Accompanying PROBLEMS for Additive Combinatorics

Many of the following problems are simply small details of some proofs of the Lecture Notes, others are simply useful to practice notions. There are more involved problems showing the limitations of some results, and finally there are a few challenging problems for fun as well.

2.1 Find $A+A$ and $A-A$ in the following cases.

$$A=\{0,1,3\}, \quad A=\{0,2,3,4,7,11,12,14\}.$$

These are examples for $|A+A|<|A-A|$ and $|A+A|>|A-A|$ respectively.

2.2 Show that $A-A=\{0\}$ implies $|A|=1$.

2.3 Show that $k\cdot A\subset kA$, where k is a positive integer, and A is a finite set of real numbers. Describe all cases when $k\cdot A=kA$.

2.4 Let A be a finite set of real numbers, and $k\geq 2$ be an integer. Prove that $|kA|\leq \binom{|A|+k-1}{k}$, and this bound is sharp.

2.5 Let A and B be finite sets of positive real numbers. Prove that $|A|+|B|-1\leq |A\cdot B|\leq |A||B|$, and prove that there is equation in the lower bound iff both A and B are geometric sequences of the same quotient. Give an example to show that the upper bound is also sharp.

2.6 Let A and B be finite sets of real numbers. Prove that $|A+B|=|A||B|$ iff $|A-B|=|A||B|$. Prove also that, in case of $0\notin B$ we have

$$\sum_x r_{A\cdot B}(x)^2=\sum_y r_{A/B}(y)^2.$$

2.7 Let $A=\{0,2,3,6\}$. Compute kA for all $k\geq 1$.

2.8 Prove that

$$|A-A|\leq |A+A|^{3/2}.$$

Hint: This follows from theorems proved in the Lecture Notes.

2.9 Let A be a finite subset of an additive group (real numbers, say), k be a positive integer, and define

$$A^{\oplus k} = \overbrace{A \times \cdots \times A}^{k} = \{(a_1, \dots, a_k) : a_j \in A\}.$$

Addition on $A^{\oplus k}$ is defined coordinatewise. Prove the following statements.

$$|A^{\oplus k}| = |A|^k,$$
$$A^{\oplus k} \pm B^{\oplus k} = (A \pm B)^{\oplus k},$$
$$A_1^{\oplus k} \pm \cdots \pm A_\ell^{\oplus k} = (A_1 \pm \cdots \pm A_\ell)^{\oplus k}.$$

This is a very important argument, the *tensor power trick*. On one hand this makes possible to construct large examples from small examples. On the other hand this can be used to eliminate unnecessary constants (see later in the Lecture Notes).

2.10 Let A be a subset of integers. Construct an injective map $\Phi : A^{\oplus k} \to \mathbb{Z}$ such that $\Phi(\mathbf{a}) \pm \Phi(\mathbf{b}) = \Phi(\mathbf{a}') \pm \Phi(\mathbf{b}')$ iff $\mathbf{a} \pm \mathbf{b} = \mathbf{a}' \pm \mathbf{b}'$. Hint: Consider the elements of $A^{\oplus k}$ as the digits of an integer on a well chosen base. (This problem shows that $\mathbb{Z}^k$ is not really richer than $\mathbb{Z}$ itself with respect to addition.)

2.11 Prove that there are arbitrarily large sets A of integers such that

$$|A + A| \geq |A - A|^{\Theta}, \quad \Theta = \frac{\log 26}{\log 25} > 1.$$

Hint: Start from the example of problem 2.1, than use problem 2.9, the power tensor trick, and finally problem 2.10 to go back to the integers. (This shows that $|A + A| \leq C|A - A|$ cannot be true with any constant $C > 0$, however large it is.)

