On Morita Contexts in Bicategories

Bertalan Pécsi

Abstract

We characterize abstract Morita contexts in several bicategories. In particular, we use heteromorphisms for the bicategory $\mathbb{Prof}$ of categories and profunctors and coreflective subcategories for $\mathbb{Cat}$ (categories and functors). In addition, we prove general statements concerning strict Morita contexts, and we give new equivalent forms to the standard notions of adjointness, category equivalence and Morita equivalence by studying the *collage* of a profunctor.

Preliminaires

This paper is to present a simple approach for studying profunctors (based on reflections and coreflections) via their so called *collage* and to study the abstract notion of *Morita context* in an arbitrary bicategory (called "right wide Morita context" in El Kaoutit[7]).
Morita contexts were first introduced in the bicategory $\mathbb{Bimod}$ [of unitary rings and bimodules] as 6-tuples $\langle A, B, M, N, \varphi, \psi\rangle$ where A and B are rings, M is an $A-B$ bimodule, N is a $B-A$ bimodule and $\varphi: M\otimes N \to A$ and $\psi: N\otimes M \to B$ are homomorphisms satisfying $\varphi\otimes M = M\otimes\psi$, $N\otimes\varphi = \psi\otimes N$. These are strongly connected with equivalences between certain subcategories (and certain quotients) of the categories of A-modules and B-modules, via *wide Morita contexts*, i.e. Morita contexts in the bicategory of Abelian categories and right exact functors. Cf. e.g. Kashu[10], Kato-Ohtake[11], Muller[14]. As it is observed in Sands[16], a Morita context in $\mathbb{Bimod}$ can be viewed as a "big ring" itself, namely a *generalized matrix ring* with entries from $\left(\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right)$.
Following this spirit, here we examine consequences of a similar observation in Grandis-Pare[9], that a profunctor is fully determined by its so called *collage*, which is a category itself [section 1]. Symmetrization of this approach leads to the notion of *bridge*, that will turn out to be exactly the Morita contexts in the bicategory $\mathbb{Prof}$ (Thm.5.1). The existence of collage (as a special double categorical colimit) for abstract *proarrows* (which generalize the 2 natural embeddings of $\mathbb{Cat}$ in $\mathbb{Prof}$) is investigated in Wood[18]. Section 2 deals with equivalence and Morita equivalence of categories in terms of bridges (see Theorems 2.2 and 2.6). In section 3 we present further properties of bridges between two fixed categories, and we investigate in what functorial manner the bicategories $\mathbb{Prof}$ [of categories and profunctors] and $\mathbb{Br}$ [of categories and bridges] are *embedded* into $\mathbb{Cosp}(\mathcal{Cat})$. Then, starting out from an arbitrary bicategory, in section 4 we define a generated bicategory with Morita contexts as arrows (construction found also in El Kaoutit[7]), and we prove results about equivalence of objects and adjunctions therein (Corollaries 4.10 and 4.9). Thm.4.4 represents Morita

contexts as special lax functors. Finally, we describe Morita contexts in the bicategories listed in section 5. Theorem 5.6 shows that a Morita context in $\mathbb{C}\mathrm{at}$ corresponds to a pair of coreflective subcategories of a certain category. Corollary 1.4 proves a result about *adjoint functors* in a similar flavor.

Thanks are due to the anonymous referee and Gabriella Böhm for the very good questions and comments, to László Márki, Ildikó Sain and Zalán Gyenis for free and useful conversations.

——————— o ———————

Throughout the paper we write composition from left to right (if $\alpha: A \to B$ and $\beta : B \to C$ are arrows in a category, then their *composition* is $\alpha\beta : A \to C$). Consequently, we usually write a^φ for the image of an element $a \in A$ under the map $\varphi: A \to B$. The restriction of a function $\varphi : A \to B$ to a subset $X \subseteq A$ is denoted by $\varphi\restriction_X$. We use the *infix* notation for binary relations $\alpha \subseteq A \times B$: writing $a[\alpha]b$ for $\langle a, b\rangle \in \alpha$.
The set of arrows $A \to B$ in the category $\mathcal{A}$ is denoted by $\mathcal{A}(A \shortmid B)$, and $\mathrm{End}_{\mathcal{A}}A$ (or simply $\mathrm{End}A$) denotes the monoid of endomorphisms of A, i.e. $\mathrm{End}A = \mathcal{A}(A \shortmid A)$. We write $\mathrm{Hom}\mathcal{A}$ or simply $\mathcal{A}$ for the class of all arrows in $\mathcal{A}$. Domain and codomain of an arrow α are written as $\mathrm{dom}\alpha$ and $\mathrm{cod}\alpha$, respectively.
We recall that $\mathcal{B} \leqslant \mathcal{A}$ is called a *reflective* subcategory, if each object $A \in \mathrm{Ob}\mathcal{A}$ has a *reflection* in $\mathcal{B}$, that is, an arrow $\alpha: A \to B$ for some $B \in \mathrm{Ob}\mathcal{B}$ such that, for any arrow $\alpha_1 : A \to B_1$ with $B_1 \in \mathrm{Ob}\mathcal{B}$, there is a unique $\beta \in \mathrm{Hom}\mathcal{B}$ satisfying $\alpha_1 = \alpha\beta$. Then mapping A to B extends naturally to a so called *reflection functor* $\mathcal{A} \to \mathcal{B}$, which is uniquely determined up to natural isomorphism. We will also use the term *reflection functor* for its restriction to a subcategory of $\mathcal{A}$. The notions of *coreflective subcategory* and *coreflection functor* are defined dually. A diagram $A \to X \leftarrow B$ in a category is called *cospan*, its dual is *span*. A full subcategory of $\mathcal{A}$ is a *skeleton* if it contains exactly one isomorphic copy of each object in $\mathcal{A}$. A morphism $e: A \to A$ is called *idempotent* if $e^2 = e$.

We will refer to a bicategory by naming its *objects* (0-cells), *arrows* (1-cells) and *2-cells* (morphisms between parallel arrows), respectively. Compositions will be understood. To distinguish, we draw 2-cells by dashed arrows, like $f \dashrightarrow g$ for arrows $f, g : A \to B$ (except for some concrete example as $\mathbb{P}\mathrm{rof}$ or $\mathbb{B}\mathrm{imod}$). We consequently omit the dot at composition and action of 1-cells.
We mostly hide the *coherence* 2-cell between two parenthesized forms of the "same" n-ary composition, as it is usual in the literature, since this is uniquely built up by the basic coherence 2-cells given by the bicategory structure, as the *coherence theorem* states (proved e.g. in MacLane[12], section VII.2). However, if one wants to follow our formulas in full details, only needs to insert these appropriate *coherence* 2-cells whenever we implicitly pass from $(fg)h$ to $f(gh)$ or f to $1_A f$, etc. Since all these are *isomorphisms* by definition, they cannot hurt the validity of the general theorems in section 4 dealing with (left, right) invertibility of certain 2-cells. For more on bicategories see Borceux[3] or Benabou[2].
We will use the word "invertible" for both arrows and 2-cells, meaning for an arrow $f : A \to B$ that there exists $g : B \to A$ such that $fg \cong 1_A$ and $gf \cong 1_B$ (i.e., f is an *equivalence arrow*).

By a diagram $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\alpha"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\beta"] & C \end{tikzcd}$ in a bicategory we will mean that α and β are 2-cells, $\alpha : f \dashrightarrow f_1$ and $\beta : g \dashrightarrow g_1$. Their *horizontal composition* $fg \dashrightarrow f_1 g_1$ is denoted by $\alpha\#\beta$. That is, $\alpha\#\beta = \alpha g \cdot f_1\beta = f\beta \cdot \alpha g_1$. (We note that $f\beta$ is the same as $1_f\#\beta$, and $\alpha g = \alpha\#1_g$.)

1 Profunctors

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. We call a category $\mathcal{H}$ a *bridge* between $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A}$ and $\mathcal{B}$ are disjoint full subcategories of $\mathcal{H}$, and $\mathrm{Ob}\mathcal{H} = \mathrm{Ob}\mathcal{A} \cup \mathrm{Ob}\mathcal{B}$. In notation: $\mathcal{H} : \mathcal{A} \rightleftharpoons \mathcal{B}$. Morphisms not belonging to $\mathcal{A}$ nor $\mathcal{B}$ are called *heteromorphisms*. We denote their class by $\mathrm{Het}\mathcal{H}$.

For example, any full embedding of categories $\mathcal{A} \hookrightarrow \mathcal{B}$ induces a bridge $\mathcal{A} \rightleftharpoons \mathcal{B}$ in which the heteromorphisms are (copies of) the original homomorphisms $A \to B$ and $B \to A$ of $\mathcal{B}$ (for $A \in \mathrm{Ob}\mathcal{A}$, $B \in \mathrm{Ob}\mathcal{B}$). Another easy example is the disjoint union (or coproduct) of categories $\mathcal{A}$ and $\mathcal{B}$ which is called the *empty bridge*: this has no heteromorphisms.

