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Balanced partitions of vector sequences ☆

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Abstract

Let $d, r \in \mathbb{N}$ and $\|\cdot\|$ be any norm on $\mathbb{R}^d$. Let B denote the unit ball with respect to this norm. We show that any sequence $v_1, v_2, \ldots$ of vectors in B can be partitioned into r subsequences $V_1, \ldots, V_r$ in a balanced manner with respect to the partial sums: For all $n \in \mathbb{N}$, $\ell \leqslant r$, we have $\|\sum_{i \leqslant k, v_i \in V_\ell} v_i - \frac{1}{r}\sum_{i \leqslant k} v_i\| \leqslant 2.0005d$. A similar bound holds for partitioning sequences of vector sets. Both results extend an earlier one of Bárány and Grinberg [I. Bárány, V.S. Grinberg, On some combinatorial questions in finite-dimensional spaces, Linear Algebra Appl. 41 (1981) 1–9] to partitions in arbitrarily many classes.

Keywords: Discrepancy; Balanced partition; Vector balancing game

1. Introduction

Let $d, N \in \mathbb{N}$. We use the short-hand $[N] := \{1, \ldots, N\}$. Let $\|\cdot\|$ be any norm on $\mathbb{R}^d$ and $B = \{v \in \mathbb{R}^d \mid \|v\| \leqslant 1\}$ its unit ball. In this paper, we give extensions of the Bárány–Grinberg theorem to partitions into more than two classes. In its most general version, this theorem states the following [1].

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Theorem 1. *Let $V_1, \ldots, V_N \subseteq B$ such that $0 \in \operatorname{conv}(V_i)$ for all $i \in [N]$. Then there are $v_i \in V_i$ such that for all $n \in [N]$,*

$$\left\| \sum_{i \in [n]} v_i \right\| \leqslant 2d.$$

The most interesting special case of Theorem 1 is that all V_i are of the form $V_i = \{v_i, -v_i\}$, cf. [2] as well. In this case, Theorem 1 yields that for any sequence $v_1, \ldots, v_N$ of vectors in B there are signs $\varepsilon_i \in \{-1, 1\}$ such that $\| \sum_{i \in [n]} \varepsilon_i v_i \| \leqslant 2d$ for all $n \in [N]$. In other words, there is a partition $[N] = I_1 \dot{\cup} I_2$ such that $\| \sum_{i \in I_j \cap [n]} v_i - \frac{1}{2} \sum_{i \in [n]} v_i \| \leqslant d$ for all $n \in [N]$ and $j \in [2]$. This partitioning version of the Bárány–Grinberg theorem was extended to partitions into $r > 2$ classes with error bound $(r-1)d$ in [3]. In the following section, we show that the factor $(r-1)$ can be replaced by a constant.

In the third section of this paper, we show that if the stronger condition $\sum_{v \in V_i} v = 0$ (instead of $0 \in \operatorname{conv}(V_i)$) holds for all $i \in [N]$, then for each $i \in [N]$ there are r distinct vectors $v_{i\ell} \in V_i$, $\ell \in [r]$, such that $\| \sum_{i \in [n]} v_{i\ell} \| \leqslant 5d$ holds for all $n \in [N]$ and $\ell \in [r]$, where $r \leqslant \max\{|V_i| | i \in [N]\}$.

It is worth mentioning here that the results hold for all norms in $\mathbb{R}^d$. This is due to the fact that proofs use linear dependences among some vectors, with the norm playing very little role. But most likely, much better bounds are valid for particular norms. For instance, it is conjectured that for $r = 2$ and Euclidean norm the best bound is of order $\sqrt{d}$. This was proved by Spencer [4] when $N = O(d)$, but the general case when N is arbitrary is open.

