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BORSUK'S THEOREM AND THE NUMBER OF FACETS OF CENTRALLY SYMMETRIC POLYTOPES

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1. Introduction

Let $C^n=\{x\in\mathbf{R}^n\colon |x_i|\leqq 1\ i=1, \dots, n\}$ be the n-dimensional cube and A be a d-dimensional subspace of $\mathbf{R}^n$ having no point in common with the $(n-d-1)$-dimensional faces of C^n. We want to find a lower bound on the number of vertices of the polytope $A\cap C^n$. More generally, given an n-dimensional centrally symmetric polytope K (whose center is at the origin) and a d-dimensional subspace $A\subset\mathbf{R}^n$, find lower bound on the number of vertices of $A\cap K$. We are going to prove two theorems concerning this question. These theorems have several interesting corollaries, for instance the following "lower bound"-type one. Every d-dimensional, centrally symmetric simplicial polytope has at least 2^d facets. (In fact this theorem is equivalent to our main result when $K=C^n$.)

This question was motivated by the following problem of Erdős [2]. Given $a_1, \dots, a_n\in\mathbf{R}^d$ vectors of at most unit length, at least how many of the 2^n vectors $\sum_{i=1}^n \varepsilon_i a_i$ ($\varepsilon_i=+1$ or -1) lie in the ball $\sqrt{d}\,B^d$, where B^d is the euclidean unit ball of $\mathbf{R}^d$. Erdős conjectured that this number is at least $c(d)2^n n^{-\frac{d}{2}}$ for some positive constant $c(d)$ depending only on d. This conjecture has been proved very recently by J. Beck [1]. In this paper we do not contribute to this problem because our results would imply only that the number in question is at least $2^{n-d}\Big/\binom{n}{d}$.

In the proofs we shall need Borsuk's theorem on antipodal maps. A continuous map $\varphi\colon S^n\to\mathbf{R}^m$ is said to be *antipodal* if $\varphi(-x)=-\varphi(x)$ for every $x\in S^n$.

BORSUK'S THEOREM. *If $m<n$, then there is no antipodal map $\varphi\colon S^n\to S^m$.*

This theorem is equivalent to the following.

If $\varphi\colon S^n\to\mathbf{R}^n$ is an antipodal map, then there exists an $x\in S^n$ with $\varphi(x)=0$.

We shall prove the following extension of Borsuk's theorem.

If $\varphi\colon S^n\to S^m$ is antipodal, then the n-dimensional measure of $\varphi(S^n)$ is not less than the (n-dimensional) measure of S^n.

2. Notation and results

Let K be a convex polytope in $\mathbf{R}^n$. The *support* of $x\in K$ is defined as the minimal face of K containing x. A face is understood to be closed. If x lies in ∂K, the boundary of K, then $t(x)=t(x, K)$ denotes the set of outer normals of unit length to K at x. It is clear that $t(x)\subset S^{n-1}$ is nonempty. The set $t(x)$ consists of one point if the boundary of K is smooth at x. The d-dimensional outer angle of K at x $(d=1,2,\dots,n)$ is defined as

$$\alpha_d(x,K)=\frac{\lambda_{d-1}(t(x))}{\lambda_{d-1}(S^{d-1})},$$

where λ_{d-1} is the $(d-1)$-dimensional Lebesgue measure in $\mathbf{R}^n$ and S^{d-1} is supposed to be isometrically imbedded into $\mathbf{R}^n$. Obviously,

$$\alpha_d(x,K)=\begin{cases}0, & \text{if the support of } x \text{ is more than } (n-d)\text{-dimensional,}\\ \infty, & \text{if the support of } x \text{ is less than } (n-d)\text{-dimensional.}\end{cases}$$

Let $\mathscr{A}^{(d)}$ denote the set of d-dimensional subspaces of $\mathbf{R}^n$. We shall consider sections of type $A\cap K$ where $K\subset\mathbf{R}^n$ is a centrally symmetric n-dimensional polytope (with center at the origin) and $A\in\mathscr{A}^{(d)}$. A section $A\cap K$ is called *regular* if A has no point in common with the $(n-d-1)$-dimensional faces of K.

