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Berge's theorem, fractional Helly, and art galleries

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Abstract

In one of his early papers Claude Berge proved a Helly-type theorem, which replaces the usual "nonempty intersection" condition with a "convex union" condition. Inspired by this we prove a fractional Helly-type result, where we assume that many $(d+1)$-tuples of a family of convex sets have a star-shaped union, and the conclusion is that many of the sets have a common point. We also investigate somewhat related art-gallery problems. In particular, we prove a $(p, 3)$-theorem for guarding planar art galleries with a bounded number of holes, completing a result of Kalai and Matoušek, who obtained such a result for galleries without holes. On the other hand, we show that if the number of holes is unbounded, then no (p, q)-theorem of this kind holds with $p \geqslant 2q-1$.

Keywords: Helly-type theorem; Art gallery; Transversal

1. Introduction

In one of his early papers [6] Claude Berge proved a Helly-type result, in which one of the usual "nonempty intersection" conditions is replaced with another, weaker condition (see [9] for a survey of Helly-type results). The theorem is cited in Berge's book [7] in the following form.

Theorem 1.1 (*Berge [6]*). *Let $C_1, C_2, \ldots, C_m$ $(m \geqslant 2)$ be closed convex sets in $\mathbf{R}^d$ whose union is convex; if the intersection of every $m-1$ of these sets is nonempty, then their intersection is nonempty.*

The interesting cases are with $m \leqslant d+1$, for otherwise, the intersection is nonempty by the (usual) Helly theorem. The following (much newer) theorem of Breen [8] easily implies Theorem 1.1:

Theorem 1.2 (*Breen [8]*). *Assume $\mathscr{C}$ is a nonempty family of closed convex sets in $\mathbf{R}^d$. If every $d+1$ or fewer sets in $\mathscr{C}$ have a starshaped union, then $\bigcap \mathscr{C} \neq \emptyset$.*

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Here a set $X \subseteq \mathbf{R}^d$ is called *starshaped* if there is a point $c \in X$ such that the segment $[x, c]$ is contained in X for every $x \in X$. To see that Breen's theorem implies Berge's result mentioned above, we suppose that the assumptions of Berge's theorem are satisfied for some $C_1, \dots, C_m$. Then $\bigcup_{i=1}^m C_i$ is starshaped since it is convex, the union of any $m-1$ or fewer sets among the C_i is starshaped because these sets have a common point, and $\bigcap C_i \neq \emptyset$ by Breen's theorem.

A fractional Helly-type theorem. Our first main result in this paper establishes a "fractional" variant of Breen's theorem. In the original fractional Helly theorem, due to Katchalski and Liu [14], not all $(d+1)$-tuples of $\mathscr{C}$ intersect, only a positive fraction of them, and the conclusion is that a positive fraction of the sets in $\mathscr{C}$ have a point in common. Precisely:

Theorem 1.3 (*Fractional Helly theorem [14]*). *For every $\alpha \in (0, 1]$ and every $d \geqslant 2$ there is a $\beta > 0$ such that the following holds. Assume $\mathscr{C}$ is a finite family of $n \geqslant d+1$ convex compact sets in $\mathbf{R}^d$ with the property that at least $\alpha\binom{n}{d+1}$ of the $(d+1)$-tuples of $\mathscr{C}$ have nonempty intersection. Then $\mathscr{C}$ contains an intersecting subfamily of size at least βn.*

Katchalski and Liu give $\beta = \alpha/(d+1)$; the best value of β is $1-(1-\alpha)^{1/(d+1)}$ as shown by Kalai [12]. It has been understood recently that the fractional Helly theorem plays a basic role in many results of combinatorial convexity; see, e.g., [3,1,2,16]. Here is our fractional Helly-type theorem:

Theorem 1.4 (*Fractional Helly with starshaped unions*). *For every $\alpha \in (0, 1)$ and every $d \geqslant 2$ there is a $\beta = \beta(\alpha, d) > 0$ such that the following holds. Assume $\mathscr{C}$ is a family of $n \geqslant d+1$ convex compact sets in $\mathbf{R}^d$ with the property that at least $\alpha\binom{n}{d+1}$* of the *$(d+1)$-tuples of $\mathscr{C}$ have a starshaped union. Then $\mathscr{C}$ contains an intersecting subfamily of size at least βn.*

It is perhaps instructive to mention that in this theorem (unlike in Breen's theorem) no assumption is made about unions of fewer than $d+1$ sets.

A (p, q)-theorem. Alon and Kleitman [3] gave a spectacular proof of a conjecture of Hadwiger and Debrunner:

Theorem 1.5 (*(p, q)-theorem [3]*). *For every integers d, p, q, $p \geqslant q \geqslant d+1 \geqslant 2$, there exists an integer T such that every finite family $\mathscr{F}$ of convex sets in $\mathbf{R}^d$ satisfying the (p, q)-property (that is, among every p sets of $\mathscr{F}$ there are some q sets with a common point) there is a set of at most T points intersecting all sets of $\mathscr{F}$ (we say that $\mathscr{F}$ has a* transversal *of size at most T).*

The Alon–Kleitman method is based mainly on the fractional Helly theorem. As is discussed in [2], a fractional Helly-type property of a set system implies a (p, q)-theorem in a quite general setting. By a standard application of this approach, Theorem 1.4 yields:

Corollary 1.6 (*A (p, q)-theorem for starshaped unions*). *For every integers d, p, q, $p \geqslant q \geqslant d+1 \geqslant 2$, there exists an integer T such that every finite family $\mathscr{F}$ of convex sets in $\mathbf{R}^d$, such that among every p sets of $\mathscr{F}$ there are some q with a starshaped union, has a transversal of size at most T.*

For completeness we briefly review the proof in Section 3.