By a *profunctor* from $\mathcal{A}$ to $\mathcal{B}$ we will mean a *directed* bridge, that is, a bridge without arrows of the form $B \to A$ (for objects $A \in \mathrm{Ob}\mathcal{A}$, $B \in \mathrm{Ob}\mathcal{B}$). If $\mathtt{T}$ is a profunctor from $\mathcal{A}$ to $\mathcal{B}$, we briefly write $\mathtt{T} : \mathcal{A} \Rightarrow \mathcal{B}$.

Remark 1.2. *In the literature a profunctor is usually defined as a functor $\mathtt{T} : \mathcal{A}^{op} \times \mathcal{B} \to \mathcal{S}et$. By considering $\mathtt{T}(A, B)$ as the set of heteromorphisms from A to B, we can see that profunctors and directed bridges are eventually the same. (In this context, the corresponding directed bridge is called the collage (or gluing) of the profunctor, cf. e.g. Grandis-Pare[9] or Wood[18].) For more on profunctors, see e.g. Borceux[3].*

The identity profunctor $\mathbf{1}_{\mathcal{A}} : \mathcal{A} \Rightarrow \mathcal{A}$ of a category $\mathcal{A}$ consists of 2 (disjoint) copies of $\mathcal{A}$, with a 3rd copy of each arrow in $\mathcal{A}$, playing the role of heteromorphisms. In the original point of view, this is just the hom functor $\mathcal{A}^{op} \times \mathcal{A} \to \mathcal{S}et$.

A functor $F : \mathcal{A} \to \mathcal{B}$ determines the profunctors $F_* : \mathcal{A} \Rightarrow \mathcal{B}$ and $F^* : \mathcal{B} \Rightarrow \mathcal{A}$ in which each object $A \in \mathrm{Ob}\mathcal{A}$ is reflected [resp. coreflected] in $\mathcal{B}$ by $A^F \in \mathrm{Ob}\mathcal{B}$. These can be realized by setting

$$\mathrm{Het}F_* := \{\langle A, \beta\rangle \mid A \in \mathrm{Ob}\mathcal{A},\ \beta \in \mathrm{Hom}\mathcal{B},\ \mathrm{dom}\beta = A^F\}$$

and defining $\alpha \cdot \langle A, \beta\rangle \cdot \beta_1 := \langle \mathrm{dom}\alpha,\ \alpha^F \beta\beta_1\rangle$ whenever they are composable. F^* is constructed similarly. Thus, we have bijections of the homsets $F_*(A \shortmid B) \simeq \mathcal{B}(A^F \shortmid B)$ and $F^*(B \shortmid A) \simeq \mathcal{B}(B \shortmid A^F)$, natural in both $A \in \mathrm{Ob}\mathcal{A}$ and $B \in \mathrm{Ob}\mathcal{B}$. In this view, it is clear that a functor $F : \mathcal{A} \to \mathcal{B}$ is left adjoint to $G : \mathcal{B} \to \mathcal{A}$ if and only if the induced profunctors F_* and G^* are isomorphic.

We note that, by the properties of reflections, the *basic heteromorphisms* (i.e., the reflection arrows $\nu_A := \langle A, 1_{A^F}\rangle$) are *epimorphic* in F_* and that a square $\begin{tikzcd} A \arrow[r, "\nu_A"] \arrow[d, "\alpha"'] & A^F \arrow[d, "\beta"'] \\ A_1 \arrow[r, "\nu_{A_1}"'] & {A_1}^F \end{tikzcd}$ commutes if and only if $\beta = \alpha^F$. Dual statements hold for F^*.

Theorem 1.3. Let $\mathtt{T}:\mathcal{A} \Rightarrow \mathcal{B}$ be a profunctor. The following statements hold:

i) $\mathcal{B}$ is a *reflective* subcategory of $\mathtt{T}$ if and only if $\mathtt{T}$ is isomorphic to F_* for some functor $F:\mathcal{A} \to \mathcal{B}$.

ii) $\mathcal{A}$ is a *coreflective* subcategory of $\mathtt{T}$ if and only if $\mathtt{T}$ is isomorphic to G^* for some functor $G:\mathcal{B} \to \mathcal{A}$.

Proof. In F_*, the basic heteromorphism $\nu_A : A \to A^F$ reflects A [respectively, $A^F \to A$ coreflects A in F^*]. Conversely, the functor F [resp. G] will be just the reflection [resp. coreflection] functor $\mathtt{T} \to \mathcal{B}$ [resp. $\mathtt{T} \to \mathcal{A}$], restricted to $\mathcal{A}$ [resp. $\mathcal{B}$]. $\square$

Corollary 1.4. Let $\mathcal{A}$ be a coreflective full subcategory and let $\mathcal{B}$ be a reflective full subcategory of a category $\mathcal{T}$. Then the reflection functor $\mathcal{T} \to \mathcal{B}$ restricted to $\mathcal{A} \to \mathcal{B}$ is left adjoint to the coreflection functor restricted to $\mathcal{B} \to \mathcal{A}$. *Conversely*, every adjoint pair of functors arises this way.

Proof. The first statement is easy to check. For the other direction, if F is left adjoint to G, define $\mathcal{T}$ as F_* (which is isomorphic to G^*). This is the *adjoint profunctor* of F and G. $\square$

Most adjoint functor pairs define reasonable directed bridges. A typical example is the profunctor $\mathcal{S}et \Rightarrow \mathcal{G}rp$ with all functions from a *set* to (the underlying set of) a *group* as heteromorphisms, hiding the adjunction of *free* and *forgetful* functors. Of course, $\mathcal{G}rp$ can be replaced to any variety, or any concrete category with *free* objects, etc. Another example is the profunctor $\mathcal{M}od_A \Rightarrow \mathcal{M}od_B$ with A-bilinear morphisms $M \times T \to N$ (for a fixed bimodule ${}_AT_B$) as heteromorphisms between *modules* $M_A \dashrightarrow N_B$, which hides the adjuntion of *tensor* and *hom* with respect to T.

Definition 1.5. Let $\mathcal{H}:\mathcal{A} \rightleftharpoons \mathcal{B}$ be a bridge. If we forget all arrows from objects of $\mathcal{B}$ to those of $\mathcal{A}$, we get a profunctor $\mathcal{H}^{>}:\mathcal{A} \Rightarrow \mathcal{B}$. The profunctor $\mathcal{H}^{<}:\mathcal{B} \Rightarrow \mathcal{A}$ is defined similarly. These are called the *parts* of $\mathcal{H}$.
We note that, regarding the class of arrows, we have $\mathcal{H} = \mathcal{H}^{>} \cup \mathcal{H}^{<}$.

Proposition 1.6. There is a bicategory naturally defined by bridges as arrows. We denote it by $\mathbb{Br}$. In our setting, the bicategory $\mathbb{Prof}$ arises as a certain sub-bicategory of $\mathbb{Br}$.

Proof. A *morphism* between bridges $\mathcal{H},\mathcal{K}:\mathcal{A} \rightleftharpoons \mathcal{B}$ is going to be just a functor $\mathcal{H} \to \mathcal{K}$ identical both on $\mathcal{A}$ and $\mathcal{B}$. Proving that natural transformations between functors $\mathcal{A}^{op} \times \mathcal{B} \to \mathcal{S}et$ straightly correspond to bridge morphisms between the corresponding directed bridges $\mathcal{A} \Rightarrow \mathcal{B}$ is left to the reader.
Composition of profunctors can be defined analogously to *tensor product* of bimodules: For $\mathtt{T}:\mathcal{A} \Rightarrow \mathcal{B}$, $\mathtt{S}:\mathcal{B} \Rightarrow \mathcal{C}$ and $A \in \mathrm{Ob}\mathcal{A}$, $C \in \mathrm{Ob}\mathcal{C}$ we set

$$\mathtt{TS}(A \shortmid C) := \{\langle t, s\rangle \mid \underset{A\to B}{t} \text{ and } \underset{B\to C}{s} \text{ for some } B \in \mathrm{Ob}\mathcal{B}\}/\sim$$

where $\sim$ is the equivalence relation generated by $\langle t\beta, s\rangle \sim \langle t, \beta s\rangle$ for all composable triples $t \in \mathrm{Het}\mathtt{T}$, $\beta \in \mathrm{Hom}\mathcal{B}$, $s \in \mathrm{Het}\mathtt{S}$. This coincides with the original definition which uses *coend* or *Kan extension*. (Cf. also paragraph 3.1 of Grandis-Pare[9] or Borceux[3].)