THEOREM 1. *Let K be a centrally symmetric, n-dimensional polytope and $A\in\mathscr{A}^{(d)}$. Then*

(1)
$$\sum_{x\in\operatorname{vert}(A\cap K)}\alpha_d(x,K)\geqq 1$$

where $\operatorname{vert}(A\cap K)$ *is the set of vertices of* $A\cap K$.

COROLLARY 1. *If $A\cap K$ is a regular section, then*

$$|\operatorname{vert}(A\cap K)|\geqq\frac{1}{\alpha_d(K)},$$

where $\alpha_d(K)=\max\{\alpha_d(x,K)$: *the support of x is $(n-d)$-dimensional*$\}$.

COROLLARY 2. *Any regular, d-dimensional section of C^n has at least 2^d vertices.*

COROLLARY 3. *Any d-dimensional, centrally symmetric, simplicial polytope has at least 2^d facets.*

COROLLARY 4. [cf. Erdős, Beck]. *If $a_1,\dots,a_n\in B^d$, then at least $2^{n-d}\Big/\binom{n}{d}$ vectors out of the 2^n vectors $\sum_{i=1}^{n}\varepsilon_i a_i$ $(\varepsilon_i=+1$ or $-1)$ lie in the ball $\sqrt{d}\,B^d$.*

Let $\mathscr{L}^{(n-d)}=\mathscr{L}^{(n-d)}(K)$ be the set of all $(n-d)$-dimensional faces of K. To present our next theorem we define a map $\varphi\colon S^{n-d}\to\operatorname{skel}_{n-d}K$ to be *special* if

(i) φ is antipodal
(ii) for each $L\in\mathscr{L}^{(n-d)}$ either $L\subset\varphi(S^{n-d})$ *or* $\operatorname{int}L\cap\varphi(S^{n-d})=\emptyset$.

Here $\operatorname{int}L$ denotes the relative interior of the face L.

We mention that some projections $\pi\colon\mathbf{R}^n\to A$ (where $A\in\mathscr{A}^{(n-d+1)}$) induce a special map $\varphi_\pi\colon S^{n-d}\to\operatorname{skel}_{n-d}K$ in a natural way. Suppose that π is a projection such that the image of every $L\in\mathscr{L}^{(n-d+1)}$ is $(n-d+1)$-dimensional. Then π, restricted to K is one-to-one on every face $L\in\mathscr{L}^{(n-d+1)}$. On the other hand, $\pi(K)$ is a convex polytope whose boundary is the "same" as S^{n-d}, and π has an inverse on this boundary. Denoting this inverse by φ_π we have the induced special map $\varphi_\pi\colon S^{n-d}\to \operatorname{skel}_{n-d} K$.

Our next theorem gives a lower bound on the number of vertices of a regular section of K through the following discrete linear program.

$$(2)\qquad \begin{cases} \text{minimize} & \sum_{L\in\mathcal{F}^{(n-d)}} x(L) \\ \text{subject to} & x(L)=0 \text{ or } 1 \qquad (\forall L), \\ & x(L)=x(-L) \qquad (\forall L), \\ & \sum_{\substack{L\in\mathcal{F}^{(n-d)}\\ L\subset\varphi(S^{n-d})}} x(L)\geqq 2 \quad (\forall \varphi \text{ special}). \end{cases}$$

Denote the minimum of this problem by M. In other words, M is the minimum size of a centrally symmetric set of $(n-d)$-faces of K meeting all special images of S^{n-d}.

THEOREM 2. *Every regular section of a centrally symmetric n-dimensional polytope K has at least M vertices.*

Corollaries 2, 3 and 4 follow from this theorem as well. Moreover we can sharpen Corollary 2 (and, similarly Corollary 3):

COROLLARY 2'. *Any regular d-dimensional section of C^n has at least 2^d vertices. Equality holds if and only if the section is a d-dimensional parallelepiped.*

Further we have

COROLLARY 5. *Every d-dimensional regular section of the d-dimensional octahedron has exactly $2\binom{n}{d-1}$ vertices.*

COROLLARY 6. *Every 2-dimensional regular section of the dodecahedron (icosahedron) has at least 6 (resp. 10) vertices.*

The proof of Theorem 1 will be based on the following extension of Borsuk's theorem.