Art galleries. Next, we consider similar theorems in a related but slightly different setting of art galleries. We first introduce some standard terminology. An (*art*) *gallery* is a compact set $X \subset \mathbf{R}^d$. A point $x \in X$ *guards* (or *sees*) a point $y \in X$ if the segment $[x, y]$ is fully contained in X. A *hole* in X is a bounded connected component of the complement $\mathbf{R}^d \setminus X$.

We are interested in (p, q) theorems for art galleries of the following kind: if X is an art gallery in which among every p points, some q can be guarded by a single point, then all of X can be guarded by at most $C = C(p, q, d)$ points.

There is a well-known positive result with $p = q = d + 1$ and $C = C(d + 1, d + 1, d) = 1$, i.e., a Helly-type result:

Theorem 1.7 (*Krasnosel'skiĭ [15]*). *Let $X \subset \mathbf{R}^d$ be a compact set, and suppose that for every $d + 1$ or fewer points of X there is a point in X that guards all of them. Then X can be guarded by a single point; that is, it is starshaped.*

Another positive result is a $(p, 3)$-theorem of Kalai and Matoušek [13], which holds for all *simply connected planar* galleries (simply connected means with no holes), with $T = \mathrm{O}(p^3 \log p)$. Here we extend it to planar galleries with a bounded number of holes:

Theorem 1.8 (*A $(p, 3)$-theorem for planar galleries with h holes*). *Assume that $p \geqslant 3$ is an integer and X is a planar art gallery with h holes having the property that among any p points of X there are some 3 that can be seen from a single point. Then X can be guarded by at most $C(h, p)$ points, where $C(h, p)$ is a suitable number depending only on h and p.*

As usual, this theorem follows from the corresponding fractional Helly-type result (see Section 3 for a short discussion of the implication):

Theorem 1.9 (*Fractional Helly for planar galleries with h holes*). *For every $\alpha \in (0, 1)$ and every integer $h \geqslant 0$ there is a $\beta > 0$ such that the following holds. Assume X is a planar art gallery with h holes and $A \subset X$ is an n-point set such that at least $\alpha\binom{n}{3}$ of the triples of points of A have common guards. Then some point can guard at least βn points of A. Quantitatively, β can be bounded from below by a polynomial in α and h^{-1}.*

In Section 4, we prove Theorem 1.9 by induction on h, using the $h = 0$ case from [13] as a basis. The proof for $h = 0$ in [13] combines a theorem of Eckhoff on f-vectors of families of sets for which every intersection is contractible or empty, the fractional Helly theorem of Katchalski and Liu, and a topological Helly-type theorem of Molnár. Our induction step goes by an elementary geometric argument.

On the negative side, Valtr [18] constructed, for every $k \geqslant 1$, a simply connected art gallery X in $\mathbf{R}^3$ of Lebesgue measure 1, in which every point sees a set of volume at least $\frac{5}{9} + \delta$ for a certain small positive constant δ, and such that X cannot be guarded by k guards. Clearly, in such a gallery, among every $\lceil \frac{9}{5} q \rceil$ points, some q can be guarded by a single point, and, therefore, no (p, q)-theorem holds in dimension $d \geqslant 3$ with $p \geqslant \frac{9}{5} q$, even for simply connected galleries.

In Section 5, we show that (p, q)-theorems do not hold for planar galleries with an unlimited number of holes for $p \geqslant 2q - 1$.

In Section 6, we prove a simple positive result, a (p, q) theorem for an unlimited number of holes in every dimension, provided that q is larger than approximately $(d/(d + 1))p$:

Theorem 1.10 (*A (p, q)-theorem for galleries in $\mathbf{R}^d$*). *Assume that d, p and q are integers with $p > q \geqslant d\lceil p/(d + 1)\rceil + 1$. Assume X is an art gallery in $\mathbf{R}^d$ (with an arbitrary number of holes) containing at least p points and having the property that among any p points of X there are some q that can be seen from a single point. Then X can be guarded by at most $p - q + 2$ points.*

It would be interesting to close the gaps between the positive results ((p, q)-theorems) and negative results (counterexamples). For *planar* art galleries with an unlimited number of holes, we have (p, q) theorems with $(p, q) = (3, 3), (4, 4), (6, 5), (9, 7), (12, 9), \ldots$ (and all cases obtainable by decreasing p or increasing q while retaining the inequality $p \geqslant q$) by Theorems 1.7 and 1.10. On the other hand, we know that no (p, q)-theorems hold for $(p, q) = (5, 3), (7, 4), (9, 5), \ldots$ (and all cases obtainable by increasing p). We do not know, however, what happens for $(p, q) = (4, 3), (5, 4), (6, 4)$, etc. That is, for example, suppose that in a planar art gallery X, possibly with holes, some 3 out of every 4 points can be guarded by a single point. Does such an X admit a transversal of size bounded by a constant, independent of the number of holes? There are also unsolved cases in higher dimensions, most notably the gap between Theorem 1.10 and Valtr's example in $\mathbf{R}^3$.

2. Proof of Theorem 1.4

Here we prove the fractional Helly theorem with starshaped unions. Consider the hypergraph H whose vertices are the sets $C \in \mathscr{C}$ and whose edges are the $(d+1)$-tuples $\{C_1, \ldots, C_{d+1}\}$ of sets from $\mathscr{C}$ with $\bigcup_{i=1}^{d+1} C_i$ starshaped. This hypergraph has edge-density α, and so by a theorem of Erdős and Simonovits [10], for every integer $t \geqslant 1$ there is a constant $c = c(t, \alpha, d) > 0$ (independent of n) such that H contains at least

$$\lfloor cn^{t(d+1)} \rfloor$$

copies of the complete $(d+1)$-partite $(d+1)$-uniform hypergraph $K_{d+1}(t)$ which has t vertices in each color class. We assume t is sufficiently large (soon it will be fixed at $t = d(d+1)$).

