use Reitzner's method from [15]. For an s-dimensional face, L, of Π_n, let $f(W_i, L)$ denote the number of vertices of L contained in W_i, and set

$$f(W_i, s) = \frac{1}{s+1} \sum_L f(W_i, L).$$

Since Π_n is simplicial and has no vertex on the boundary of any W_i with probability one, $f_s(\Pi_n) = \sum_{i=1}^m f(W_i, s)$. The expected number of $|X(n) \cap W_i| = \Theta(\log\log n)$, which, in turn, shows that the expectation of $f(W_i, s)$ is $\Omega(\log\log n)$. But there is an extra difficulty here: we need a bound M on each $f(W_i, s)$ when applying the Baldi–Rinott theorem. The condition $B(r) \subset \Pi_n$ is not enough and we have to introduce a new condition, to be denoted by B_i:

$$|X(n) \cap W_i| \le c_2 \log\log n \qquad \text{for each } i,$$

where c_2 is a large constant. It is straightforward to check that for any $C > 0$, c_2 can be chosen so large that

$$\mathbf{P}(B_i \text{ holds}) \ge 1 - (\log n)^{-C}.$$

Then the union bound shows that

$$\mathbf{P}(B_i \text{ fails for some } i) = O\big((\log n)^{-C+(d-1)/2}\big).$$

It is clear that if L is an s-face of Π_n contributing to $F(W_i, s)$, then all vertices of L belong to a cell W_j with i, j connected in G or to W_i. There are at most D such cells. So under condition B_i, there are at most $c_2 D \log\log n$ vertices in the union of these cells. This shows that $M = (\log\log n)^{d^2}$ works and the application of the Baldi–Rinott theorem goes through.

Again we have to remove the conditions $B, B_1, \dots, B_m$. This is done in the same way as in Section 12.

REMARK 13.4. This is where the careful choice of r (in fact, ρ) pays off. With the more generous selection $\rho^2 = 2\log n - \log(c\log n)^3$, we would only have $f(W_i, s) = O((\log n)^{d/2})$, and the right-hand side in the estimate of the Baldi–Rinott theorem does not tend to zero.

14. Concluding remarks. Our plan can be used for many other parameters. In certain cases, one merely has to repeat the proof. In others, however, there are substantial technical difficulties. Let us present two representative examples.

The surface area of K_n. The proof is more or less the same as the proof for the volume. The reader is invited to work out the details. In fact, the result holds for all intrinsic volumes, but the estimate for variance is not straightforward.

The probability content of K_n. The probability content of K_n is $\Psi(K_n)$. For this parameter, the general plan still works, but there is a nonnegligible difficulty. In the proof of the second coupling, we used the fact that the expectation and variance of the random variable under study (such as the volume, number of faces, or even the surface area) are both polylogarithmic in n. Thus, the error term $n^{-1/2+o(1)}$ is dominating and one can finish the proof easily. For the case of the probability content, it is no longer true, as the variance is $n^{-2+o(1)}$. To overcome this obstacle, we can follow [20] and start by proving a sharp concentration result, which gives a tight control on the tail $Y(P) - \mu$ and $Y(P') - \mu'$. Such a concentration result is available thanks to the method developed in [19]. The details will appear elsewhere.

REFERENCES

[1] AFFENTRANGER, F. (1991). The convex hull of random points with spherically symmetric distributions. *Rend. Sem. Mat. Univ. Politec. Torino* **49** 359–383. MR1231058

[2] AFFENTRANGER, F. and SCHNEIDER, R. (1992). Random projections of regular simplices. *Discrete Comp. Geom.* **7** 219–226. MR1149653

[3] BALDI, P. and RINOTT, Y. (1989). On normal approximations of distributions in terms of dependency graphs. *Ann. Probab.* **17** 1646–1650. MR1048950

[4] BÁRÁNY, I. and DALLA, L. (1997). Few points to generate a random polytope. *Mathematika* **44** 325–331. MR1600549

[5] BÁRÁNY, I. and REITZNER, M. (2006). Central limit theorems for random polytopes in convex polytopes. *Probab. Theory Related Fields* **133** 483–507. MR2197111

[6] BARYSHNIKOV, Y. M. and VITALE, R. A. (1994). Regular simplices and Gaussian samples. *Discrete Comp. Geom.* **11** 141–147. MR1254086

[7] CABO, A. J. and GROENEBOOM, P. (1994). Limit theorems for functionals of convex hulls. *Probab. Theory Related Fields* **100** 31–55. MR1292189

[8] GROENEBOOM, P. (1988). Limit theorems for convex hulls. *Probab. Theory Related Fields* **79** 327–368. MR0959514

[9] HSING, T. (1994). On the asymptotic distribution of th area outside a random convex hull in a disk. *Ann. Appl. Probab.* **4** 478–493. MR1272736

[10] HUETER, I. (1994). The convex hull of a normal sample. *Adv. in Appl. Probab.* **26** 855–875. MR1303866

[11] HUETER, I. (1999). Limit theorems for the convex hull of random points in higher dimensions. *Trans. Amer. Math. Soc.* **351** 4337–4363. MR1670156

[12] HUG, D., MUNSONIUS, G. O. and REITZNER, M. (2004). Asymptotic mean values of Gaussian polytopes. *Beiträge Algebra Geom.* **49** 531–548. MR2093024

[13] HUG, D. and REITZNER, M. (2005). Gaussian polytopes: Variances and limit theorems. *Adv. in Appl. Probab.* **37** 297–320. MR2144555

[14] RAYNAUD, H. (1970). Sur l'enveloppe convexe des nuages de points aléatoires dans $\mathbf{R}^n$. *J. Appl. Probab.* **7** 35–48. MR0258089

[15] REITZNER, M. (2005). Central limit theorems for random polytopes. *Probab. Theory Related Fields* **133** 483–507. MR2197111

[16] RÉNYI, A. and SULANKE, R. (1963). Über die konvexe Hülle von n zufällig gewählten Punkten. *Z. Wahrsch. Verw. Gebiete* **2** 75–84. MR0156262

[17] RINOTT, Y. (1994). On normal approximation rates for certain sums of dependent random variables. *J. Comput. Appl. Math.* **55** 135–143. MR1327369

[18] STEIN, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **2** 583–602. Univ. California Press, Berkeley. MR0402873

[19] VU, V. H. (2005). Sharp concentration of random polytopes. *Geom. Funct. Anal.* **15** 1284–1318. MR2221249

[20] VU, V. H. (2006). Central limit theorems for random polytopes in a smooth convex set. *Adv. in Math.* **207** 221–243. MR2264072

[21] WEIL, W. and WIEACKER, J. (1993). Stochastic geometry. In *Handbook of Convex Geometry* **A, B** 1391–1438. North-Holland, Amsterdam. MR1243013

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