3.2 Let $k \geq 2$ be an integer. If $A \subset \{1, \dots, N\}$ satisfies that all k–terms sums from A are different, that is $a_1 + \cdots + a_k = a_1' + \cdots + a_k'$ implies that $\{a_1, \dots, a_k\} = \{a_1', \dots, a_k'\}$, then

$$|A| \leq (k \cdot k! \cdot N)^{1/k}.$$

3.3 Prove the following inequalities (they are used in the Lecture Notes)

$$(1 + x)^\alpha \leq 1 + \alpha x, \text{ if } x \geq -1,\ 1 \geq \alpha \geq 0,$$
$$(1 + x)^\alpha \geq 1 + \alpha x, \text{ if } x \geq 0,\ \alpha \geq 1.$$

3.4 Let $k \geq 2$ be an integer and prove that there is an infinite set of positive integers A, such that all k–terms sums from A are different (as in Problem 3.2) and

$$A(x) =: \sum_{\substack{a \leq x \\ a \in A}} 1 > x^{\frac{1}{2k-1}}.$$

3.5 Show that there is an infinite Sidon set A of positive integers such that

$$\limsup_{x\to\infty} \frac{A(x)}{\sqrt{x}} > 0.$$

(Hint: in the Lecture Notes we constructed a Sidon set of $p-1$ elements up to $2p^2$, realize that in this set $2p < a_1 < a_2 < \cdots < a_{p-1} < 2p^2$ and also $a_{k+1} - a_k > p$. For a sufficiently sparse set of primes, you can join together these sets.)

3.6 Prove that the construction of Theorem 2.3 can be extended by putting p^2 into the set, such getting a Sidon set of $p+1$ elements up to p^2. (Notice that this provides a Sidon set with $\sqrt{N}+1$ elements up to N, though for special N only, and so the right hand side of Theorem 2.1 cannot be simplified to $\sqrt{N}$. It is an open question if every Sidon subset of $\{1,\dots,N\}$ has $\le \sqrt{N}+C$ elements, with some absolute constant C.)

3.7 Let p be a (large) prime, $k \ge 2$ be an integer. Fix a primitive root $g \pmod p$. Show that the set

$$A = \{1 \le m < p \,:\, g^m \equiv q \pmod p,\ q \le p^{1/k},\ q \text{ prime}\}$$

satisfies that all k–terms sums from A are different (as in Problem 3.2). By the prime number theorem, formula (2.1) in the lecture notes, this provides a set of $(1-\epsilon)kN^{1/k}/\log N$ elements up to N with the given property. (Compare to Problem 3.2.)

3.8 In class we constructed a Sidon set of $p-1$ elements up to $2p^2$. Generalize that construction to improve the statement of Problem 3.7, at least for $k=3$. (Hint:

$$a_\ell = 9p^2\ell + 3ps_\ell + t_\ell, \text{ where } s_\ell \equiv \ell^2 \pmod p,\ t_\ell \equiv \ell^3 \pmod p,\ 1 \le \ell, s_\ell, t_\ell \le p-1.)$$

3.9 Generalize the construction of Theorem 2.3 to get a set A of p elements up to p^k-1, such that all k–terms sums from A are different (as in Problem 3.2). (Hint: you must understand the structure of $\mathbb{F}_{p^k}$.)

4.1 Give an example for $A \subset \mathbb{Z}_m$ for which $|A+A| = |A| < 2|A|-1$. (You choose m as well.)

4.2 Let $p > 2$ be an odd prime, and $(ab,p) = 1$. Prove that

$$\{ax^2 + by^2 \,:\, x, y \in \mathbb{Z}_p\} = \mathbb{Z}_p.$$

4.3 Let G be a finite (non abelian) group, and let $A, B \subset G$. Prove that if $|A| + |B| > |G|$ then $A + B = G$.

4.4 $A \subset \mathbb{Z}_p$ is a k–term arithmetic progression if there are residue classes $a, d \in \mathbb{Z}_p$ such that $A = \{a + \ell d : 0 \le \ell < k\}$. Prove that if $A \subset \mathbb{Z}_p$ is an arithmetic progression then so is $\mathbb{Z}_p \setminus A$.