Lemma 2.1. *Each such hypergraph $K_{d+1}(t)$ contains at least t/d vertices such that the corresponding convex sets have a point in common.*

Proof. Let $V_1, V_2, \ldots, V_{d+1}$ be the color classes of $K_{d+1}(t)$. If for each transversal $C_1 \in V_1, \ldots, C_{d+1} \in V_{d+1}$ the intersection $\bigcap_{i=1}^{d+1} C_i$ is nonempty, then by the colored Helly theorem of Lovász (see [5] or [16]), the sets in one of the classes have a point in common, and we are finished with the proof.

We now assume that some transversal $C_1, \ldots, C_{d+1}$ has an empty intersection, and let $k \leqslant d+1$ be the size of a smallest nonintersecting subset of $\{C_1, \ldots, C_{d+1}\}$. (Of course, $k \geqslant 2$.) To simplify notation we assume that this is the subset $\{C_1, \ldots, C_k\}$. Further, let $v \in \mathbf{R}^d$ be the center of the Euclidean ball B of minimum radius satisfying, for each $i = 1, 2, \ldots, k$,

$$B \cap C_i \neq \emptyset.$$

By a suitable translation we can make sure that $v = 0$, which makes notation simpler. We write r for the radius of B. Let b_i be the point of C_i closest to 0. The halfspace $H_i = \{x : \langle b_i, x - b_i \rangle \geqslant 0\}$ (with boundary passing through b_i and perpendicular to $[0, b_i]$) clearly contains C_i. The following claim describes some of the properties of the b_i.

Claim 2.2.

(i) *The b_i are all distinct and $\|b_i\| = r$.*
(ii) *The point* 0 *lies in the relative interior of* $\operatorname{conv}\{b_1, \ldots, b_k\}$.

Proof. If 0 were not in $\operatorname{conv}\{b_1, \ldots, b_k\}$, then its projection onto $\operatorname{conv}\{b_1, \ldots, b_k\}$ would be closer to each b_i which is impossible. Thus 0 lies in the relative interior of the convex hull of some subset of $\{b_1, \ldots, b_k\}$. We may suppose that 0 lies in the relative interior of $\operatorname{conv}\{b_1, \ldots, b_j\}$ with $j \leqslant k$. But this yields $\bigcap_1^j H_i = \emptyset$, implying, in turn, that $\bigcap_1^j C_i = \emptyset$. This contradicts the minimality of k unless $k = j$. This proves part (ii).

As for part (i), if $b_1 = b_2$, say, then $H_1 = H_2$, and so $\bigcap_1^k H_i = \bigcap_2^k H_i = \emptyset$. Thus $C_2, \ldots, C_k$ have no point in common, again contradicting the minimality of k. Assume, finally, that $\|b_1\| < r$ (say). Then one can easily check that $b_1, b_2, \ldots, b_k$ are all contained in a ball of radius strictly than r. Namely, it suffices to move the center of the ball a little, from 0 towards the point of $\operatorname{conv}\{b_2, b_3, \ldots, b_k\}$ that lies nearest to 0. $\square$

Now we return to the proof of Lemma 2.1. We distinguish two cases: when $k = d+1$ and when $k \leqslant d$.

First, we suppose $k = d+1$. Then $C = \bigcup_{i=1}^k C_i$ is starshaped; that is, there is a point $c \in C$ with $[c, x] \subset C$ for each $x \in C$. In particular, each $[c, b_i] \subset C$. Now C_i is contained in the halfspace H_i and Claim 2.2(ii) implies that these halfspaces do not have a point in common. In particular, c cannot be contained in all of them: $c \notin H_1$, say. Then $[c, b_1)$ is disjoint from C_1, and a small part of this segment near b_1 lies in B. Since we have $B \cap C_i = \{b_i\}$, this part cannot be contained in any of $C_2, \ldots, C_k$ either; a contradiction.

Assume now that $k \leqslant d$. For each choice of $D_{k+1} \in V_{k+1}, \ldots, D_{d+1} \in V_{d+1}$, the union of the transversal $C_1, \ldots, C_k, D_{k+1}, \ldots, D_{d+1}$ is starshaped, and thus there is point c such that the segment $[c, b_i]$ is contained in $C_1 \cup \cdots \cup C_k \cup D_{k+1} \cup \cdots \cup D_{d+1}$. Again, the halfspaces H_i for $i = 1, \ldots, k$ have no point in common. Thus c is not contained in one of the halfspaces, the jth, say. Then a small piece of the segment $[c, b_j]$ near b_j is contained in B, which is disjoint from $\bigcup_{i=1}^{k} C_i$. Thus one of the points $b_1, \ldots, b_k$ is contained in the union of $\bigcup_{i=k+1}^{d+1} D_i$ for every transversal $C_1, \ldots, C_k, D_{k+1}, \ldots, D_{d+1}$. Consequently, some b_j is contained in at least $t/k \geqslant t/d$ of the sets D_j. The lemma is proved. □

The proof of Theorem 1.4 is finished by averaging. We first choose t so large that $t/d \geqslant d+1$; namely, we let $t = d(d+1)$. Each copy of $K_{d+1}(t)$ in the hypergraph H contains an intersecting $(d+1)$-tuple. Thus at least $c(t, \alpha, d)n^{t(d+1)}$ of the sets of size $t(d+1)$ contain an intersecting $(d+1)$-tuple, and each such $(d+1)$-tuple appears in at most $c'(t, d)n^{t(d+1)-(d+1)}$ of them. So the number of intersecting $(d+1)$-tuples in $\mathscr{C}$ is a positive fraction of all $(d+1)$-tuples of $\mathscr{C}$, and the fractional Helly theorem 1.3 implies the result.