THEOREM 3. *If $\varphi\colon S^k\to S^n$ is an antipodal map, then $\lambda_k(\varphi(S^k))\geqq\lambda_k(S^k)$. Here λ_k is the k-dimensional Lebesgue measure (both in $\mathbf{R}^{k+1}$ and $\mathbf{R}^{n+1}$) normalized so that $\lambda_k(S^k)$ equals the k-dimensional mesaure of any copy of S^k isometrically imbedded into S^k.*

Let us mention two open problems: The first one arises from an attempt to find an alternative proof of Theorem 3. Let $K\subset\mathbf{R}^n$ be a symmetric convex polytope and $\varphi\colon \operatorname{vert} K\to\mathbf{R}^m-\{0\}$ such that for every vertex v, if $v_1, \dots, v_r$ are the neighbours of v then there exist coefficients $\lambda_1, \dots, \lambda_r>0$ such that

$$\varphi(v)=\lambda_1\varphi(v_1)+\dots+\lambda_r\varphi(v_r).$$

Then we conjecture that $\varphi(\operatorname{vert} K)$ lies in an n-dimensional subspace of $\mathbf{R}^m$. This conjecture would imply Theorem 3.

To present the second problem write $f_k(P)$ for the number of k-dimensional faces of the polytope P. Suppose P is symmetric, simple and d-dimensional with

$2n$ facets. The lower bound theorem would say that $f(P)$ is not less than a function of d, n and k. An obvious guess for that function is

$$f_0(P) \geqq 2^d+2(n-d)(d-1),$$

$$f_k(P) \geqq 2^{d-k}\binom{d}{k}+2(n-d)\binom{d}{k+1} \quad \text{for} \quad 1 \leqq k \leqq d-1.$$

This is supported by a kind communication of P. McMullen [4]. If the guess is correct, the minimal polytopes would be obtained from the cube by successive centrally symmetric truncations of vertices.

3. Proofs

PROOF OF THEOREM 1. Let us choose an $\varepsilon>0$ such that if L is a face of K and $A\cap L=\emptyset$, then $A\cap(L+\varepsilon B^n)=\emptyset$. Such an ε exists because each face of K is compact.

Put now $K_\varepsilon=K+\varepsilon B^n$ and let S^{d-1} be the unit sphere of the subspace A. The map $\pi\colon A\cap\partial K\to S^{d-1}$ defined by $\pi(y)=\frac{y}{\|y\|}$ is one-to-one and antipodal. We define a map $\varphi\colon S^{d-1}\to S^{n-1}$ by $\varphi(z)=t(\pi^{-1}(z), K_\varepsilon)$. Since K_ε is smooth at every point of its boundary, φ is well defined, continuous and antipodal. Theorem 3 then implies

$$\lambda_{d-1}(S^{d-1}) \leqq \lambda_{d-1}(\varphi(S^{d-1})) = \lambda_{d-1}(t(A\cap\partial K_\varepsilon, K_\varepsilon)).$$

Claim. $t(A\cap\partial K_\varepsilon, K_\varepsilon)\subseteqq \bigcup t(\operatorname{int} L, K)$, where the union is taken over all faces L of K with $L\cap A\neq\emptyset$.

Suppose $z\in t(y, K_\varepsilon)$ for some $y\in A\cap\partial K_\varepsilon$. Then $y=x+\varepsilon z$ where $x\in\partial K$ and $z\in t(x, K)$, as one can check easily. Write L for the support of x (in K), then $x\in\operatorname{int} L$ and $z\in t(\operatorname{int} L, K)$. All we have to show is that $L\cap A\neq\emptyset$. Suppose that $L\cap A=\emptyset$, then by the choice of ε, $A\cap(L+\varepsilon B^n)=\emptyset$, too. But $y\in A$ and $y=x+\varepsilon z\in L+\varepsilon B^n$, a contradiction.

From this we have

$$\lambda_{d-1}(S^{d-1}) \leqq \lambda_{d-1}(t(A\cap\partial K_\varepsilon, K_\varepsilon)) \leqq \sum_{L\cap A\neq\emptyset} \lambda_{d-1}(t(\operatorname{int} L, K)).$$

Clearly $\lambda_{d-1}\,(t(\operatorname{int} L, K))=0$ if $\dim L>n-d$. Suppose $A\cap K$ a regular section, then $L\cap A=\emptyset$ for every face L with $\dim L<n-d$. Thus

$$1 \leqq \sum_{\substack{L\cap A\neq\emptyset\\ L\in\mathscr{L}^{(n-d)}}} \frac{\lambda_{d-1}(t(\operatorname{int} L, K))}{\lambda_{d-1}(S^{d-1})} = \sum_{x\in\operatorname{vert} A\cap K} \alpha_d(x, L),$$

because $t(\operatorname{int} L, K)$ coincides with $t(x, K)$ for every $x\in\operatorname{int} L$ and $L\cap A=\emptyset$ for some $L\in\mathscr{L}^{(n-d)}$ if and only if $A\cap L$ is a vertex of $A\cap K$.