In the proofs of both results below we invoke the recursive method of [3], which states, roughly speaking, that if one can guarantee the existence of a 2-partition with good bound on its discrepancy, then one can guarantee the existence of an r-partition with a slightly weaker bound on its discrepancy. Precisely, we have the following:

Theorem 2. *Let $r \geqslant 2$ be an integer. Let $v_1, \ldots, v_n$ be a sequence of vectors and $\mathscr{E}$ be a set of subsets of $[n]$. Assume that for all integers $1 \leqslant r_1 < r_0 \leqslant r$ and all $V_0 \subseteq [n]$ there is a $V_1 \subseteq V_0$ such that for all $E \in \mathscr{E}$,*

$$\left\| \sum_{i \in V_1 \cap E} v_i - \frac{r_1}{r_0} \sum_{i \in V_0 \cap E} v_i \right\| \leqslant K.$$

Then there is a partition $V = V_1 \dot{\cup} \ldots \dot{\cup} V_r$ such that for all $\ell \in [r]$ and $E \in \mathscr{E}$ we have

$$\left\| \sum_{i \in V_\ell \cap E} v_i - \frac{1}{r} \sum_{i \in V \cap E} v_i \right\| \leqslant C(r)K,$$

where $C(r)$ is an absolute constant satisfying $C(r) \leqslant 2.0005$ for all $r \in \mathbb{N}$.

Note that the assumption of the theorem is equivalent to saying that for all integer $1 \leqslant r'$, $r'' < r_0 \leqslant r$ with $r' + r'' = r_0$ and all $V_0 \subseteq [n]$ there is a partition $V' \dot{\cup} V'' = V_0$ such that for all $E \in \mathscr{E}$

$$\left\| \sum_{i \in V' \cap E} v_i - \frac{r'}{r_0} \sum_{i \in V_0 \cap E} v_i \right\| \leqslant K \quad \text{and} \quad \left\| \sum_{i \in V'' \cap E} v_i - \frac{r''}{r_0} \sum_{i \in V_0 \cap E} v_i \right\| \leqslant K.$$

The proof of Theorem 2 starts with setting $V_0 = V$ and proceeds by partitioning V' and V'' further. Details can be found in [3].

Theorem 2 was worked out only in the context of hypergraph coloring (Theorem 3.6 in [3]), which in our language means $v_i = \mathbf{1}_d$ for all $i \in [n]$. However, the proofs easily reveal that all results hold as well for the general setting of Theorem 2.

2. Vector partitioning

Assume V is a finite or infinite sequence of vectors $v_1, v_2, \ldots$ We introduce the (non-standard) notation $\sum_k V = \sum_{i=1}^k v_i$. Further, for a subsequence X of V we define $\sum_k X = \sum_{i \leqslant k, v_i \in X} v_i$.

Theorem 3. *For every sequence $V \subset B$, and for every integer $r \geqslant 2$, there is a partition of V into r subsequences $X_1, \ldots, X_r$ such that for all k and j*

$$\sum_k X_j \in \frac{1}{r} \sum_k V + C(r) d B.$$

Proof. Assume $r_0 = r_1 + r_2$ (with positive integers r_1, r_2). We are going to construct a partition of V into subsequences Y_1 and Y_2 such that for each k and for $j = 1, 2$,

$$\sum_k Y_j \in \frac{r_j}{r_0} \sum_k V + dB.$$

This implies the theorem via Theorem 2.

For the construction of Y_1, Y_2 we use a modified version of the method of "floating variables" as given in [1]. Define $V_k = \{v_1, v_2, \ldots, v_{k+d}\}, k = 0, 1, 2, \ldots$ We are going to construct mappings $\beta_k : V_k \to [-r_1, r_2]$ and subsets $W_k \subset V_k$ with the following properties (for all k):

(i) $\sum_{V_k} \beta_k(v) v = 0$,
(ii) $\beta_k(v) \in \{-r_1, r_2\}$ whenever $v \in W_k$,
(iii) $|W_k| = k$ and $W_k \subset W_{k+1}$.