Remark. A method similar to the proof of Lemma 2.1 also yields a quick proof of Theorem 1.2. Indeed, assume $\bigcap \mathscr{C} = \emptyset$, and let $\{C_1, \ldots, C_k\}$ be the smallest size subfamily of $\mathscr{C}$ with no common point. By Helly's theorem $k \leqslant d+1$. Then the Euclidean ball B of minimum radius intersecting each C_i has properties as in Claim 2.2(i) and (ii), and the same argument shows that $\bigcup_{i=1}^{k} C_i$ is not starshaped.

3. (p, q)-theorems from fractional Helly theorems: the Alon–Kleitman method

Here we review the derivation of (p, q)-theorems from fractional Helly-type theorems, focusing on the particular two cases considered in this paper. (The argument can be found in a number of sources by now, but it seems simpler to repeat it briefly than to refer to the sources and point out the necessary modifications.)

We recall the fractional matching number and the fractional transversal number of a set system. First let $\mathscr{F}$ be a set system on a finite ground set X. The *fractional transversal number* $\tau^*(\mathscr{F})$ is the infimum of $\sum_{x \in X} f(x)$ over all functions $f: X \to [0, 1]$ such that $\sum_{x \in F} f(x) \geqslant 1$ for all $F \in \mathscr{F}$. The *fractional matching number* $\nu^*(\mathscr{F})$ is the supremum of $\sum_{F \in \mathscr{F}} \psi(F)$ over all functions $\psi: \mathscr{F} \to [0, 1]$ with $\sum_{F \in \mathscr{F}: x \in F} \psi(F) \leqslant 1$ for all $x \in X$. We have $\nu^*(\mathscr{F}) = \tau^*(\mathscr{F})$ for all $\mathscr{F}$, by the duality of linear programming or a similar argument (see [16] for a discussion in this setting, or [1] for a proof of an equivalent but differently phrased statement).

Now we assume that X is a compact metric space and $\mathscr{F}$ is a (possibly infinite) family of compact subsets of X. We can define $\nu^*(\mathscr{F})$ in the same way as above, with the additional condition that the considered functions ψ must be finitely supported (i.e., $\psi(F) \neq 0$ only for finitely many $F \in \mathscr{F}$). The fractional transversal number $\tau^*(\mathscr{F})$ can be defined as the infimum of $\mu(X)$ over all Borel measures μ on X with $\mu(F) \geqslant 1$ for all $F \in \mathscr{F}$. Then we again have $\nu^*(\mathscr{F}) = \tau^*(\mathscr{F})$; see [13].

Proof of Corollary 1.6 (*sketch*). We may assume $q = d+1$. Let $\mathscr{F}$ be a finite system of convex sets in $\mathbf{R}^d$ such that among every p sets, some $d+1$ have a starshaped union. It suffices to verify that $\nu^*(\mathscr{F})$ is bounded by a function of d and p, since $\tau^*(\mathscr{F}) = \nu^*(\mathscr{F})$ and, as noted by Alon and Kleitman, the transversal number of $\mathscr{F}$ is bounded by a function of $\tau^*(\mathscr{F})$ for any finite system of convex sets in $\mathbf{R}^d$.

Let $\psi: \mathscr{F} \to [0, 1]$ be an optimal fractional packing ($\sum_{F \in \mathscr{F}: x \in F} \psi(F) \leqslant 1$ for all x). Since ψ is an optimal solution to a linear program with integer coefficients, we may assume that the values of ψ are rational. We write $\psi(F) = m(F)/D$, where D and the $m(F)$ are integers. We form a new collection (multiset) $\mathscr{F}_m$ of sets by putting $m(F)$ copies of each $F \in \mathscr{F}$ into $\mathscr{F}_m$. Thus $N = |\mathscr{F}_m| = \sum_{F \in \mathscr{F}} m(F)$ and $\nu^*(\mathscr{F}) = N/D$.

Among every $p' = d(p-1)+1$ sets of $\mathscr{F}_m$, some $d+1$ have a starshaped union (if there are p distinct sets among our p' sets, then the assumption on $\mathscr{F}$ applies, and otherwise, some set is repeated at least $d+1$ times). So each p'-tuple of sets in $\mathscr{F}_m$ contributes a $(d+1)$-tuple with a starshaped union, and each $(d+1)$-tuple can be extended to a p'-tuple in $\binom{N-d-1}{p'-d-1}$ ways. So there are at least $\binom{N}{p'}/\binom{N-d-1}{p'-d-1}$ $(d+1)$-tuples with a starshaped union. This number is no less than $\alpha\binom{N}{d+1}$ for some $\alpha = \alpha(d, p) > 0$, and Theorem 1.4 yields a point x contained in at least βN sets of $\mathscr{F}_m$ for some $\beta = \beta(p, d) > 0$. Then

$$1 \geqslant \sum_{F \in \mathscr{F}: x \in F} \psi(F) = \sum_{F \in \mathscr{F}: x \in F} \frac{m(F)}{D} \geqslant \frac{1}{D} \cdot \beta N = \beta \nu^*(\mathscr{F}),$$

and so $\nu^*(\mathscr{F}) \leqslant 1/\beta$. □

Proof of Theorem 1.8 (*sketch*). Suppose that X is a planar art gallery with h holes such that among every p points, some 3 admit a single guard. For every $x \in X$, let $V(x)$ be the set of points of X visible from x, and let $\mathscr{V} = \{V(x): x \in X\}$. A transversal of $\mathscr{V}$ is exactly a set that guards all of X.