Finally, if $A\cap K$ is not a regular section, then some member of the left hand side of (1) equals $+\infty$.

Corollary 1 is an immediate consequence.

PROOF OF COROLLARY 2. It is easy to see that $\alpha_d(x, C^n)=2^{-d}$ if the support of x is $(n-d)$-dimensional. Using Corollary 1 this fact implies the result.

PROOF OF COROLLARY 3. It is easy to check and actually well known [3] that every d-dimensional, centrally symmetric and simple polytope is a regular section of C^n for some n. So Corollary 2 says that every d-dimensional, centrally symmetric and simple polytope has at least 2^d vertices. Dualizing this statement we get Corollary 3.

Here we mention that Corollary 2 does not hold for non-regular sections. This follows from the fact that every d-dimensional, symmetric polytope with $2n$ facets is a section of C^n. For instance, the d-dimensional octahedron is a (non-regular) section of $C^{2^{d-1}}$ and it has only $2d$ vertices.

PROOF OF COROLLARY 4. We may clearly suppose that the vectors $a_1, \dots, a_n \in B^d$ are in general position, say their entries are algebraically independent over the rationals. Put

$$A = \left\{x \in \mathbf{R}: \sum_{i=1}^{n} x_i a_i = 0\right\} \in \mathscr{A}^{(n-d)}.$$

$P = A \cap C^n$ is a regular section because the points $a_1, \dots, a_n$ are in general position. By Corollary 2, $|\mathrm{vert}\, P| \geqq 2^{n-d}$. To each vertex x^0 of P there corresponds a sign sequence $\varepsilon_1, \dots, \varepsilon_n$ such that $\varepsilon_i = x_i^0$ if $|x_i^0| = 1$ and $\left\|\sum_1^n \varepsilon_i a_i\right\| \leqq \sqrt{d}$. This is a simple geometric fact the proof of which is left to the reader. On the other hand any sign sequence can correspond to at most $\binom{n}{d}$ vertices of P. (One can slightly improve this bound, but it would not influence the order of magnitude. It is easy to construct an example where a sign sequence corresponds to $\binom{n-1}{d}$ vertices of P.) This shows that at least $2^{n-d} \Big/ \binom{n}{d}$ vectors out of the 2^n vectors $\sum_{i=1}^{n} \varepsilon_i a_i$ $(\varepsilon_i = \pm 1)$ lie in the ball $\sqrt{d} B^d$.

PROOF OF THEOREM 2. Suppose that $A \in \mathscr{A}^d$ and that the section $A \cap K$ is regular. For $L \in \mathscr{L}^{(n-d)}$ put

$$x_A(L) = \begin{cases} 1 & \text{if } A \cap L \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\sum_{L \in \mathscr{L}^{(n-d)}} x_A(L) = |\mathrm{vert}\, A \cap K|$. We show $x_A(L)$ satisfies the conditions of the discrete linear program (2). All we have to check is the condition

$$\sum_{\substack{L \subseteqq \varphi(S^{n-d}) \\ L \in \mathscr{L}^{(n-d)}}} x_A(L) \geqq 2 \tag{3}$$

for each special map $\varphi: S^{n-d} \to s\,\mathrm{kel}_{n-d} K$. Now let φ be a special map, then, for $L \subseteqq \varphi(S^{n-d})$ $x_A(L) = 1$ iff $L \cap A \neq \emptyset$. So (3) holds iff $A \cap \varphi(S^{n-d})$ consists of at least two pints. Consider the orthogonal complement, $A^\perp$, of A and let $\pi: \mathbf{R}^n \to A^\perp$ be the orthogonal projection. Since φ is antipodal, $A \cap \varphi(S^{n-d})$ contains two points iff $0 \in \pi \circ \varphi(S^{n-d})$. But $\pi \circ \varphi: S^{n-d} \to A^\perp$ $(\cong \mathbf{R}^{n-d})$, so by Borsuk's theorem there exists a $z \in S^{n-d}$ with $\pi \circ \varphi(z) = 0$.