Then, composition of two *bridges* is defined using composition of profunctors on both sides: for $\mathcal{H} : \mathcal{A} \rightleftharpoons \mathcal{B}$ and $\mathcal{K} : \mathcal{B} \rightleftharpoons \mathcal{C}$, we set

$$\mathcal{H}\mathcal{K} := \mathcal{H}^{>}\mathcal{K}^{>} \ \cup \ \mathcal{K}^{<}\mathcal{H}^{<}$$

with the straightforward compositions, e.g., for heteromorphisms $A \xrightarrow{h} B \xrightarrow{k} C \xrightarrow{k_1} B_1 \xrightarrow{h_1} A_1$, let $\langle h, k\rangle \cdot \langle k_1, h_1\rangle := h(kk_1)h_1$ where (kk_1) is a composition in $\mathcal{K}$ but already lies in $\mathcal{B}$.
Let $\mathtt{T}, \mathtt{T}_1 : \mathcal{A} \Rightarrow \mathcal{B}$ and $\mathtt{S}, \mathtt{S}_1 : \mathcal{B} \Rightarrow \mathcal{C}$ be profunctors, and let $\varphi : \mathtt{T} \to \mathtt{T}_1$ and $\psi : \mathtt{S} \to S_1$ be bridge morphisms, then their *horizontal composition* $\varphi\#\psi : \mathtt{TS} \to \mathtt{T}_1\mathtt{S}_1$ is defined by mapping $\langle t, s\rangle \in \mathrm{Het}(\mathtt{TS})$ to $\langle t^{\varphi}, s^{\psi}\rangle \in \mathrm{Het}(\mathtt{T}_1\mathtt{S}_1)$. We write $\mathtt{T}\psi$ for $1_{\mathtt{T}}\#\psi$ and $\varphi\mathtt{S}$ for $\varphi\#1_{\mathtt{S}}$. Similarly for bridges. By that, we obtain the desired bicategories $\mathbb{Br}$ and $\mathbb{Prof}$. $\square$

Remark 1.7. *Observe that the composition in a bridge $\mathcal{H}$ can be given by morphisms of profunctors $\mu : \mathcal{H}^{>}\mathcal{H}^{<} \to \mathbf{1}_{\mathcal{A}}$ and $\nu : \mathcal{H}^{<}\mathcal{H}^{>} \to \mathbf{1}_{\mathcal{B}}$. Then associativity in $\mathcal{H}$ turns to the identities $\mu\mathcal{H}^{>} = \mathcal{H}^{>}\nu$ and $\mathcal{H}^{<}\mu = \nu\mathcal{H}^{<}$.*

This directly proves that a bridge is none other than a *Morita context* in the bicategory $\mathbb{Prof}$, as will be defined in Def.4.1.

2 Equivalence bridges

Definition 2.1. We call a bridge $\mathcal{H} : \mathcal{A} \rightleftharpoons \mathcal{B}$ an *equivalence bridge*, if each object $A \in \mathrm{Ob}\mathcal{A}$ is isomorphic to some $B \in \mathrm{Ob}\mathcal{B}$, and each object $B \in \mathrm{Ob}\mathcal{B}$ is isomorphic to some $A \in \mathrm{Ob}\mathcal{A}$ in $\mathcal{H}$.

We warn that "equivalence bridge" does not exactly coincide with "invertible arrow" in $\mathbb{Br}$ since, as we will see, by Thm.5.1 and Cor.4.10, the parts of an invertible bridge make an adjoint equivalence in $\mathbb{Prof}$, not in $\mathbb{Cat}$, i.e., a Morita equivalence instead of category equivalence. See also Thm.2.6.

Theorem 2.2. Categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if there is an equivalence bridge between them.

Proof. First, let us assume that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Choose skeletons $\mathcal{A}_0 \leqslant \mathcal{A}$ and $\mathcal{B}_0 \leqslant \mathcal{B}$, together with an isomorphism arrow $\xi_A : A \to A_0$ for each object $A \in \mathrm{Ob}\mathcal{A}$, and fix a category isomorphism $F : \mathcal{A}_0 \to \mathcal{B}_0$. Then, let $\mathcal{H} :=$ "$(F_* \cup F^*)$", that is, for objects $A \in \mathrm{Ob}\mathcal{A}$ and $B \in \mathrm{Ob}\mathcal{B}$ we set

$$\mathcal{H}(A \shortmid B) := \{\langle A, \beta\rangle \mid \beta : {A_0}^F \to B\} \ \text{ and } \ \mathcal{H}(B \shortmid A) := \{\langle \beta, A\rangle \mid \beta : B \to {A_0}^F\}$$

where A_0 is the isomorphic copy of A in $\mathcal{A}_0$. When composing these, $\langle A, \beta\rangle$ and $\langle \beta, A\rangle$ play the role of formal compositions "$\xi_A \cdot \beta$" and "$\beta \cdot {\xi_A}^{-1}$", respectively, passing from $\mathcal{A}_0$ to $\mathcal{B}_0$ via F. In particular we define $\langle \beta, A\rangle \cdot \langle A, \delta\rangle := \beta \cdot \delta$ and $\langle A, \beta\rangle \cdot \langle \delta, C\rangle := \xi_A \cdot (\beta\delta)^{F^{-1}} \cdot {\xi_C}^{-1}$. Then, it is straightforward to check that $\mathcal{H}$ is an equivalence bridge.
For the other direction, we can construct an equivalence functor $\mathcal{A} \to \mathcal{B}$ from an equivalence bridge $\mathcal{H} : \mathcal{A} \rightleftharpoons \mathcal{B}$, by fixing an isomorphism $A \to B$ of $\mathcal{H}$ for each $A \in \mathrm{Ob}\mathcal{A}$. $\square$

We note that the existence of an equivalence bridge between categories is the *weakest* requirement of category equivalence in view of the *axiom of choice*. Moreover, Thm.2.2 is itself equivalent to the axiom of choice. The proofs are left to the reader. Cf. also Makkai[13] and Thm.1.364 in Freyd-Scedrov[8].

Instead of using skeletons, we could also have constructed the equivalence bridge from an equivalence pair of functors, using the following fundamental lemma (proved in e.g. Pare[15] or Baez-Lauda[1]) which states that, in a bicategory, any equivalence between objects A and B induces an *adjoint* equivalence between them.

Lemma 2.3. Let $f : A \to B$ and $g : B \to A$ be 1-cells in a bicategory, and suppose there are isomorphism 2-cells $\varphi : fg \dashrightarrow 1_A$ and $\psi : gf \dashrightarrow 1_B$. Then there exists an isomorphism $\varphi' : fg \dashrightarrow 1_A$, such that $\varphi' f = f\psi$ (passing $fgf \dashrightarrow f$) and $g\varphi' = \psi g$ (passing $gfg \dashrightarrow g$).[1]

Proposition 2.4. Let $\mathcal{H} : \mathcal{A} \rightleftharpoons \mathcal{B}$ be a bridge, such that the multiplication morphism $\mu : \mathcal{H}^{>}\mathcal{H}^{<} \to \mathbf{1}_{\mathcal{A}}$ is *surjective*. Then μ is immediately an *isomorphism*.

Proof. By surjectivity, there are heteromorphisms h_A and k_A for every object $A \in \mathrm{Ob}\mathcal{A}$ such that $h_A \cdot k_A = 1_A$. Now, if $\langle f, g\rangle^{\mu} = \alpha \in \mathrm{Hom}\mathcal{A}$ and $A = \mathrm{dom} f$, then $\langle f, g\rangle = \langle h_A \cdot k_A f, g\rangle = \langle h_A, k_A f \cdot g\rangle = \langle h_A, k_A \cdot \alpha\rangle$ in $\mathcal{H}^{>} \cdot \mathcal{H}^{<}$, depending only on α. So, if also $\langle f_1, g_1\rangle^{\mu} = \alpha$, then we have $\langle f_1, g_1\rangle = \langle h_A, k_A\alpha\rangle = \langle f, g\rangle$, so μ is injective. □

An analogous general assertion is proved in Prop.4.6. Cf. also Prop.1.10 in Chifan-Dasc.-Nast.[4].

By the *idempotent* (or *Cauchy*) *completion* of a category $\mathcal{A}$ we mean the category $\mathcal{A}^{id}$ whose objects are the idempotent morphisms of $\mathcal{A}$, and whose arrows between idempotents $e, f \in \mathrm{Hom}\mathcal{A}$ are those morphisms $\alpha \in \mathrm{Hom}\mathcal{A}$ which satisfy $e\alpha f = \alpha$.[2] In particular, if $e : A \to A$ is idempotent, we have $e : e \to e$ (this is the identity arrow of e), $e : A \to e$ and $e : e \to A$ in $\mathcal{A}^{id}$. To distinguish these we write $1_e : e \to e$, $e^{\diamond} : A \to e$ and $e_{\diamond} : e \to A$.

Note that $\mathcal{A}$ fully embeds in $\mathcal{A}^{id}$ by sending an object $A \in \mathrm{Ob}\mathcal{A}$ to its identity morphism.

Definition 2.5. We call a bridge $\mathcal{M} : \mathcal{A} \rightleftharpoons \mathcal{B}$ a *Morita bridge*, if each identity morphism of $\mathcal{M}$ can be factorized as composition of heteromorphisms.