As shown in [13,17], the VC-dimension of $\mathscr{V}$ is bounded by a function of the number of holes h. The transversal number $\tau(\mathscr{F})$ of any set system $\mathscr{F}$ of finite VC-dimension is bounded by a function of the VC-dimension and $\tau^*(\mathscr{F})$ (see, e.g., [16]), so it suffices to bound $\tau^*(\mathscr{V})$. A double-counting argument as in the previous proof, this time using Theorem 1.9 instead of Theorem 1.4, shows that $v^*(\mathscr{V})$ is bounded by a function of p, and we have $\tau^*(\mathscr{V}) = v^*(\mathscr{V})$ as mentioned above. □

4. Proof of Theorem 1.9

Here we prove the fractional Helly theorem for planar art galleries with h holes. As was remarked in the introduction, the case $h = 0$ of the theorem is proved in [13], and it is also the starting point of our induction argument.

Write X^c for the complement of $X \subset \mathbf{R}^2$. If X is an art gallery with h holes, then X^c consists of $h + 1$ connected components, h of them bounded and one unbounded. Each component is simply connected.

Definition 4.1. A segment $[x, y]$ is called a *bridge* in X if $[x, y] \subset X$ and x and y belong to the boundary of distinct connected components of X^c.

Any art gallery with $h > 0$ has bridges. Let $A \subset X$ be the given set of n points. A *tripod* is a fourtuple $(x; a, b, c)$ of points such that $x \in X$, $a, b, c \in A$, and the segments $[x, a]$, $[x, b]$, $[x, c]$ are all contained in X. A tripod simply represents a triple of A that is guarded by x. For the proof of Theorem 1.9, we assume that there are $m = \alpha\binom{n}{3}$ tripods.

Proof of Theorem 1.9. We show that there is a bridge that is missed by at least γm tripods where $\gamma = \alpha^2/27^3$. With this bridge one can connect two distinct components of the complement of X (to arrive at an art gallery with one fewer holes) while keeping a positive fraction of the tripods.

We first observe that if $x, y \in \mathbf{R}^2$ with x and y belonging to distinct components of the complement of X, then $[x, y]$ contains a bridge $[x', y']$ with x and x' from the same component.

Now let C be a hole of X, and let S be a horizontal line and T a vertical line that intersect in C. By the above observation, S contains two bridges S_1, S_2 connecting C to other components of X^c, with S_1 to the left and S_2 to the right of C; see Fig. 1. Similarly, T contains two bridges T_1 and T_2, T_1 above C and T_2 below C. If any of these four bridges is disjoint from at least γm tripods, then we are finished. So each bridge is avoided by no more than γm of the tripods, and then at least $(1 - 4\gamma)m$ of the tripods meet each of the bridges. The lines S and T split the plane into four sectors. So at least one sector contains the center of at least $(\frac{1}{4} - \gamma)m$ of tripods that meet all of S_1, S_2, T_1, T_2.

Let $\mathscr{P}$ be the set of these tripods, and assume that their centers are in the top right sector. A tripod in $\mathscr{P}$ is of the form $(x; a, b, c)$, where a and b come in clockwise order around x, and the hole C comes between a and b in this order. Thus $[a, x]$ intersects S_2 and T_2, and $[b, x]$ intersects S_1 and T_1.

Let $E \subseteq A \times A$ be the set of all pairs (a, b) such that $\mathscr{P}$ contains at least $\delta(n - 2)$ tripods of the form $(x; a, b, c)$, where

$$\delta = \left(\frac{1}{4} - \gamma\right)\frac{\alpha}{6}.$$

We note that

$$|E| \geqslant \delta\binom{n}{2}.$$

Indeed, each pair (a, b) takes part in at most $n - 2$ tripods from $\mathscr{P}$, and the pairs not in E take part in fewer than $\delta(n - 2)$ tripods. Thus

$$\left(\frac{1}{4} - \gamma\right)\alpha\binom{n}{3} \leqslant |\mathscr{P}| < (n - 2)|E| + \delta(n - 2)\binom{n}{2}$$

and $|E| \geqslant \delta\binom{n}{2}$ follows.

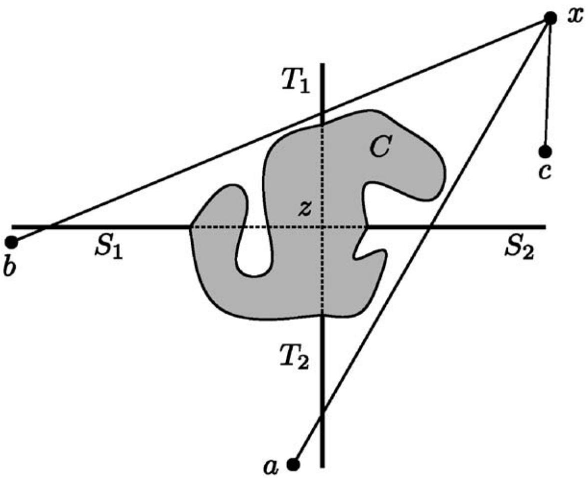

Fig. 1. A tripod of $\mathscr{P}$.

Let $z \in C$ be the intersection point of the lines S and T. For every tripod $(x; a, b, c) \in \mathscr{P}$, $[a, x]$ is below C and $[b, x]$ is above C, and a, z, b come in clockwise order around x. Consequently, a and b come in clockwise order around z as well. We consider the segment L connecting the endpoints (not lying in C) of S_1 and T_2. Let a^* be the intersection of L with the line spanned by z and $a \in A$. Each pair $(a, b) \in E$ is represented by the segment, or interval, $[a^*, b^*] \subset L$. There are $|E| \geqslant \delta\binom{n}{2}$ such intervals with at most n endpoints.