4.5 Let $\emptyset \neq A, B \subset \mathbb{Z}_p$. Prove that if

$$|A+B| = |A| + |B| - 1 = p - 1$$

then there is a unique $c \in \mathbb{Z}_p \setminus (A+B)$ such that

$$B = \mathbb{Z}_p \setminus (c - A).$$

4.6 Let $A \subset \mathbb{Z}_p$, $|A| \geq 2$ and $B = \{0, 1, \ldots, k-1\}$ with $|B| = k \geq 2$. Prove that if

$$|A+B| = |A| + |B| - 1 < p - 1$$

then A is an interval as well (an arithmetic progression with difference 1).

4.7 Let $A, B \subset \mathbb{Z}_p$ such that $|A|, |B| \geq 2$ and $|A+B| = |A| + |B| - 1 < p - 1$. Prove that if B is an arithmetic progression then A is an arithmetic progression with the same difference.

4.8 Recall the division algorithm for polynomials over a field (for all $f(x), g(x)$ there are $q(x), r(x)$ such that $f(x) = q(x)g(x) + r(x)$, where deg $r(x) <$ deg $g(x)$). Use this to give a simple proof to Lemma 3.8.

4.9 Let $\emptyset \neq A, B \subset \mathbb{Z}_p$ and let $C = \{a + b : a \in A,\, b \in B,\, ab \not\equiv 1 \pmod{p}\}$ Prove that

$$|C| \geq \min(p,\, |A| + |B| - 3).$$

4.10 Let $m \geq 2$ be an integer, $(u - v, m) = 1$ be two integers and A be the (multi)set consists of $m-1$ copies of u and $m-1$ copies of v. Prove that no m elements of A add up to a multiple of m.

4.11 Let p be a prime, and $A = \{0 \leq a_1 \leq a_2 \leq \cdots \leq a_{2p-1} \leq p-1\}$ be a (multi)set of integers (or residue classes mod p, ordered according to their least non negative representative). Prove that you can always select p elements of A such that they add up to a multiple of p. (Hint: first consider if there is an i such that $a_i = a_{i+p-1}$, next use the Cauchy–Davenport theorem to the sets $\{a_i, a_{i+p-1}\}$ for $i = 1, \ldots, p-1$.)

4.12 Let $m \geq 2$ be an integer, and A be a (multi)set of $2m-1$, not necessarily different integers. Prove that you can always select m elements of A such that they add up to a multiple of m. (This is the so called Erdős–Ginzburg–Ziv theorem. Problem 4.10 shows that you cannot relax the size of A in general.)

5.1 Show that Theorem 4.1 (van der Waerden) implies Theorem 4.6 (Grünwald), that is the latter is not more general, than the former.

5.2 We say that B is an *affine cube* of real numbers with dimension d, if there are real numbers $a_0, a_1, \ldots, a_d$ such that

$$B = \{a_0 + \epsilon_1 a_1 + \cdots + \epsilon_d a_d : \epsilon_j = 0, 1 \text{ for all } j\}.$$

If all of these elements are different then $|B| = 2^d$. Prove that if $|B| < 2^d$ then B contains a 3–term arithmetic progression. (An affine cube is a generalized arithmetic progression of length 2 in every direction, and $|B| < 2^d$ means, it is not proper.)

5.3 Let $A = \{a_1 < a_2 < \ldots\}$ be an infinite set of positive integers, $k \geq 3$ be a fixed integer, and suppose there is a constant $C > 0$ such that $a_{n+1} - a_n \leq C$ for all n. Prove that A contains a k–term arithmetic progression. (Hint: apply van der Waerden's theorem, do not refer Szemerédi's theorem.)

5.4 Let $r \geq 2$ be an integer and let

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

be a system of linear equations with integer coefficients satisfying

$$\sum_{j=1}^{n} a_{ij} = 0, \quad i = 1, \ldots, m.$$

Prove that if this system has non trivial solutions (not $x_1 = \cdots = x_n$) in integers, then coloring the integers with r color, the system also has non trivial monochromatic solutions.