The construction is by induction on k. For $k = 0$, $W_0 = \emptyset$ and $\beta_0 = 0$ clearly suffice. Now assume that β_k and W_k have been constructed and satisfy (i) to (iii). The $d + 1$ vectors in $V_{k+1} \setminus W_k$ are linearly dependent, so there are $\alpha(v) \in \mathbb{R}$ not all zero such that

$$\sum_{V_{k+1} \setminus W_k} \alpha(v) v = 0.$$

Putting $\beta_k(v_{k+d+1}) = 0$, we have

$$\sum_{W_k} \beta_k(v) v + \sum_{V_{k+1} \setminus W_k} (\beta_k(v) + t\alpha(v)) v = 0$$

for all $t \in \mathbb{R}$. For $t = 0$ all coefficients lie in $[-r_1, r_2]$. Hence for a suitable $t = t^*$, all coefficients still belong to $[-r_1, r_2]$, and $\beta_k(v) + t\alpha(v) \in \{-r_1, r_2\}$ for some $v = v^* \in V_{k+1} \setminus W_k$. Set now $W_{k+1} = W_k \cup \{v^*\}$ and $\beta_{k+1}(v) = \beta_k(v)$, if $v \in W_k$, and $\beta_{k+1}(v) = \beta_k(v) + t^*\alpha(v)$, if $v \in V_{k+1} \setminus W_k$. Now W_{k+1} and β_{k+1} satisfy the requirements. Moreover, $\beta_{k+1}(v) = \beta_k(v)$ for all $v \in W_k$.

We now define the subsequences Y_1 and Y_2. Put v_i into Y_1 if $v_i \in W_k$ and $\beta_k(v_i) = r_2$ for some k, and put v_i into Y_2 if $v_i \in W_k$ and $\beta_k(v_i) = -r_1$ for some k. As $\beta_k(v) = \beta_{k+1}(v)$ once $v \in W_k$, this definition is correct for all vectors that appear in some W_k. The remaining (at most d) vectors can be put into Y_1 or Y_2 in any way. Set $\gamma(v) = r_2$, if $v \in Y_1$, and $\gamma(v) = -r_1$, if $v \in Y_2$.

Clearly, $r_2 \sum_k Y_1 - r_1 \sum_k Y_2 \in r_0 dB$ for all $k \leqslant d$. For $k > d$ we have, with $k = h + d$,

$$\begin{aligned} r_2 \sum_k Y_1 - r_1 \sum_k Y_2 &= \sum_{V_h} \gamma(v)v = \sum_{V_h} \gamma(v)v - \sum_{V_h} \beta_h(v)v \\ &= \sum_{V_h} (\gamma(v) - \beta_h(v))v = \sum_{V_h \setminus W_h} (\gamma(v) - \beta_h(v))v. \end{aligned}$$

The last sum contains at most d non-zero terms, each having norm at most r_0. Thus

$$r_2 \sum_k Y_1 - r_1 \sum_k Y_2 \in r_0 dB$$

for every k. Adding this to the trivial equation $r_1 \sum_k Y_1 + r_1 \sum_k Y_2 = r_1 \sum_k V$ (expressing that Y_1, Y_2 form a partition of V), we obtain

$$\sum_k Y_1 \in \frac{r_1}{r_0} \sum_k V + dB$$

for every k. □

3. Vector selection

Let now $V_1, \ldots, V_N$ be a sequence of finite subsets of B such that $|V_i| \geqslant r$ for all $i \in [N]$. An r-selection of (V_i) is a mapping $\chi : [N] \times [r] \to \mathbb{R}^d$ such that $\chi(i, [r])$ is an r-element subset of V_i for all $i \in [N]$. For such a χ, we define its discrepancy with respect to (V_i) by

$$\operatorname{disc}(\chi, (V_i)_{i \in [N]}) = \max_{n \in [N]} \max_{\ell \in [r]} \left\| \sum_{i \in [n]} \left(\chi(i, \ell) - \frac{1}{|V_i|} \sum_{v \in V_i} v \right) \right\|.$$

Theorem 4. *There is an r-selection with discrepancy at most $5d$.*

We mention that this theorem also holds for infinite sequences of finite subsets of B.