We use the following simple fact [4]: given $\delta\binom{n}{2}$ distinct open intervals on a line, having n endpoints altogether, there is a point common to at least

$$\frac{\delta^2}{4}\binom{n}{2}$$

of the intervals.

So in our case there is a point p common to the interior of $(\delta^2/4)\binom{n}{2}$ intervals of the form (a^*, b^*) with $(a, b) \in E$. Let $\mathscr{P}' \subseteq \mathscr{P}$ be the set of all tripods with $(a, b) \in E$ and such that the interval (a^*, b^*) is intersected by the ray starting at z and containing p. This ray contains a bridge M connecting C to another component of X^{c} and disjoint from all the tripods in $\mathscr{P}'$. This finishes the proof since

$$|\mathscr{P}'| \geqslant \delta(n-2) \cdot \frac{\delta^2}{4}\binom{n}{2} = \frac{3}{4}\left(\left(\frac{1}{4}-\gamma\right)\frac{\alpha}{6}\right)^3\binom{n}{3} \geqslant \gamma\alpha\binom{n}{3}. \qquad \square$$

This proof shows that a hole can be eliminated by reducing the number of tripods from $\alpha\binom{n}{3}$ to $(\alpha/27)^3\binom{n}{3}$. It gives the estimate $\beta = \Omega(\alpha^{3^h}/27^{3^{h+1}})$. Next, we present a better bound, polynomial in α and h^{-1}. However, we believe that the proper value of β is much smaller.

An improved quantitative bound. We first choose one point c_i, called *the center of* C_i, from the interior of the hole C_i, $i = 1, \ldots, h$. The three lines of the three segments of a tripod split the plane into six cells, and the centers c_i into six groups, one group in each cell. Two tripods are considered *equivalent* if their two triples of lines split the centers into the same six groups. There are $O(h^6)$ equivalence classes of tripods, and so we can select $\Omega(m/h^6)$ tripods that partition the centers into at most six classes.

Now we keep these $\bar{m} = \Omega(m/h^6)$ tripods. Thus $\bar{m} = \bar{\alpha}\binom{n}{3}$ where $\bar{\alpha} = \Omega(\alpha/h^6)$ The c_i lie in six cells $D_1, \ldots, D_6$. We will find a bridge between two holes (or rather two connected components of X^{c}) that avoids the remaining $\bar{m}$ tripods, until each cell contains at most one c_i.

Assume D_1 contains c_1 and c_2. The segment $[c_1, c_2]$ contains a bridge of the form $[x, y]$, where x is on the boundary of C_1 and y is on the boundary of another connected component of X^{c}. If this component is another hole C_3, then C_1

and C_3 are connected with the bridge $[x, y]$ that avoids the remaining $\bar{m}$ tripods. Thus the two holes C_1 and C_3 and the bridge $[x, y]$ form a new hole with new center c_3, and the hole C_1, together with its center c_1 has disappeared. (We note that c_3 may be in a cell different from D_1.) If this component is the unbounded one, then $[x, y]$ is a bridge between C_1 and the outer boundary of X that avoids the remaining $\bar{m}$ tripods. Again, the hole C_1 and its center c_1 have disappeared.

We can continue this bridging process as long as there are two c_i in the same cell. We end up with a new art gallery having at most six holes but still $\bar{m} = \bar{\alpha}\binom{n}{3}$ tripods. The previous proof finds a bridge while keeping $(\bar{\alpha}/27)^3\binom{n}{3}$ of the tripods. After six repetitions, we find a new art gallery without holes having at least

$$\Omega\left(\bar{\alpha}^{3^6}\right) = \Omega\left(\frac{\alpha^{729}}{h^{4374}}\right)$$

tripods. This improves on the earlier bound for all $h > 6$.

5. A counterexample to (p, q)-theorems for an unbounded number of holes

We begin by observing that visibility in planar art galleries with an unlimited number of holes can represent essentially any finite set system. That is, given a set system $\mathscr{F}$ on $[n]$, we can place n points $a_1, a_2, \ldots, a_n$ in a gallery and arrange holes so that the set $A_F = \{a_i \colon i \in F\}$ has a common guard iff $F \in \mathscr{F}$. Of course, there are some trivial restrictions that have to be observed: First, if A_F has a common guard then A_G has a common guard for all $G \subseteq F$. Second, it is impossible to prevent one-point subsets from having a common guard, and it is problematic to do so for two-point subsets. So the next lemma, which partially formalizes this observation, speaks about a system $\mathscr{F}$ of sets of size at least 3 and incomparable by inclusion.

Lemma 5.1. *Let $n \geqslant 2$ be an integer, let $\varepsilon > 0$ be a real number, and let $\mathscr{F} \subseteq 2^{[n]}$ be a system of subsets of $[n]$, each of size at least 3, such that no two sets in $\mathscr{F}$ are comparable by inclusion. Then there exist points $a_1, \ldots, a_n \in \mathbf{R}^2$ and sets $L_1, L_2, \ldots, L_n$ of lines with the following properties:*

- *Each a_i lies in the ε-neighborhood of the point $(i, 0)$.*
- *The lines of L_i all pass through a_i.*
- *We have $L_i = \{\ell_{i,F} \colon F \in \mathscr{F}, i \in F\}$.*
- *For every $F \in \mathscr{F}$, there is a point p_F lying in the ε-neighborhood of the point $(1, -1)$ such that the lines of $L = L_1 \cup \cdots \cup L_n$ passing through p_F are exactly $\ell_{i,F}$ with $i \in F$.*
- *No point different from the a_i and the p_F is common to more than two lines of L.*

Assuming this lemma, a fractional Helly theorem for art galleries with arbitrarily many holes in the plane (with the “fractional Helly number” $k \geqslant 3$) can be refuted as follows: let $\mathscr{F} = \binom{[n]}{k}$ be the system of all k-tuples on $[n]$, construct the a_i and the L_i as above, and let X be the union of the lines of $\bigcup_{i=1}^{n} L_i$ intersected with a sufficiently large ball. Then every k-tuple of the a_i has a common guard but no $k + 1$ of them have a common guard, which is an ultimate form of invalidity of a fractional Helly theorem.