We note that a bridge $\mathcal{M} : \mathcal{A} \rightleftharpoons \mathcal{B}$ is a Morita bridge if and only if *each morphism* of $\mathcal{M}$ is a composition of heteromorphisms. In other words, both multiplication morphisms are surjective, hence, by Prop.2.4, are profunctor isomorphisms.

In this terminology we find an elementary proof of the classical result which asserts that Theorem 2.6.a) and c) are equivalent.

Theorem 2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. The following conditions are equivalent:

a) There are profunctors $\mathtt{T} : \mathcal{A} \Rightarrow \mathcal{B}$ and $\mathtt{S} : \mathcal{B} \Rightarrow \mathcal{A}$ such that both compositions $\mathtt{TS}$ and $\mathtt{ST}$ are isomorphic to the identity. (That is, $\mathcal{A}$ and $\mathcal{B}$ are equivalent in $\mathbb{P}\mathrm{rof}$).

[1] Here we have hidden the coherence 2-cells $(fg)f \dashrightarrow f(gf)$ and $(gf)g \dashrightarrow g(fg)$.

[2] To keep precisity, we should write $\langle e, \alpha, f\rangle$ for morphisms $e \xrightarrow{\alpha} f$ in $\mathcal{A}^{id}$.

b) There is a Morita bridge $\mathcal{M}$ between $\mathcal{A}$ and $\mathcal{B}$.
c) Categories $\mathcal{A}^{id}$ and $\mathcal{B}^{id}$ are equivalent.

Proof.
a)⇒b): By Lemma 2.3, there exist isomorphisms $\varphi:\mathtt{TS} \to \mathbf{1}_\mathcal{A}$ and $\psi:\mathtt{ST} \to \mathbf{1}_\mathcal{B}$ such that $\varphi\mathtt{T} = \mathtt{T}\psi$ and $\mathtt{S}\varphi = \psi\mathtt{S}$. These define an associative composition on the class of morphisms $\mathtt{T} \cup \mathtt{S}$ yielding a bridge $\mathcal{M}:\mathcal{A} \rightleftharpoons \mathcal{B}$. This $\mathcal{M}$ is readily seen to be a Morita bridge.
b)⇒c): We can form the idempotent completion $\mathcal{M}^{id}$ of the category $\mathcal{M}$. Since each identity morphism $1_A \in \mathrm{Hom}\mathcal{A}$ can be written in the form $1_A = hk$ for heteromorphisms $h, k \in \mathrm{Het}\mathcal{M}$, we obtain that $A \in \mathrm{Ob}\mathcal{A}$ is isomorphic to $kh \in \mathrm{Ob}\mathcal{B}^{id}$. From this we can easily conclude that $\mathcal{M}^{id}$ is an equivalence bridge between $\mathcal{A}^{id}$ and $\mathcal{B}^{id}$.
c)⇒b): Let $\mathcal{H} : \mathcal{A}^{id} \rightleftharpoons \mathcal{B}^{id}$ be an equivalence bridge, and consider its full subcategory $\mathcal{M}$ on $\mathrm{Ob}\mathcal{A} \cup \mathrm{Ob}\mathcal{B}$. Then $\mathcal{M}$ is a Morita bridge: for an object $A \in \mathrm{Ob}\mathcal{A}$ there is an idempotent $f : B \to B$ in $\mathrm{Ob}\mathcal{B}^{id}$ and an isomorphism $\alpha : A \to f$ in $\mathcal{H}$. By the structure of $\mathcal{B}^{id}$ we have arrows $f^\diamond : B \to f$ and $f_\diamond : f \to B$ with $f_\diamond \cdot f^\diamond = 1_f$. Then $1_A = (\alpha f_\diamond)\cdot(f^\diamond\alpha^{-1})$.
b)⇒a): Consider the parts $\mathcal{M}^> : \mathcal{A} \Rightarrow \mathcal{B}$ and $\mathcal{M}^< : \mathcal{B} \Rightarrow \mathcal{A}$ of $\mathcal{M}$. These are profunctors satisfying $\mathcal{M}^>\mathcal{M}^< \cong \mathbf{1}_\mathcal{A}$ and $\mathcal{M}^<\mathcal{M}^> \cong \mathbf{1}_\mathcal{B}$, according to Prop.2.4. □

3 More on bridges

By a *proper bridge* we mean a bridge that has heteromorphisms in both directions.

Proposition 3.1. There exist categories $\mathcal{A}$ and $\mathcal{B}$ such that there is no proper bridge between them.

Proof. Let $\mathit{1}$ denote the trivial category on one object (denoted by 1), and let $\mathcal{A}$ be a nontrivial monoid without idempotents (for example the free category on one morphism $A \to A$). Then, in a proper bridge $\mathcal{H} : \mathit{1} \rightleftharpoons \mathcal{A}$, we would have heteromorphisms $f : 1 \to A$ and $g : A \to 1$, and since $\mathit{1}$ contains only the identity, $fg = 1_1$, then gf should be idempotent. But $\mathcal{A}$ does not have idempotents other than 1_A, so we have that $1 \cong A$ in $\mathcal{H}$, contradicting that $\mathrm{End}1 \not\cong \mathrm{End}A$. □

We note that, though $\mathbb{P}\mathrm{rof}(\mathcal{A}|\mathcal{B})$ is always complete and cocomplete, we could not straightforwardly define even products or coproducts of bridges:

Proposition 3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Then the category $\mathbb{B}\mathrm{r}(\mathcal{A}|\mathcal{B})$ has initial object but not necessarily a terminal object nor finite coproducts.

Proof. The initial object is the empty bridge (disjoint union of $\mathcal{A}$ and $\mathcal{B}$), and of course, it is preserved by bridge composition. On the other hand, if we choose $\mathcal{A}$ and $\mathcal{B}$ as in Prop.3.1, and arbitrary nonempty profunctors $\mathtt{F} : \mathcal{A} \Rightarrow \mathcal{B}$ and $\mathtt{G} : \mathcal{B} \Rightarrow \mathcal{A}$ (these always exist if $\mathcal{A}$ and $\mathcal{B}$ are nonempty), regarding both as bridges $\mathcal{A} \rightleftharpoons \mathcal{B}$, then they do not even have a common span $\mathtt{F} \to \mathcal{H} \leftarrow \mathtt{G}$ in $\mathbb{B}\mathrm{r}(\mathcal{A}|\mathcal{B})$. This also proves that $\mathtt{F}$ and $\mathtt{G}$ does not have a coproduct. □

Observe that, based on Def.1.1, a bridge $\mathcal{H} : \mathcal{A} \rightleftharpoons \mathcal{B}$ (and therefore also a profunctor) can be represented as a special cospan of its full embedding functors $\mathcal{A} \hookrightarrow \mathcal{H} \hookleftarrow \mathcal{B}$. Next we study the functorial behaviour of this correspondence.
For that we recall the followings: a category $\mathcal{C}$ with pushouts induces the bicategory $\mathbb{C}\mathrm{osp}(\mathcal{C})$ whose arrows are the cospans of $\mathcal{C}$, with composition defined by pushout. A *colax functor* Φ between bicategories $\mathbb{A} \longrightarrow \mathbb{B}$ is defined as a mapping on objects, arrows and 2-cells of $\mathbb{A}$ to those of $\mathbb{B}$, preserving composition of 2-cells, together with the so called *colax* (or *comparison*) 2-cells $(fg)^{\Phi} \dashrightarrow f^{\Phi}g^{\Phi}$ for all composable pairs of arrows f, g in $\mathbb{A}$ and ${1_A}^{\Phi} \overset{\varepsilon_A}{\dashrightarrow} 1_{A^{\Phi}}$, which make the induced diagrams with the corresponding coherence 2-cells commutative. Cf. e.g. Street[17], section 9. Φ is said to be *normalized* if ε_A is identity (thus ${1_A}^{\Phi} = 1_{A^{\Phi}}$) for all objects A in $\mathbb{A}$.

Definition 3.3. We say that $\Phi : \mathbb{A} \longrightarrow \mathbb{B}$ is a *coreflective colax embedding*, if Φ is a colax functor such that, for all $A, A_1 \in \mathrm{Ob}\mathbb{A}$, it is a coreflective full embedding of categories $\mathbb{A}(A \shortmid A_1) \hookrightarrow \mathbb{B}(A^{\Phi} \shortmid {A_1}^{\Phi})$, and for any composable pair of arrows $A \overset{f}{\to} A_0 \overset{g}{\to} A_1$, the corresponding colax 2-cell $(fg)^{\Phi} \dashrightarrow f^{\Phi}g^{\Phi}$ is just a coreflection arrow (that is, $f^{\Phi}g^{\Phi} \in \mathrm{Ob}\mathbb{B}(A^{\Phi} \shortmid {A_1}^{\Phi})$ is coreflected by $fg \in \mathrm{Ob}\mathbb{A}(A \shortmid A_1)$).