To prove the theorem, we apply the following lemma twice.

Lemma 5. *Let $r \in \mathbb{N}$, $r \geqslant 2$. Let $V_1, \ldots, V_N \subseteq B$ such that $|V_i| \geqslant r$ for all $i \in [N]$. Then for all $k \in [r]$ there are $U_i \subseteq V_i$ such that $|U_i| = k$ for all $i \in [N]$ and* $\max_{n \in [N]} \| \sum_{i \in [n]} (\sum_{v \in U_i} v - \frac{k}{|V_i|} \sum_{v \in V_i} v) \| \leqslant 2d$.

Proof. We give an algorithm for the construction of the sets U_i. For each $i \in [N]$, $v \in V_i$ put $x_{iv} = \frac{k}{|V_i|}$. We iteratively change these numbers to zeros and ones in such a way that $U_i := \{v \in V_i | x_{iv} = 1\}$ gives the desired solution. For the start let $n = 1$. What we do is the following: View those x_{iv} such that $x_{iv} \notin \{0, 1\}$ and $i \leqslant n$ as variables. If there is exactly one solution to the linear system

$$\sum_{i \in [N]} \sum_{v \in V_i} \left(x_{iv} - \frac{k}{|V_i|} \right) v = 0, \tag{1}$$

$$\sum_{v \in [|V_i|]} x_{iv} = k, \quad i \in [N], \tag{2}$$

$$x_{iv} \in [0, 1], \quad i \in [N], \quad v \in V_i,$$

then increase n by one and try again. Otherwise our existing solution may be changed in such a way that at least one more variable x_{iv} becomes 0 or 1. If n reaches N and no solution can be found, then stop and change the remaining non-integral values of x_{iv} to 0 or 1 in such a way that (2) is still fulfilled.

Assume that in some step of this iteration no solution can be found. Then there are at least as many constraints containing variables as there are variables. Let q be the number of constraints of type (2) that contain a variable. Then the total number of constraints containing variables is at most $d + q$, and the number of variables is at least $2q$. Hence $q \leqslant d$ holds if no non-trivial solution can be found, and at most $q + d \leqslant 2d$ of the x_{iv}, $i \leqslant n$, are not in $\{0, 1\}$. Denote the set of these pairs (i, v) by I. Since the remaining x_{iv}, $i \leqslant n$, are not changed anymore, our final solution $\tilde{x}$ satisfies

$$\sum_{i\in[n]}\sum_{v\in V_i}\left(\tilde{x}_{iv} - \frac{k}{|V_i|}\right)v = \sum_{i\in[n]}\sum_{v\in V_i}\left(x_{iv} - \frac{k}{|V_i|}\right)v + \sum_{(i,v)\in I}(\tilde{x}_{iv} - x_{iv})v$$
$$= \sum_{(i,v)\in I}(\tilde{x}_{iv} - x_{iv})v.$$

Since $|I| \leqslant 2d$, we conclude $\| \sum_{i\in[n]}\sum_{v\in V_i}(\tilde{x}_{iv} - \frac{k}{|V_i|})v\| \leqslant 2d$ for all $n \in [N]$. Since $\tilde{x}_{iv} \in \{0, 1\}$, putting $U_i := \{v \in V_i | \tilde{x}_{iv} = 1\}$ gives the desired solution. □

Proof of the theorem. Let us assume first that $|V_i| = r$ for all $i \in [N]$. Then, by the above lemma, for all integers r_1, r_2 such that $r = r_1 + r_2$ there are $U_i^{(1)} \dot\cup U_i^{(2)} = V_i$ such that $|U_i^{(j)}| = r_j$ and $\| \sum_{i\in[n]}(\sum_{v\in U_i^{(j)}} v - \frac{r_j}{r}\sum_{v\in V_i} v)\| \leqslant 2d$. Hence from Theorem 2, we obtain an r-selection (actually an r-partition) of (V_i) such that

$$\left\| \sum_{i\in[n]}\left(\chi(i,\ell) - \frac{1}{r}\sum_{v\in V_i} v\right)\right\| \leqslant 2C(r)d$$

for all $n \in [N]$, $\ell \in [r]$.