5.5 Let A be a finite set of positive real numbers with $|A| > 1$. Prove that $A + A$ contains at least $\binom{|A|}{2}$ three term arithmetic progressions.

5.6 Let $A \subset \{1, \ldots, N\}$, where $N \geq 3$. Prove that if $|A| > \frac{2N+2}{3}$ then A contains a non trivial three term arithmetic progression. (Do not refer Szemerédi's theorem.)

6.1 Let A be a finite set of integers. Prove that for any integers $k \geq 1$ and x we have $r_{kA-kA}(x) \leq r_{kA-kA}(0)$. (Hint: $r_{mA-nA}(x)$ denotes the number of representations $x = a_1 + \cdots + a_m - a_{m+1} - \cdots - a_{m+n}$, express $r_{kA-kA}(x)$ by the circle method.)

6.2 Let A be a finite set of integers. Prove that for any integers $m, n \geq 0$, $m + n$ is even, and any integer x, we have $r_{mA-nA}(x) \leq r_{kA-kA}(0)$, where $k = (m + n)/2$.

6.3 Let A and B be finite sets of integers. Prove that for any integers $m, n \geq 0$ and x we have $r_{mA+nB}(x) \leq \sqrt{r_{mA-mA}(0) r_{nB-nB}(0)}$.

6.4 Let A be a finite set of integers, satisfying $|A+A| \le C|A|$ with some $C \ge 2$. Let $\hat{A}(\alpha) = \sum_{a\in A} e(a\alpha)$. Prove that

$$\int_0^1 |\hat{A}(\alpha)|^4 \, d\alpha \ge \frac{1}{C}\,|A|^3.$$

(Note that $|A|^3$ is an obvious upper bound for the above integral.)

6.5 Prove that for real numbers

$$\sum_{j=1}^{n}(a_j+b_j)^2 \le 2\sum_{j=1}^{n}(a_j^2+b_j^2),$$

and equality holds iff $a_j = b_j$ for all $j = 1,\dots,n$.

6.6 Let q, k, and r be positive integers, and define

$$A = \{n_0 + n_1 q + \cdots + n_k q^k : 0 \le n_j < q/2 \text{ integers}, n_0^2 + \cdots + n_k^2 = r\}.$$

Show that A does not contain any non trivial 3–term arithmetic progressions.

6.7 (Behrend, 1946) For any (large) integer N, there is a set $A \subset \{1,\dots,N\}$ such that A does not contain any non trivial 3–term arithmetic progressions, and $|A| > Ne^{-c\sqrt{\log N}}$ with an absolute constant $c > 0$. (Hint: use the previous problem with optimal choices of the parameters k, q, r.)

6.8 Let A be a finite set of real numbers. Prove that

$$\max_{x\neq 0} r_{A-A}(x) \ge \frac{|A|^2}{|A \pm A|} - 1.$$

(Hint: Consult the proof of Corollary 1.11.)

6.9 Let A be a finite set of real numbers, and suppose that $|A-A| \le |A|^{1+\delta}$, (or $|A+A| \le |A|^{1+\delta}$), with some $0 < \delta < 1$. Show that A contains an affine cube of dimension d for all $d < \log_2(1+\delta^{-1})$, if $|A|$ is sufficiently large. (We say that B is an affine cube of real numbers with dimension d, if there are real numbers $x_0, x_1, \dots, x_d$ such that

$$B = \{x_0 + \epsilon_1 x_1 + \cdots + \epsilon_d x_d : \epsilon_j = 0, 1 \text{ for all } j\}.$$

Alternatively one can first fix a $d \ge 1$ and than conclude the existence of an affine cube with dimension d, if $|A \pm A| \le |A|^{1+\delta}$ with a $0 < \delta < 1/(2^d-1)$.)

6.10 (Hilbert, 1892) Let $r, d \ge 1$ be integers. For any r–coloring of the integers, prove that a color class contains an affine cube of dimension d. (Hint: this is true for density reasons, and follows from the previous problem. At least one color class must be dense, and for a dense set $|A-A|$ is small.)