The dual notions (*lax functor*, *normalized lax functor*, *reflectie lax embedding*) are defined by reverting the 2-cells.

Lemma 3.4. Let $\mathcal{H} : \mathcal{A} \rightleftharpoons \mathcal{B}$ and $\mathcal{K} : \mathcal{B} \rightleftharpoons \mathcal{C}$ be bridges, then the union of all given arrows, $\mathcal{H} \cup \mathcal{K} \cup \mathcal{HK}$ (including also $\mathcal{B}$) is a pushout of $\mathcal{H} \hookleftarrow \mathcal{B} \hookrightarrow \mathcal{K}$ in $\mathcal{C}at$.

Proof. If the functors $F : \mathcal{H} \to \mathcal{S}$ and $G : \mathcal{K} \to \mathcal{S}$ equal on $\mathcal{B}$, they uniquely define a functor from this "threefold composition" $\mathcal{H} \cup \mathcal{K} \cup \mathcal{HK}$ to $\mathcal{S}$, by mapping $h \mapsto h^F$, $k \mapsto k^G$, $\langle h, k \rangle \mapsto h^F \cdot k^G$ and $\langle k, h \rangle \mapsto k^G \cdot h^F$. ☐

Theorem 3.5. Let $\mathcal{C}at^{\text{full emb.}}$ denote the category of categories and full embeddings. The correspondence sketched above Def.3.3 yields the following coreflective colax embeddings: $\mathbb{B}\mathrm{r} \longrightarrow \mathbb{C}\mathrm{osp}(\mathcal{C}at^{\text{ full emb.}})$ and $\mathbb{P}\mathrm{rof} \longrightarrow \mathbb{C}\mathrm{osp}(\mathcal{C}at)$.

Proof. A cospan of functors $\mathcal{A} \overset{F}{\to} \mathcal{T} \overset{G}{\leftarrow} \mathcal{B}$ is coreflected in $\mathbb{P}\mathrm{rof}(\mathcal{A} \shortmid \mathcal{B})$ by the profunctor defined by $\mathtt{T}(A \shortmid B) := \mathcal{T}(A^F \shortmid B^G)$ for objects A, B. Moreover, if F and G are both full embeddings, then the full restriction of $\mathcal{T}$ to objects of $\mathrm{Ob}\mathcal{A} \cup \mathrm{Ob}\mathcal{B}$ is a coreflection of this cospan in $\mathbb{B}\mathrm{r}(\mathcal{A} \shortmid \mathcal{B})$.
Let $\mathcal{H} : \mathcal{A} \rightleftharpoons \mathcal{B}$ and $\mathcal{K} : \mathcal{B} \rightleftharpoons \mathcal{C}$ be bridges [resp. profunctors]. By lemma 3.4, the composition of the corresponding cospans $\mathcal{A} \hookrightarrow \mathcal{H} \hookleftarrow \mathcal{B}$ and $\mathcal{B} \hookrightarrow \mathcal{K} \hookleftarrow \mathcal{C}$ is just the "threefold composition" $\mathcal{H} \cup \mathcal{K} \cup \mathcal{HK}$. This is coreflected by $\mathcal{HK}$ in $\mathbb{B}\mathrm{r}(\mathcal{A} \shortmid \mathcal{C})$ [resp. in $\mathbb{P}\mathrm{rof}(\mathcal{A} \shortmid \mathcal{C})$]. ☐

It is worth to mention that, in the same way (through "threefold compositions"), we also have the following *reflective lax embeddings*: $\mathbb{R}\mathrm{el} \longrightarrow \mathbb{S}\mathrm{pan}(\mathcal{S}et)$ and $\mathbb{P}\mathrm{rof} \longrightarrow \mathbb{S}\mathrm{pan}(\mathcal{C}at)$, this latter maps a profunctor $\mathtt{F} : \mathcal{A} \Rightarrow \mathcal{B}$ to the span $\mathcal{A} \leftarrow \mathcal{H}et\mathtt{F} \to \mathcal{B}$ where $\mathcal{H}et\mathtt{F}$ stands for the arrow category of $\mathtt{F}$ restricted only to heteromorphisms (i.e., its morphisms are the commutative squares of $\mathtt{F}$ with heteromorphisms as top and bottom arrows). See also Grandis-Pare[9].

4 Morita contexts in general

Definition 4.1. A *Morita context* or, in our previous terminology, an *abstract bridge* in a bicategory $\mathbb{B}$ is defined to be a tuple $\langle A, B, f^>, f^<, \mu^\times, \mu^\Diamond \rangle$ where A and B are objects, $f^> : A \to B$ and $f^< : B \to A$ are arrows, and $\mu^\times : f^> f^< \dashrightarrow 1_A$ and $\mu^\Diamond : f^< f^> \dashrightarrow 1_B$ are 2-cells in $\mathbb{B}$ (called the *constructor 2-cells*), subject to the identities[3]

$$\mu^\times f^> \;=\; f^> \mu^\Diamond \qquad \text{and} \qquad f^< \mu^\times \;=\; \mu^\Diamond f^<. \tag{1}$$

The dual notion (with reversed 2-cells) is called *cobridge* or *left Morita context*.

We sometimes abbreviate a Morita context $\langle A, B, f^>, f^<, \mu^\times, \mu^\Diamond \rangle$ as $\mathbb{f} : A \rightleftharpoons B$. A Morita context is *strict* if both $\mu^\times$ and $\mu^\Diamond$ are isomorphisms. Strict Morita contexts and adjoint equivalences are basically the same (we can switch between them by inverting the unit of the adjunction). Moreover, any adjunction $\langle \underset{A \to B}{f}, \underset{B \to A}{g}, \underset{1_A \dashrightarrow fg}{\varepsilon}, \underset{gf \dashrightarrow 1_B}{\eta} \rangle$ immediately becomes a Morita context whenever its unit, ε is invertible.

Remark 4.2. *Unlike at adjoints, the parts of a Morita context do not determine each other up to equivalence. To illustrate this, we can take any proper bridge $\mathcal{H}$ as a Morita context in $\mathbb{Prof}$ (cf. Remark 1.7), and keep one of its parts, say $\mathcal{H}^>$. That still remains a bridge, but now goes with the empty profunctor as its other part.*

Theorem 4.3. For a given bicategory $\mathbb{B}$, its Morita contexts as arrows naturally define another bicategory [denoted by $\mathbf{Mrt}\mathbb{B}$] on the same objects, which is symmetrical on 1-cells.

Proof. Let $\mathbb{f} = \langle A, B, f^>, f^<, \mu^\times, \mu^\Diamond \rangle$ and $\mathbb{g} = \langle A, B, g^>, g^<, \vartheta^\times, \vartheta^\Diamond \rangle$ be Morita contexts in $\mathbb{B}$. By a *morphism* (or *2-cell*) from $\mathbb{f}$ to $\mathbb{g}$ we mean a pair of 2-cells $\alpha^> : f^> \dashrightarrow g^>$ and $\alpha^< : f^< \dashrightarrow g^<$ such that $(\alpha^> \# \alpha^<) \cdot \vartheta^\times = \mu^\times$ and $(\alpha^< \# \alpha^>) \cdot \vartheta^\Diamond = \mu^\Diamond$ (where $\#$ denotes horizontal composition).
Composition of Morita contexts $\mathbb{f} : A \rightleftharpoons B$ and $\mathbb{g} : B \rightleftharpoons C$ is defined on the arrows $\underset{A \longrightarrow C}{f^> g^>}$ and $\underset{C \longrightarrow A}{g^< f^<}$ with constructor 2-cells

$$\underset{f^> g^> g^< f^< \dashrightarrow f^> f^< \dashrightarrow 1_A}{(f^> \vartheta^\times f^<) \cdot \mu^\times} \qquad \text{and} \qquad \underset{g^< f^< f^> g^> \dashrightarrow g^< g^> \dashrightarrow 1_B}{(g^< \mu^\Diamond g^>) \cdot \vartheta^\Diamond} \;.$$

Note that, for $\mathbb{B} = \mathbb{Prof}$, this coincides with the bridge composition defined at Prop.1.6. These constructor 2-cells satisfy the identities of Def.4.1, because, $(f^> \vartheta^\times f^< f^> g^>) \cdot (\mu^\times f^> g^>) = (f^> \vartheta^\times f^< f^> g^>) \cdot (f^> \mu^\Diamond g^>) = f^> (\vartheta^\times \# \mu^\Diamond) g^>$, and $(f^> g^> g^< \mu^\Diamond g^>) \cdot (f^> g^> \vartheta^\Diamond)$ amounts the same. $\mathbf{Mrt}\mathbb{B}$ is *symmetrical* on 1-cells in the sense that each arrow $\mathbb{f}$ in $\mathbf{Mrt}\mathbb{B}$ (as above) has a *reverse* arrow: $\mathbb{f}^\smile := \langle B, A, f^<, f^>, \mu^\Diamond, \mu^\times \rangle$. All the detailed constructions are also found in El Kaoutit[7]. $\square$

[3] Let α, λ and ϱ denote the coherence 2-cells $(f^> f^<) f^> \overset{\alpha}{\dashrightarrow} f^> (f^< f^>)$, $1_A f^> \overset{\lambda}{\dashrightarrow} f^>$ and $f^> 1_B \overset{\varrho}{\dashrightarrow} f^>$. Then, the left hand side of (1) abbreviates the following equality: $\mu^\times f^> \cdot \lambda = \alpha \cdot f^> \mu^\Diamond \cdot \varrho$. All similar abbreviations are hidden throughout the paper.

