Loops on a punctured disk and knotted tubes in $\mathbb{R}^4$

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Homotopy classes of loops on a twice-punctured disk (or a surface with boundary, more generally) admit a Lie bi–algebra structure called the Goldman-Turaev Lie bi–algebra. The Lie bracket and co-bracket are defined in terms of intersections and self-intersections of curves. Combining results of Alekseev–Kawazumi–Kuno–Naef with results of the speaker and Bar-Natan, one obtains a surprising statement: "well-behaved universal finite type invariants" of the Goldman–Turaev Lie bi-algebra are in bijection with the well-behaved universal finite type invariants invariants of a very different topological structure: a class of knotted tubes in $\mathbb{R}^4$. However, the bijection goes through representing both classes of invariants as solutions to certain algebraic equations (the Kashiwara–Vergne equations). This is clearly the wrong proof of a worthwhile theorem. But what is the right proof?