7.1 Let A be a finite set of positive real numbers, and $B = \log A = \{\log a : a \in A\}$. Prove that

$$\frac{|A|^4}{2\log|A|} \le |A+A|^2\,|B+B|.$$

7.2 Let A be a subset of a field F (either $\mathbb{R}$ or $\mathbb{Z}_p$). We define

$$Q[A] = \frac{A-A}{(A-A)\setminus\{0\}} = \left\{\frac{a-b}{c-d};\, a,b,c,d \in A, c \ne d\right\}.$$

Prove the following elementary properties.

$$0, 1 \in Q[A],$$
$$Q[A+x] = Q[A], \text{ for all } x \in F,$$
$$Q[x\cdot A] = Q[A], \text{ for all } 0 \ne x \in F,$$
$$Q[A] = -Q[A],$$
$$Q[A]\setminus\{0\} = \frac{1}{Q[A]\setminus\{0\}}.$$

7.3 Let A be a finite subset of a field F, $|A| > 1$, and let $x \in F$ be arbitrary. Prove that $|A + (x\cdot A)| = |A|^2$ iff $x \notin Q[A]$. (Hint: check if the map $(a, a') \mapsto a + xa'$ is injective.)

7.4 Let A be a subset of $\mathbb{Z}_p$ with $|A| > \sqrt{p}$. Prove that $Q[A] = \mathbb{Z}_p$. (Hint: prove first 7.3.)

7.5 Let A be a non–empty finite subset of the positive real numbers. Prove that $|A\cdot A + A| \ge |A|^{3/2}$.

7.6 Prove that $A\cdot B + C + D = \mathbb{Z}_p$ whenever A, B, C, D are subsets of $\mathbb{Z}_p$ with $|A||B||C||D| > p^3$, especially $A\cdot A + A\cdot A + A\cdot A = \mathbb{Z}_p$ whenever A is a subset of $\mathbb{Z}_p$ with $|A| > p^{3/4}$.

7.7 Prove that $(A+A)\cdot A + A\cdot A + A\cdot A = \mathbb{Z}_p$ whenever A is a subset of $\mathbb{Z}_p$ with $|A| > (2p)^{2/3}$. (Hint at the end of the Lecture Notes, Ch 6.)

7.8 Prove that $A\cdot A + A\cdot A + A\cdot A = \mathbb{Z}_p$ whenever A is a subset of $\mathbb{Z}_p$ with $|A| > (2p)^{2/3}$ and $|A+A| < |A|^2/(2\sqrt{p})$.

8.1 Let $\alpha \ne 0$ be a real number and Λ be the lattice in $\mathbb{R}^2$ with basis $\mathbf{a}_1 = (\alpha, 1)$ and $\mathbf{a}_2 = (1, 0)$. Compute $\det(\Lambda)$.

8.2 Let α, β be real numbers and Λ be the lattice in $\mathbb{R}^3$ with basis $\mathbf{a}_1 = (\alpha, \beta, 1)$, $\mathbf{a}_2 = (1, 0, 0)$, and $\mathbf{a}_3 = (0, 1, 0)$. Compute $\det(\Lambda)$.

8.3 Let Λ be the subgroup of $\mathbb{R}^2$ generated by the vectors $\mathbf{a}_1 = (1, 0)$, $\mathbf{a}_2 = (0, 1)$, and $\mathbf{a}_3 = (1/3, 2/3)$. Show that Λ is a lattice, find a basis for Λ, and compute $\det(\Lambda)$.

8.4 Prove the following version of Theorem 7.2 (Minkowski's first theorem) Let Λ be a lattice in $\mathbb{R}^d$, and let K be a symmetric convex body in $\mathbb{R}^d$ with volume $\mathrm{vol}(K) \geq 2^d \det(\Lambda)$. Then $\overline{K}$ (the closure of K) contains a lattice point different from the origin.