Observe that, if $\mathbb{B}$ has *initial arrows* (i.e. initial objects in all $\mathbb{B}(A \mid B)$ for $A, B \in \mathrm{Ob}\mathbb{B}$, preserved by the actions $x \mapsto fx$ and $x \mapsto xf$ for all adequate arrow f), then it admits natural embeddings $\mathbb{B} \hookrightarrow \mathbf{Mrt}\mathbb{B}$ and $\mathbb{B}^{op} \hookrightarrow \mathbf{Mrt}\mathbb{B}$.

The following theorem is analogous to the observation that a lax functor from the terminal 2-category $\mathbb{1}$ to any bicategory $\mathbb{B}$ is just a *monad* in $\mathbb{B}$. (Cf. Example 8 in Street[17], section 9.) We recall the notions defined before Def.3.3 and that a 2-category is a strictly associative bicategory (i.e., where all coherence 2-cells are identities).

Theorem 4.4. Let $\mathbb{2}$ denote the 2-category on the category $\begin{tikzcd} \cdot \arrow[r, bend left, "s"] & \cdot \arrow[l, bend left, "s^{-1}"] \end{tikzcd}$ with only identity 2-cells. Then, a *normalized lax functor* $\mathbb{2} \longrightarrow \mathbb{B}$ amounts exactly to a Morita context in $\mathbb{B}$. Dually, the *normalized colax functors* $\mathbb{2} \longrightarrow \mathbb{B}$ are the left Morita contexts.

Proof. Let $\Phi : \mathbb{2} \longrightarrow \mathbb{B}$ be a normalized lax functor and set $f := s^{\Phi} : A \to B$, $g := (s^{-1})^{\Phi} : B \to A$. Further, let μ and ν denote the lax 2-cells for ss^{-1} and $s^{-1}s$, respectively, that is, $\mu : fg \dashrightarrow 1_A$ and $\nu : gf \dashrightarrow 1_B$. The rest of structural 2-cells are predefined for Φ since it is normalized. For example, its colax 2-cells $1_A f \dashrightarrow f$ and $f 1_B \dashrightarrow f$ are just the coherence 2-cells of $\mathbb{B}$.

The commutativity of the combined diagrams of μ and ν with the coherence 2-cells $(fg)f \dashrightarrow f(gf)$ and $(gf)g \dashrightarrow g(fg)$, required for lax functoriality, means exactly the equalities of (1). ☐

Remark 4.5. *Morita contexts may also be defined in double categories. A double category consists of objects, horizontal and vertical arrows and squares surrounded by them. Squares are composed both horizontally and vertically, satisfying* $\left(\frac{\alpha\ \beta}{\varphi\ \psi}\right) = \left(\begin{array}{c|c} \alpha & \beta \\ \varphi & \psi \end{array}\right)$ *and associativities.* Grandis-Pare[9] *considers so called "pseudo-double categories" (or "vertically weak double categories") in which the composition of vertical arrows is not strict: defined only up to isomorphism. This perfectly suits our work (differing only that we rather deal with "horizontally weak" double categories).*
Such a (horizontally weak) double category $\mathbf{D}$ *determines its horizontal "underlying" bicategory* $\mathbb{H}\mathbf{D}$*, keeping only the identity vertical arrows (and taking the squares as 2-cells). Then, applying the construction above (which prepares* $\mathbf{Mrt}(\mathbb{H}\mathbf{D})$*) and allowing again all original vertical arrows, we get the (horizontally still weak) double category where horizontal arrows are replaced to horizontal Morita contexts. Let us denote it by* $\mathbf{Mrt}_{\mathbb{H}}(\mathbf{D})$*. See also sections 5.1 and 5.4. For more on double categories the reader is referred e.g. to* Ehresmann[6] *or* Grandis-Pare[9].

Proposition 4.6. Let $\langle f^{>}, f^{<}, \mu^{\mathsf{X}}, \mu^{\Diamond} \rangle$ be a Morita context. Then, if μ^{X} is *left invertible* then it is *invertible* at once. Same holds for $\mu^{\Diamond}$.

Proof. Let $\varrho : 1_A \dashrightarrow f^{>}f^{<}$ be a left inverse of μ^{X}. Then we have

$$1_{f^{>}f^{<}} = f^{>}f^{<}1_{1_A} = f^{>}f^{<}(\varrho{\cdot}\mu^{\mathsf{X}}) = (f^{>}f^{<}\varrho){\cdot}(f^{>}f^{<}\mu^{\mathsf{X}}) =$$
$$= (f^{>}f^{<}\varrho){\cdot}(f^{>}\mu^{\Diamond}f^{<}) = (f^{>}f^{<}\varrho){\cdot}(\mu^{\mathsf{X}}f^{>}f^{<}) = \mu^{\mathsf{X}}\#\varrho = \mu^{\mathsf{X}}{\cdot}\varrho\ .$$

☐

Lemma 4.7. Let $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\alpha"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\beta"] & C \end{tikzcd}$ be a diagram in the bicategory $\mathbb{B}$ such that all arrows f, f_1, g, g_1 are invertible and $\alpha\#\beta$ is an isomorphism, as well. Then, α and β are invertible, too.

Proof. Knowing that $\alpha\#\beta = f\beta{\cdot}\alpha g_1 = \alpha g{\cdot} f_1\beta$ is invertible, we get that $f\beta$ and αg are right invertible, while αg_1 and $f_1\beta$ are left invertible. Since f, f_1, g, g_1 are invertible, the actions $\xi \mapsto f\xi,\ \xi \mapsto \xi g$, etc. are category equivalences, reflecting left and right invertibility. Hence, α and β are both left and right invertible. $\square$

Theorem 4.8. The following conditions are equivalent for a Morita context $\mathbb{f}$ in a bicategory $\mathbb{B}$:
a) $\mathbb{f}$ is *strict*.
b) There exists a 2-cell from a strict Morita context $\mathbb{h}$ to $\mathbb{f}$ in $\mathbf{Mrt}\mathbb{B}$. (Such a 2-cell must be an *isomorphism*.)
c) $\mathbb{f}$ is an *invertible arrow* in $\mathbf{Mrt}\mathbb{B}$ (and then, together with $\mathbb{f}^{\smile}$ they constitute an *adjoint equivalence*).

Proof. *a)⇒b)*: Set $\mathbb{h} := \mathbb{f}$.
b)⇒a): Let $\mathbb{h} = \langle A, B, h^>, h^<, \eta^\times, \eta^\Diamond\rangle$ be a strict Morita context and let $(\alpha^>, \alpha^<)$ be a morphism $\mathbb{h} \dashrightarrow \mathbb{f}$. Then, by definition, $(\alpha^>\#\alpha^<)\cdot\mu^\times = \eta^\times$, which is invertible. Thus, $\mu^\times$ is *left invertible*, so, by Prop.4.6, it is *invertible*. Similarly for $\mu^\Diamond$, proving that $\mathbb{f}$ is strict, indeed. In addition, we get that the horizontal composition $\alpha^>\#\alpha^<$ is invertible, too, implying that both $\alpha^>$ and $\alpha^<$ are so (by Lemma 4.7).
a)⇒c): If $\mathbb{f} = \langle A, B, f^>, f^<, \mu^\times, \mu^\Diamond\rangle$ is strict, then $(\mu^\times, \mu^\Diamond)$ and $(\mu^\Diamond, \mu^\times)$ are invertible 2-cells $\mathbb{f}\mathbb{f}^{\smile} \dashrightarrow \mathbf{1}_A$ and $\mathbb{f}^{\smile}\mathbb{f} \dashrightarrow \mathbf{1}_B$, respectively.
c)⇒a): Let $\mathbb{g} = \langle B, A, g^>, g^<, \vartheta^\times, \vartheta^\Diamond\rangle$ be an inverse of $\mathbb{f}$, so that $\mathbf{1}_A \cong \mathbb{f}\mathbb{g}$ and $\mathbf{1}_B \cong \mathbb{g}\mathbb{f}$. Then, using *b)⇒a)*, we get that $\mathbb{f}\mathbb{g}$ and $\mathbb{g}\mathbb{f}$ are strict, i.e. their constructor 2-cells are invertible, in particular $(f^>\vartheta^\times f^<)\cdot\mu^\times$ and $(f^<\vartheta^\Diamond f^>)\cdot\mu^\Diamond$ are so, concluding that $\mu^\times$ and $\mu^\Diamond$ are *left invertible* again. $\square$

Corollary 4.9. Let $\langle \mathbb{f}, \mathbb{g}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}\rangle$ be an *adjunction* in $\mathbf{Mrt}\mathbb{B}$. Then its unit, $\boldsymbol{\varepsilon}$ is necessarily an isomorphism 2-cell. (Thus, every adjunction becomes a Morita context in $\mathbf{Mrt}\mathbb{B}$.)