Proof of Lemma 5.1. The construction goes by induction and it is quite simple. Let us call an intersection of the lines in L *proper* if it is not one of the a_i.

We suppose that $a_1, a_2, \ldots, a_{k-1}$ and $L_1, \ldots, L_{k-1}$ have already been constructed, such that for every $F \in \mathscr{F}$ with $|[k-1] \cap F| \geqslant 2$, the lines $\ell_{i,F}$ with $i \in [k-1] \cap F$ pass through a common point p_F lying very near to $(1, -1)$, and there are no other proper intersections of the lines of multiplicity greater than two. We choose a point a_k sufficiently near to $(k, 0)$ that lies on no line determined by two proper intersections of the lines in $L_1 \cup \cdots \cup L_{k-1}$. Then for every $F \in \mathscr{F}$ with $k \in F$, we choose a line $\ell_{k,F} \in L_k$. If $|F \cap [k-1]| \geqslant 2$, then p_F has already been defined, and we let $\ell_{k,F}$ connect a_k and p_F. Otherwise, we choose $\ell_{k,F}$ through a_k so that it passes near $(1, -1)$ and avoids all intersections of the lines in $L_1 \cup \cdots \cup L_{k-1}$. If $F \cap [k-1]$ consists of a single index j (and $k \in F$), then the intersection $\ell_{k,F}$ with $\ell_{j,F}$ becomes p_F. It is easy to check that the induction hypothesis holds for k as well, and thus the result of the construction is as claimed in the lemma. $\square$

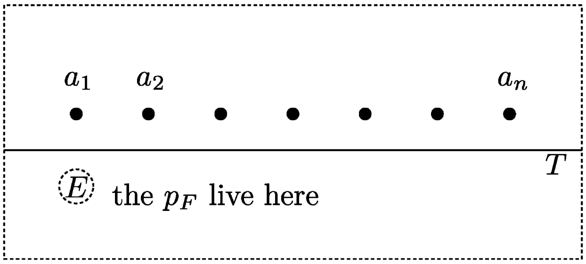

Fig. 2. The construction of X; a global view.

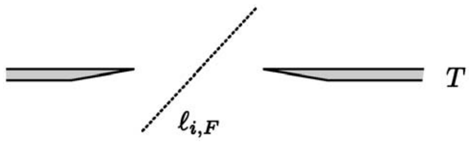

Fig. 3. Detail of an opening in T.

In order to refute a (p, q) theorem, we have to work harder, since we must make sure that for *every* p-tuple of points there are some q with a common guard, not only for p-tuples in a suitably chosen set A.

Proposition 5.2. *For every $q \geqslant 3$ and every m, there exists a planar art gallery X (a closed polygonal set with holes) such that for every set A of $p = 2q - 1$ points of X, there is a point of X that can see at least q points of A, and no m points of X together can see all of X (i.e., X cannot be guarded by m guards).*

Proof. We choose $n = qm + 1$ and let $\mathscr{F}$ consist of all q-element subsets of $[n]$. Then we apply the construction from Lemma 5.1 with a small $\varepsilon > 0$, obtaining points $a_1, \ldots, a_n$ and sets $L_1, \ldots, L_n$ of lines as in the lemma.

The construction of the gallery X is indicated in Fig. 2. The gallery is obtained from a large rectangle by deleting suitable holes. First we remove from X a very thin horizontal rectangle T as indicated in the picture. Then we choose a very small positive $\eta > 0$, and for every line $\ell_{i,F}$ in each L_i we make an opening of width η near the intersection of $\ell_{i,F}$ with T, as indicated in Fig. 3 (so the gallery now has many holes, namely, the remaining pieces of T).

For reasons to be soon seen, we need to introduce another auxiliary parameter, a large number K. The construction so far has the following property. For every n and K there exist positive numbers ε, η, δ satisfying the condition below:

> Let y be a point in the $(K\varepsilon)$-neighborhood E' of the point $(1, -1)$, and let G be the set of all indices $i \in [n]$ such that y can see some point in the δ-neighborhood of a_i. Then $|G| \leqslant q$.

In the last step of the construction, we add a very small hole H_i near a_i, for each $i \in [n]$, as shown in Fig. 4. In reality, H_i is very small compared to δ (which is hard to illustrate in a picture). The V-shaped opening on the bottom of H_i is such that, in the absence of any other hole, a_i can see just the ε-neighborhood E of $(1, -1)$ but no points left or right of it. Then the "visibility cones" of any two points among the a_i intersect only in E', the $(K\varepsilon)$-neighborhood of $(1, -1)$, for K sufficiently large in terms of n. Finally, the top of H_i is sufficiently flat so that all of it is visible from all points of the top side of the gallery. This finishes the description of the construction, and it remains to check that it has the desired properties.

First, the points $a_1, \ldots, a_n$ require at least $n/q > m$ guards. Indeed, anywhere except for E', no point can see more than one a_i, and within E' no point can see more than q of the a_i by the condition mentioned above.