8.5 Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis of $\mathbb{R}^n$, $\lambda_1, \ldots, \lambda_n$ be positive real numbers, and $K \subset \mathbb{R}^n$ be a symmetric convex body. Prove that

$$K^* = \{\sum_{k=1}^{n} \lambda_k x_k \mathbf{a}_k : \sum_{k=1}^{n} x_k \mathbf{a}_k \in K\}$$

is a symmetric convex body.

8.6 Let $\mathbb{Z}[i] = \{u+iv : u, v \in \mathbb{Z},\ i^2 = -1\}$ be the ring of *Gaussian integers*, and $0 \neq z \in \mathbb{C}$ be a complex number. Prove that $\Lambda = \{zw : w \in \mathbb{Z}[i]\}$ is a lattice (with the natural embedding of $\mathbb{C}$ into $\mathbb{R}^2$), find a basis for Λ and compute $\det(\Lambda)$.

8.7 Derive Lemma 5.2, (Dirichlet's approximation theorem) from Minkowski's first theorem. *Let $k \geq 1$ be an integer and $\alpha_1, \ldots, \alpha_k$ be real numbers. For any integer $Q \geq 1$ there is an integer $1 \leq q \leq Q^k$ such that*

$$\|q\alpha_j\| \leq \frac{1}{Q}, \text{ for all } j = 1, \ldots, k.$$

(Hint: Use the lattices of Problem 8.1 and 8.2, generalize them for arbitrary k, and study the convex body

$$K = \{x_1\mathbf{e}_1 + \cdots + x_{k+1}\mathbf{e}_{k+1} : |x_1| < \frac{1}{Q}, \ldots, |x_k| < \frac{1}{Q}, |x_{k+1}| < Q^k + 1\}.)$$

8.8 Consider the lattice of Problem 8.1. Study the romboid with four vertices $(\pm U, 0)$ and $(0, \pm V)$, where U, V are positive real numbers, and $UV = 2$. Prove the following approximation theorem. *Let α be a real irrational number. There are infinitely many rational numbers a/q such that $(a, q) = 1$, and*

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{2q^2}.$$

8.9 Prove, if $\ell_1, \dots, \ell_k$ be homogeneous linear forms in the variables $x_1, \dots, x_k$, say

$$\ell_j(x_1, \dots, x_k) = a_{j1}x_1 + \cdots + a_{jk}x_k, \quad j = 1, \dots, k,$$

with real coefficients

$$A = \begin{pmatrix} a_{11} \ \dots \ a_{1k} \\ \vdots \\ a_{k1} \ \dots \ a_{kk} \end{pmatrix},$$

and determinant $\det(A) \neq 0$, then the points $(\ell_1, \dots, \ell_k)$ corresponding to integer values of $x_1, \dots, x_k$ form a lattice Λ with $\det(\Lambda) = |\det(A)|$, and present a basis for Λ. (In other words $\Lambda = \{A\mathbf{x};\, \mathbf{x} \in \mathbb{Z}^k\}$, where vectors are considered as column vectors.)

8.10 Let $\ell_1, \dots, \ell_k$ be homogeneous linear forms with real coefficients in the variables $x_1, \dots, x_k$, as in Problem 8.9. If $L_1, \dots, L_k$ are positive real numbers with $L_1 \cdots L_k \geq |\det(A)|$ then there are integers $x_1, \dots, x_k$, not all zero, such that

$$|\ell_1| \leq L_1, \dots, |\ell_k| \leq L_k.$$

8.11 Let $\ell_1, \dots, \ell_k$ be homogeneous linear forms with real coefficients in the variables $x_1, \dots, x_k$, as in Problem 8.9. Prove that there are integers $x_1, \dots, x_k$, not all zero, such that

$$|\ell_1| + \cdots + |\ell_k| \leq (k!|\det(A)|)^{1/k}.$$

(Compare to Problem 8.8.)