Corollary 4.10. Two objects are equivalent in $\mathbb{B}$ if and only if they are equivalent in $\mathbf{Mrt}\mathbb{B}$.

Proof. In the presence of lemma 2.3, this is just Thm.4.8.*a)* ⇔ *c)*. $\square$

5 Examples

Next we investigate Morita contexts in several well known bicategories.

5.1 In $\mathbb{P}\mathrm{rof}$

Recall that $\mathbb{B}\mathrm{r}$ denotes the bicategory of *categories*, *bridges* and *bridge morphisms* and $\mathbb{P}\mathrm{rof}$ stands for its restriction to *directed bridges*. Then, as it has been previously indicated, we have the following

Theorem 5.1. The bicategories $\mathbf{Mrt}\mathbb{Prof}$ and $\mathbb{Br}$ coincide.

Proof. According to Remark 1.7 and the proofs of Prop.1.6 and Thm.4.3, we only have to prove that the 2-cells (and their composition) coincide. Let $F : \mathcal{H} \to \mathcal{K}$ be a bridge morphism between bridges $\mathcal{H}, \mathcal{K} : \mathcal{A} \rightleftharpoons \mathcal{B}$. Let η^{X}, $\eta^{\Diamond}$ and κ^{X}, $\kappa^{\Diamond}$ denote their constructor 2-cells, (i.e., for instance, $\langle h, h_1 \rangle^{\eta^{\mathsf{X}}} = hh_1 \in \mathcal{A}$ for heteromorphisms $h : A \to B$ and $h_1 : B \to A_1$ in $\mathcal{H}$).
Then F is determined by its *parts*, $F^{>} := F\restriction_{\substack{\mathcal{H}^{>}\\ \mathcal{H}^{>} \to \mathcal{K}^{>}}}$ and $F^{<} := F\restriction_{\substack{\mathcal{H}^{<}\\ \mathcal{H}^{<} \to \mathcal{K}^{<}}}$ which are 2-cells in $\mathbb{Prof}$, and the conditions $(F^{>}\#F^{<})\cdot\kappa^{\mathsf{X}} = \eta^{\mathsf{X}}$ and $(F^{<}\#F^{>})\cdot\kappa^{\Diamond} = \eta^{\Diamond}$ express exactly that F is *functor*. $\square$

Strict Morita contexts are the *Morita bridges* (Def.2.5), and $\mathbb{Prof}$ admits *initial arrows*, namely the empty bridges. We note that Remark 4.5 applies here for the (horizontally weak) double category $\mathbf{CAT}$ with categories as objects, profunctors as horizontal and functors as vertical arrows, and with squares

$$\begin{tikzcd} \mathcal{A} \arrow[r, Rightarrow, "\mathtt{T}"] \arrow[d, "F"'] \arrow[dr, phantom, "K"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A}_1 \arrow[r, Rightarrow, "\mathtt{T}_1"'] & \mathcal{B}_1 \end{tikzcd}$$

where K is a functor $\mathtt{T} \to \mathtt{T}_1$ such that $K\restriction_{\mathcal{A}} = F$ and $K\restriction_{\mathcal{B}} = G$. Similarly, squares of $\mathbf{Mrt}_{\mathbb{H}}(\mathbf{CAT})$ are just functors between the horizontal arrows (i.e., bridges), obeying the same restriction property.

5.2 In $\mathbb{Biact}$

Theorem 5.2. In the bicategory $\mathbb{Biact}$ of monoids, biacts (i.e. bimodules between monoids) and biact morphisms, a Morita context between monoids A and B is just a category with 2 objects, say X and Y, satisfying $A = \mathrm{End}X$ and $B = \mathrm{End}Y$.

Proof. It is obvious by Thm.5.1, using that a monoid is just a category with 1 object. $\square$

5.3 In $\mathbb{Rel}$

The bicategory $\mathbb{Rel}$ consists of all sets (as objects), all relations between them (as arrows), and the inclusions of relations (as 2-cells), i.e. the homcategory $\mathbb{Rel}(A \shortmid B)$ is the usual partial order on the power set of $A \times B$.

A Morita context in $\mathbb{Rel}$ is a pair of relations $A \overset{\alpha}{-} B \overset{\beta}{-} A$ such that $\alpha\beta \leqslant 1_A$ and $\beta\alpha \leqslant 1_B$. Dually, the pair $\langle \alpha, \beta \rangle$ is a cobridge (left Morita context) iff $\alpha\beta \geqslant 1_A$ and $\beta\alpha \geqslant 1_B$.

For a pair $\langle \alpha, \beta \rangle$ of relations we draw an undirected *coloured graph* by putting a *blue* edge between elements $a \in A$ and $b \in B$ iff $a[\alpha]b$ and a *red* edge iff $b[\beta]a$. By a *correspondence* in such a graph we mean a pair of elements connected by both blue and red edges, like $a \overset{\text{red}}{\underset{\text{blue}}{\frown\!\smile}} b$.

Theorem 5.3. Let $\alpha : A - B$ and $\beta : B - A$ be relations.
1) The pair $\langle \alpha, \beta \rangle$ is a Morita context in $\mathbb{Rel}$ if and only if each connected component of its coloured graph uses only one colour, or is a correspondence.
2) The pair $\langle \alpha, \beta \rangle$ is a left Morita context in $\mathbb{Rel}$ if and only if its coloured graph contains a correspondence on each $a \in A$ and each $b \in B$.

Proof. 1) In any component with 2 colours we find an element, say $a \in A$, covered by a blue and a red edge. That is, $a[\alpha]b$ and $b_1[\beta]a$ for some $b, b_1 \in B$, but then $b_1[\beta\alpha]b$, yielding $b_1 = b$. So, $\langle a, b \rangle$ lies in a correspondence, whence, by the same argument, no more edges can cover neither a nor b.
2) We require that $\forall a \exists b : a[\alpha]b[\beta]a$ and $\forall b \exists a : b[\beta]a[\alpha]b$. $\square$

5.4 In $\mathbb{Bimod}$

Now we consider the bicategory $\mathbb{Bimod}$ of unitary rings, bimodules, and bimodule morphisms. A Morita context in $\mathbb{Bimod}$ is a pair of bimodules $\langle {}_AM_B \, , \, {}_BN_A \rangle$ equipped with associative "scalar multiplication" morphisms: $\varphi : M \underset{B}{\otimes} N \to A$ and $\psi : N \underset{A}{\otimes} M \to B$.

An illustrative example is the *(strict)* Morita context between an aribtrary unitary ring R and its full matrix ring $R^{n \times n}$ for an $n \in \mathbb{N}$: Let M be the bimodule of all row vectors ($R^{1 \times n}$), and let N be the bimodule of all column vectors ($R^{n \times 1}$) with the usual actions of R and $R^{n \times n}$, and let φ and ψ be also given by matrix multiplication. Then these are bimodule isomorphisms (they are easily seen to be surjective, and the analogous version of Prop.2.4 holds for $\mathbb{Bimod}$, cf. Cohn[5]).

Let $\mathcal{Ab}$ denote the category of Abelian groups. Recall that idempotent elements e, f of a ring R are called *orthogonal* if $ef = 0 = fe$.
We note that the following theorem is essentially proved in Sands[16], section 3.

Theorem 5.4. A Morita context in $\mathbb{Bimod}$ induces and is fully determined by
a) a 2 objected category enriched over $\mathcal{Ab}$, or, by
b) a unitary ring R together with orthogonal idempotent elements $e, f \in R$ such that $e + f = 1$, or, by
c) a unitary ring R with a distinguished idempotent $e \in R$.