Next, we consider arbitrary $p = 2q - 1$ points in X. We divide them into two groups:

1. those contained in the lower halfplane (below the top side of T) plus those contained in the regions $B_1, B_2, \ldots, B_n$ indicated in Fig. 4: B_i is the part of the δ-neighborhood of a_i below the dotted line (extending the bottom side of H_i) plus the V-shaped opening on the bottom of H_i; and
2. those contained in the upper halfplane (above the top side of T or on it) but not in any of the B_i.

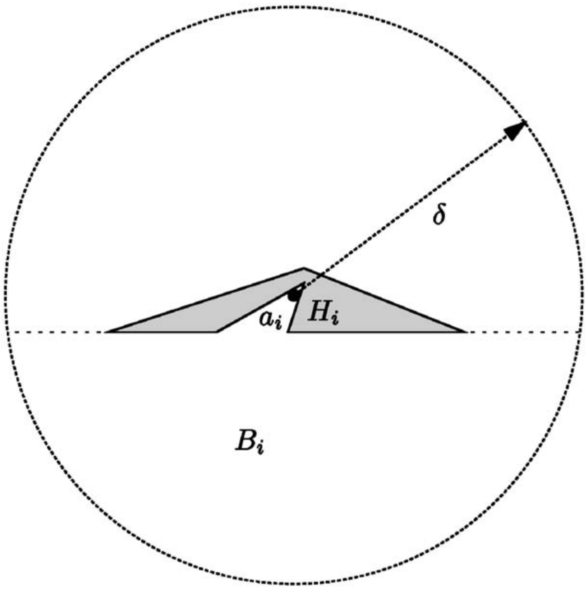

Fig. 4. Detail of the hole H_i near a_i.

One of these groups contains at least q points.

Any q points of group 1 can be seen from a single point. Indeed, all of the lower halfplane can be seen from any point of E. There are at most q distinct i such that one of the considered q points is in B_i, and by the construction, there is a point in E that can see all of these B_i.

All of group 2 (upper halfplane minus the B_i) can also be seen from a single point. For any particular point on the top side of the gallery, the holes H_i throw "shadows," which can be made arbitrarily narrow by making the H_i sufficiently small compared to δ. So by moving the observation point along the top side of X, we can find a position where the shadows avoid given q points of group 2. □

6. Proof of Theorem 1.10

This proof is similar to one by Hadwiger and Debrunner in [11] for the original (p,q)-problem under the condition $p(d-1)<(q-1)d$.

Assume $p=t(d+1)-r$ where t,r are positive integers with $0\leqslant r\leqslant d$. Then $q\geqslant td+1$.

Call a subset T from X *unguarded* if there is no point in X guarding all points of T. Let s be the maximal number of pairwise disjoint unguarded $(d+1)$-tuples from X and let $T_1,\dots,T_s$ be such a system of $(d+1)$-tuples. We claim that $s<t$. If not, then let T be the union of the first t $(d+1)$-tuples, $|T|=t(d+1)\geqslant p$. Every subset $S\subset T$ with q elements contains one of the $(d+1)$-tuples $T_1,\dots,T_t$, and no such S can be guarded by a single point form X, contradicting the (p,q)-property.

So $s\leqslant t-1$ and no $(d+1)$-tuple which is disjoint from $T:=\bigcup_1^s T_i$ is unguarded. That is, all $(d+1)$-tuples in $X'=X\setminus T$ have a guard in X. At this point Krasnosel'skiĭ's theorem (Theorem 1.7) almost applies: only X' is not compact and the guards may lie in X but not in X'. But the proof of Krasnosel'skiĭ's theorem applies with minute and straightforward modifications. We outline this modified proof later, for the reader's convenience and also because Krasnosel'skiĭ's neat proof deserves it.

So X' is guarded by a single point. We need guards for the points in T. Since $|T|\leqslant t(d+1)<p$ and since $|X|\geqslant p$ we choose a subset V from X' with $|T|+|V|=p$. By the (p,q) condition, $T\cup V$ contains a subset Q of size q that can be guarded from a single point. The points in $T\setminus Q$ can guard themselves; this is altogether

$$1+1+|T\setminus Q|\leqslant 1+1+|(T\cup V)\setminus Q|=2+p-q$$

guards for the whole art gallery. This proves Theorem 1.10. □

Proof of Krasnosel'skiĭ's theorem (*sketch and modifications*). First we consider the theorem itself, with any $d + 1$ points in a compact $X \subset \mathbf{R}^d$ visible from a single point. For $x \in X$, let $V(x)$ be the set of points visible from x. Then by Helly's theorem the sets $\mathrm{conv}(V(x))$ $(x \in X)$ have a point in common, say y. We show that $y \in V(x)$ for every $x \in X$. If not, then for some $x \in X$, the segment $[y, x]$ contains a subsegment $[u, z)$ disjoint from X but with $z \in X$. Let δ be the distance of u from X, and let v be on $[u, z]$ at distance $\delta/2$ from z. Now the distance of $[u, v]$ from X is attained on a pair (w, x_0) with $w \in [u, v]$ and $x_0 \in X$. It is easily checked that $w \neq u$. Now $V(x_0)$ is contained in the halfspace H whose boundary is orthogonal to $[w, x_0]$ and contains x_0. But H must contain y, and so the angle $ywx_0 < \pi/2$. Thus some point of $[u, v]$ is closer to x_0 than w, a contradiction proving Krasnosel'skiĭ's theorem.

In our proof of Theorem 1.10 above, the situation is similar but x_0 is only guaranteed to lie in X. But if we can find a point $x' \in X'$ close to x_0, then the same contradiction is reached. So we are done unless $x_0 \in X \setminus X'$ is isolated in X, and this case is easy to discuss. $\square$

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