Corollary 2 follows from Theorem 2 as well. In order to see this take the special map $\varphi\colon S^{n-d}\to \mathrm{s\,kel}_{n-d}\,C^n$ which is induced by some projection and consider the set of special maps $\{g\circ\varphi\colon g\in G\}$ where G is the group generated by the reflections of C^n. Clearly $L\subseteqq g\circ\varphi(S^{n-d})$ for exactly 2^{n-d+1} elements $g\in G$ (for each fixed $L\in\mathscr{L}^{(n-d)}$) and $|G|=2^n$. So summing up the inequalities

$$\sum_{L\subseteqq g\circ\varphi(S^{n-d})} x_A(L) \geqq 2$$

for every $g\in G$ we get $\sum_L x_A(L)\geqq 2^d$. This implies $M\geqq 2^d$. The same method gives Corollary 2′ as well. Indeed, if the set $\{L\in\mathscr{L}^{(n-d)}\colon L\cap A\neq\varnothing\}$ contains two faces, L_1 and L_2 that are not parallel, then one can find a special map φ (induced by same projection) so that both $L_1, L_2\subseteqq\varphi(S^{n-d})$. Consequently

$$\sum_{L\subseteqq\varphi(S^{n-d})} x_A(L) \geqq 4 > 2.$$

This implies $M>2^d$.

To see that Corollary 5 holds we use the method of proof of Theorem 2. The $(n-d+1)$-dimensional subspace $x_{i_1}=\ldots=x_{i_{d-1}}=0$ $(1\leqq i_1<i_{d-1}\leqq n)$ intersects the octahedron

$$O^n = \left\{x\in\mathbf{R}^n\colon \sum_{i=1}^{n}|x_i| \leqq 1\right\}$$

in an $(n-d+1)$-dimensional octahedron $O^{n-d+1}_{i_1,\ldots,i_{d-1}}$ whose boundary is clearly the image of a special map $\varphi\colon S^{n-d}\to\mathrm{s\,kel}_{n-d}\,O^n$. Since the section $A\cap O^n$ is regular and $O^{n-d+1}_{i_1,\ldots,i_{d-1}}$ lies in a subspace,

$$\sum_{L\in\varphi(S^{n-d})} x_A(L) = 2.$$

Summing up these equalities for each such φ we get

$$|\mathrm{vert}\, A\cap O^n| = \sum_{L\in Z^{(n-d)}} x_A(L) = 2\binom{n}{d-1},$$

because every $L\in Z^{(n-d)}$ lies on the boundary of exactly one octahedron $O^{n-d+1}_{i_1,\ldots,i_{d-1}}$.

We mention that Corollary 1 does not imply Corollary 5 (for $n\geqq 4$ and $d=2$ for instance). And in general, Theorem 2 seems to be stronger than Theorem 1.

Corollary 6 can be proven using a suitable set of special maps.

PROOF OF THEOREM 3. We can suppose that $n\geqq k$. We are going to use the following formula which is a consequence of the Fubini theorem. If $X\subseteqq S^n$ is λ_k measurable, then

$$\lambda_k(X) = \int_{\mathscr{A}} |X\cap A|\, d\mu \tag{4}$$

where μ is the invariant measure on the set $\mathscr{A}$ of all $(n+1-k)$-dimensional subspaces of $\mathbf{R}^{n+1}$, normalized suitably. Applying this formula to $X=\varphi(S^k)$,

$$\lambda_k(\varphi(S^k)) = \int |\varphi(S^k)\cap A|\, d\mu \geqq \int 2 d\mu,$$

because $|\varphi(S^k)\cap A|\geqq 2$ for every $A\in\mathscr{A}$ as we have seen in the proof of Theorem 2. Let φ_0: $S^k\to S^n$ be an isometric imbedding of S^k into S^n. Then $|\varphi_0(S^k)\cap A|=2$ for μ-almost every $A\in\mathscr{A}$. Applying (4) again with $X=\varphi_0(S^k)$

$$\lambda_k(\varphi_0(S^k)) = \int 2d\mu,$$

and this proves the theorem.

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