Proof. a): Following the ideas given at the case of $\mathbb{Biact}$, we construct a category $\mathcal{H}$ with objects, say, X and Y, such that $\mathrm{End}X$ and $\mathrm{End}Y$ are just the given rings, while the homsets $\mathcal{H}(X|Y)$ and $\mathcal{H}(Y|X)$ are the given bimodules. Composition is coming from the bridge structure.
b): For a category $\mathcal{H}$ as in a), we define R as the direct sum of Abelian groups $\mathrm{End}X \oplus \mathcal{H}(X|Y) \oplus \mathrm{End}Y \oplus \mathcal{H}(Y|X)$. Multiplication of composable elements is defined just as in $\mathcal{H}$, and let the rest result in 0. The distinguished idempotents are 1_X and 1_Y.
Conversely, for a given ring R as in b), we set $A := eRe$ and $B := fRf$ (these are rings with units e and f, respectively), and $M := eRf$ and $N := fRe$ (these are $A-B$ and $B-A$ bimodules, respectively). The homomorphisms $\varphi : M \underset{B}{\otimes} N \to A$ and $\psi : N \underset{A}{\otimes} M \to B$ are given by the multiplication of R.
c): The idempotent e determines f by $f = 1 - e$, and $1 - e$ is always idempotent, orthogonal to e. $\square$

Besides the bicategory constructed in general below Def.4.1, according to Remark 4.5, as $\mathbb{Bimod}$ is the horizontal underlying bicategory of a convenient double category **RING** [in which vertical arrows are bimodules, horizontal arrows are ring homomorphisms, and squares are adequate bimodule morphisms], we can allow arrows between Morita contexts in $\mathbb{Bimod}$ with distinct base rings, too: Let $\langle A, B, M, N, \varphi, \psi\rangle$ and $\langle A_1, B_1, M_1, N_1, \varphi_1, \psi_1\rangle$ be Morita contexts. Then a 4-tuple $\langle f, g, p, q\rangle$ will be a *morphism* between them if $\underset{A\to A_1}{f}$ and $\underset{B\to B_1}{g}$ are unitary ring homomorphisms, $\underset{M\to M_1}{p}$ and $\underset{N\to N_1}{q}$ are Abelian group homomorphisms satisfying $a^f m^p b^g = (amb)^p$ and $b^g n^q a^f = (bna)^q$ for all elements $a \in A$, $b \in B$, $m \in M$, $n \in N$, and making the following diagrams commute.

$$\begin{tikzcd} M\otimes N \arrow[r, "\varphi"] \arrow[d, "p\otimes q"'] & A \arrow[d, "f"] \\ M_1\otimes N_1 \arrow[r, "\varphi_1"'] & A_1 \end{tikzcd} \qquad \begin{tikzcd} N\otimes M \arrow[r, "\psi"] \arrow[d, "q\otimes p"'] & B \arrow[d, "g"] \\ N_1\otimes M_1 \arrow[r, "\psi_1"'] & B_1 \end{tikzcd}$$

Corollary 5.5. The category of Morita contexts and morphisms defined above[4] is equivalent to the category of unitary rings with a distinguished idempotent (i.e. pairs $\langle R, e\rangle$ with $e^2 = e \in R$) and unitary ring homomorphisms respecting the given idempotents.

Proof. Using Theorem 5.4.c) we can easily see that the required properties of a morphism mean exactly that the induced mapping between the corresponding rings is a ring homomorphism. □

We note that the duality of Morita contexts $\langle M, N, \varphi, \psi\rangle \leftrightsquigarrow \langle N, M, \psi, \varphi\rangle$ corresponds to the *duality* $\langle R, e\rangle \leftrightsquigarrow \langle R, (1-e)\rangle$ of "idempointed rings".

5.5 In $\mathbb{Cat}$

We call Morita contexts in the bicategory $\mathbb{Cat}$ (of categories, functors and natural transformations) *functor bridges*. So, they are pairs of functors $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] & \mathcal{B} \arrow[l, bend left, "G"] \end{tikzcd}$ together with natural transformations $\varphi : FG \to 1_{\mathcal{A}}$ and $\psi : GF \to 1_{\mathcal{B}}$ such that $\varphi F = F\psi$ and $G\varphi = \psi G$.

As we already mentioned, an adjoint pair of functors with invertible unit always makes a functor bridge. (E.g., F is a coreflective embedding and G is the coreflection functor.) In particular, it applies for any (adjoint) equivalence. For another example, if both $\mathcal{A}$ and $\mathcal{B}$ have initial objects ($0_{\mathcal{A}}$ and $0_{\mathcal{B}}$), then the constant functors $A \mapsto 0_{\mathcal{B}}$ and $B \mapsto 0_{\mathcal{A}}$ constitute a Morita context in $\mathbb{Cat}$.

The next theorem gives a characterization of functor bridges in terms of coreflective subcategories, in a very similar way as Cor.1.4.

Theorem 5.6. Let $\mathcal{A}$ and $\mathcal{B}$ be coreflective full subcategories of a category $\mathcal{C}$. Then the coreflection functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ constitute a functor

[4]Note that this is just the underlying category of $\mathbf{Mrt}_{\mathbb{H}}(\mathbf{RING})$ with all its horizontal arrows as objects and squares as morphisms between them.

bridge. *Conversely*, every functor bridge arises this way.
Dually, cobridges in $\mathbb{Cat}$ correspond to pairs of reflective full subcategories.

Proof. Consider the natural embedding of bicategories $\mathbb{Cat} \hookrightarrow \mathbb{Prof}$ which is contravariant on functors (sending $F\colon\mathcal{A} \to \mathcal{B}$ to $F^*\colon\mathcal{B} \Rightarrow \mathcal{A}$). It is covariant on natural transformations, so, a functor bridge $\langle F, G, \varphi, \psi\rangle$ appears in $\mathbb{Prof}$ as an ordinary bridge with parts G^* and F^*, that is, by Thm.1.3 the base categories $\mathcal{A}$ and $\mathcal{B}$ are coreflective subcategories of this bridge. Conversely, since the embedding $F \mapsto F^*$ is also *full* on natural transformations, a pair of coreflective subcategories induces a functor bridge with the (adequate restrictions of the) coreflection functors $F\colon\mathcal{A} \to \mathcal{B}$ and $G\colon\mathcal{B} \to \mathcal{A}$, again by Thm.1.3. □

As an example, consider the category of relation structures $\langle X, \varrho\rangle$ where ϱ is a binary relation on the set X (a function $f : X \to Y$ is a morphism from $\langle X, \varrho\rangle$ to $\langle Y, \sigma\rangle$ if $x[\varrho]x_1$ implies $x^f[\sigma]{x_1}^f$ for every pair $x, x_1 \in X$). Then its full subcategories $\mathcal{Sym}$ of symmetric and $\mathcal{Rfl}$ of reflexive relation structures are both reflective and coreflective, so, by Thm.5.6 and Cor.1.4 both coreflection functors $\mathcal{Sym} \to \mathcal{Rfl}$ and $\mathcal{Rfl} \to \mathcal{Sym}$ have a left adjoint and both make part of a functor bridge.

References

[1] J. C. Baez and A. D. Lauda. Higher-Dimensional Algebra V: 2-Groups. *Theory and Applications of Categories*, vol.**12**, (2004), 423–491.

[2] J. Bénabou. Introduction to Bicategories Reports of the Midwest Category Seminar, *Lecture Notes in Mathematics* vol.**47**, (1967), 1–77.

[3] F. Borceux. Handbook of Categorical Algebra: Vol. 1. Basic Category Theory, *Encycl. of Math. and its Appl.*, vol.**50**, (1994).

[4] N. Chifan, S. Dăscălescu, and C. Năstăsescu. Wide Morita Contexts, Relative Injectivity and Equivalence Results. *Journal of Algebra*, vol.**284**, (2005), 705–736.

[5] P. Cohn. Morita Equivalence and Duality. *Queen Mary College Math. Notes, London*, (1976).

[6] C. Ehresmann. Catégories Structurées III: Quintettes et Applications Covariantes. *Cahiers Top. Géom. Diff. Cat.*, vol.**5**, (1963), 1–22.

[7] L. El Kaoutit. Wide Morita Contexts in Bicategories. *Arab. J. Sci. Eng.* vol.**33**, (2008), 153–173.

[8] P. J. Freyd and A. Scedrov. Categories, Allegories. North-Holland, Amsterdam, (1990).

[9] M. Grandis and R. Paré. Limits in Double Categories. *Cahiers Top. Géom. Diff. Cat.*, vol.**45**, (1999), 162–220.

[10] A. I. Kashu. On Equivalence of Some Subcategories of Modules in Morita Context. *Algebra and Discrete Mathematics*, vol.**3**, (2003), 46–53.

[11] T. Kato and K. Ohtake. Morita Contexts and Equivalences. *Journal of Algebra*, vol.**61**, (1979), 360–366.

[12] S. MacLane Categories for the Working Mathematician, *(second edition)*. Graduate Texts in Mathematics, vol.**5**, Springer, (1998).

[13] M. Makkai. Avoiding the Axiom of Choice in General Category Theory. *Journal of Pure and Applied Algebra*, vol.**108**, (1996), 109–173.

[14] B. J. Müller. The Quotient Category of a Morita Context. *Journal of Algebra*, vol.**28**, (1974), 389–407.

[15] R. Houston. Paré's Observation. (2006) http://bosker.wordpress.com/2006/06/15/pares-observation/

[16] A. D. Sands. Radicals and Morita Contexts. *Journal of Algebra*, vol.**24**, (1973), 335–345.

[17] Ross Street. Categorical Structures. Handbook of Algebra Vol. 1. *Elsevier Science, Amsterdam*, (1996), 529–577.

[18] R. J. Wood. Proarrows II. *Cahiers Top. Géom. Diff. Cat.*, vol.**26**, (1985), 135–168.