8.12 Let $\ell_1, \dots, \ell_k$ be homogeneous linear forms with real coefficients in the variables $x_1, \dots, x_k$, as in Problem 8.9. Prove that there are integers $x_1, \dots, x_k$, not all zero, such that

$$|\ell_1| \cdots |\ell_k| \leq k^{-k}k!|\det(A)|.$$

(Hint: Use the theorem of arithmetic and geometric means,

$$(|\ell_1| \cdots |\ell_k|)^{1/k} \leq \frac{|\ell_1| + \cdots + |\ell_k|}{k},$$

and refer to the previous problem.)

9.1 Let $h \geq 1$ be an integer, and consider the following directed graph G (best to imagine on a diagonally cutted checker board, where edges are directed to up or to right).

$$V = \{(x, y) :\ x + y \leq h,\ 0 \leq x, y\},$$

$$E = \{(x, y) \to (x, y + 1)\} \cup \{(x, y) \to (x + 1, y)\}.$$

Prove that G is a commutative layered graph of level h.

9.2 Let G be a directed graph, and define the inverse graph G^{-1} by changing the direction of all edges to the opposite. Prove that if G is a commutative graph on h layers, then so is G^{-1}. For the graph of Problem 9.1 prove also that $\mu_h(G^{-1}) = 1/\mu_h(G)$. (Note that $\mu_h(G)$ is the magnification ratio $D(V_0, V_h)$ in G, while in G^{-1} edges run from V_i to V_{i-1} and $\mu_h(G^{-1})$ is the magnification ratio $D(V_h, V_0)$.) Find an example to show that $\mu_h(G^{-1}) = 1/\mu_h(G)$ is not true in general.

9.3 Let $A, B \subset \{1, \dots, N\}$, and for $i = 0, \dots, h$ we define disjoint vertex sets $V_i = (A + iB) \cap \{1, \dots, N\}$, finally we define an edge from $x \in V_{i-1}$ to $y \in V_i$ if $y - x \in B$. Prove that the graph defined this way (called the *truncated addition graph*) is commutative (on h layers).

9.4 Let A and B be finite subsets of integers, and G is the addition graph of level h over $A + hB$. Prove without using Plünnecke's inequality that $D(A, A + hB) \geq 1$.

9.5 Let A be a finite set of integers. Prove that $|A + A + A|^2 \leq |A + A|^3$. (Hint: Find the right application of Plünnecke's inequality.)

9.6 Let $|A| = n$, $2 \leq i \leq h$, and suppose that $|iA| \leq cn$. Prove that $|hA| \leq c^{h/(i-1)}n$.

10.1 Prove that a k–isomorphism is an ℓ–isomorphism for all $2 \leq \ell \leq k$.

10.2 Prove that $\{1, 2, 4\}$ is 2–isomorphic to $\{1, 2, 5\}$, but not 2-isomorphic to $\{1, 2, 3\}$.

10.3 Prove that a 2–isomorphic image of a generalized arithmetic progression is a generalized arithmetic progression.

10.4 Let $A \subset Z_1$, $a \in Z_1$, $B \subset Z_2$, and $b \in Z_2$ be given, where Z_1 and Z_2 are (additive) Abelian groups. Prove that if A and B are k–isomorphic then $A + a$ and $B + b$ are also k–isomorphic.

10.5 Let A be a finite set of real numbers, $|A| = n$, and suppose that $|A + A| \leq Cn$. Prove that there are $0 < \delta = \delta(C) < 1$ and $0 < c = c(C) < 1$ such that there is an arithmetic progression P of length $k \geq n^c$ with $|A \cap P| > \delta k$. (Hint: Freiman's theorem.)

10.6 If A is $k(m + n)$–isomorphic to B then $mA - nA$ is k–isomorphic to $mB - nB$.

10.7 Prove that any two finite Sidon–sets of equal size are 2–isomorphic (refer to Chapter 2 for the definition of Sidon sets).

10.8 Suppose that $\rho_p : \mathbb{Z}_p \to \mathbb{Z}$ maps any residue class to its least positive residue. If $A \subset \mathbb{Z}_p$ and $\rho_p(a) \leq p/k$ for all $a \in A$ then A is k–isomorphic to $\rho_p(A)$.