If $|V_i| > r$ for some i, apply the Lemma 5 (with $k = r$) to obtain $\tilde{V}_i \subseteq V_i$ such that $|\tilde{V}_i| = r$ and $\| \sum_{i\in[n]}(\sum_{v\in\tilde{V}_i} v - \frac{r}{|V_i|}\sum_{v\in V_i} v)\| \leqslant 2d$. By the above, there is an r-selection for $(\tilde{V}_i)$ such that

$$\left\| \sum_{i\in[n]}\left(\chi(i,\ell) - \frac{1}{r}\sum_{v\in \tilde{V}_i} v\right)\right\| \leqslant 2C(r)d$$

for all $n \in [N]$, $\ell \in [r]$. Note that, trivially, χ is also an r-selection for (V_i). It satisfies

$$\left\| \sum_{i\in[n]}\left(\chi(i,\ell) - \frac{1}{|V_i|}\sum_{v\in V_i} v\right)\right\|$$
$$\leqslant \left\| \sum_{i\in[n]}\left(\chi(i,\ell) - \frac{1}{r}\sum_{v\in \tilde{V}_i} v\right)\right\| + \left\| \sum_{i\in[n]}\left(\frac{1}{r}\sum_{v\in \tilde{V}_i} v - \frac{1}{|V_i|}\sum_{v\in V_i} v\right)\right\|$$
$$\leqslant 2C(r)d + \frac{1}{r}2d$$

for all $n \in [N]$ and $\ell \in [r]$. By noting that $C(2) = 1$ and $C(r) \leqslant 2.0005$ for all $r \in \mathbb{N}$, we obtain the constant of 5. □

We may remark that a closer inspection of $C(r)$ for small r yields better constants. For example, easy calculations by hand or Lemma 3.5 in [3] show that $C(r) + \frac{1}{r} \leqslant 2.1$ for $r \leqslant 10$ (for $r = 7$ observe that $C(7) \leqslant \max\{\frac{1}{3} + C(3), \frac{1}{4} + C(4)\}$). Hence the bound $C(r) \leqslant 2.0005$ implies $C(r) + \frac{1}{r} \leqslant 2.1$ for all $r \in \mathbb{N}$, leading to a constant of 4.2 instead of 5.

The following is an immediate consequence of Theorem 4.

Corollary 6. *Let $r, N \in \mathbb{N}$. For $i \in [N]$ let $V_i \subseteq B$ such that $\sum_{v \in V_i} v = 0$ and $|V_i| \geqslant k$. Then there is a k-selection of (V_i) such that*

$$\left\| \sum_{i \in [n]} \chi(i, \ell) \right\| \leqslant 5d$$

for all $n \in [N]$ and $\ell \in [r]$.

This answers a question of Emo Welzl concerning multi-class extensions of Theorem 1 posed at the Oberwolfach Seminar on "Discrepancy Theory and its Applications" in March 2004. It is clear that the stronger assumption $\sum_{v \in V_i} v = 0$ is necessary. Already for $d = 1$ and $r = 2$, the sequence $V_i = \{-\frac{1}{2}, 1\}$ shows that $0 \in \text{conv}(V_i)$ does not suffice.

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References

[1] I. Bárány, V.S. Grinberg, On some combinatorial questions in finite-dimensional spaces, Linear Algebra Appl. 41 (1981) 1–9.

[2] J. Beck, V.T. Sós, Discrepancy theory, in: R. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, Elsevier, 1995, pp. 1405–1446.

[3] B. Doerr, A. Srivastav, Multicolour discrepancies, Combinatorics, Probability and Computing 12 (2003) 365–399.

[4] J. Spencer, Balancing vectors in the max norm, Combinatorica 6 (1986) 55–65.