Invariant Random Subgroups and their Applications

Miklós Abért

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March 28, 2014
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What is an IRS

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- Let $\Gamma$ act on $(X, \mu)$ by p.m.p. maps. Then the random subgroup
  \[ \operatorname{Stab}_{\Gamma}(x) \leq \Gamma \quad (x \in X \text{ is } \mu\text{-random}) \]
  is an IRS of $\Gamma$. 

Lemma (A-Glasner-Virag)
Every IRS of $\Gamma$ arises as the stabilizer for a p.m.p. action of $\Gamma$. 

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**Lemma (A-Glasner-Virag)**

*Every IRS of $\Gamma$ arises as the stabilizer for a p.m.p. action of $\Gamma$.***
Invariant random subgroups:

- Tend to behave like normal subgroups, rather than arbitrary subgroups

\[ \text{IRS}(\Gamma), \text{the set of IRS's of } \Gamma, \text{endowed with the weak topology, is compact. So, every sequence has a convergent subsequence.} \]
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- Translates to Benjamini-Schramm convergence of the quotient spaces
- Tends to carry over spectral information (spectral measure, \(L^2\) Betti numbers, Plancherel measure, etc)
How to approximate the universal cover

Let $M$ be a compact space (complex, manifold). Let $\Gamma = \pi_1(M)$.

- Want to approximate the universal cover $\tilde{M}$ with compact covers $M_n$ of $M$.

- Usual solution: Take a chain $(\Gamma_n)$ of normal subgroups of finite index in $\Gamma$ with $\bigcap \Gamma_n = 1$ (*). Let $M_n = \tilde{M}/\Gamma_n$.

- Problems: Normal. Chain $(SL_2(\mathbb{Z}) \mod p)$. Not a natural convergence notion (can not merge).

- Suggestion: For $H \leq \Gamma$ of finite index let $\mu_H$ denote a uniform random conjugate of $H$ in $\Gamma$. Use weak convergence of IRS' s.

- A sequence $H_n$ is approximating if $\mu_{H_n} \to \mu_1$ where $\mu_1 = 1$ a.s.

- This convergence notion is equivalent to local sampling convergence: from a typical place in $M_n$, and looking at a bounded distance, we won't be able to distinguish $M_n$ and $\tilde{M}$.

- Typically, whatever is continuous for normal chains, is expected to be continuous for this convergence notion.
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The Lück Approximation Theorem for IRS’s

\( b_k \): \( k \)-th Betti number over \( \mathbb{Q} \); \( \beta_k^2 \): \( k \)-th \( L^2 \) Betti number.

**Theorem (Lück Approximation)**

Let \( M \) be a finite complex and let \( H_n \leq \pi_1(M) \) be finite index subgroups such that \( \mu_{H_n} \to \mu_1 \). Then for all \( k \) we have

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\lim_{n \to \infty} \frac{b_k(M_n)}{|\pi_1(M) : H_n|} = \beta_k^2(\tilde{M}).
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Gaboriau: \( L^2 \) Betti numbers of a p.m.p. action only depend on its IRS.

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- Minimal number of generators (converges on chains, but does the limit depend on the chain?). Fixed Price Problem of Gaboriau.
A countable group \( \Gamma \) is *sofic* if it admits a sequence of maps \( \phi_n : \Gamma \to \text{Sym}(n_k) \) such that for every finite subset \( S \subseteq \Gamma \), \( \phi_n \) restricted to \( S \) behaves like an injective group homomorphism with ratio of error tending to 0 (Gromov, Weiss).
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Sofic groups and IRS

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**Lemma**

Let $\Gamma = F / N$ where $F$ is a free group. Then $\Gamma$ is sofic if and only if there exist subgroups $H_n \leq F$ of finite index such that

$$\mu_{H_n} \to \delta_N$$

where $\delta_N$ is the Dirac measure on $N$. 

**Generalized sofic question (Aldous-Lyons):** is every IRS in a free group the weak limit of finite index IRS' s? Also open.
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A theorem on IRS’s.

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**Theorem (Kesten’s thesis)**

Let $\Gamma = \langle S \rangle$ and let $N \triangleleft \Gamma$ be a normal subgroup of infinite index. Then $\rho(\text{Cay}(\Gamma, S)) = \rho(\text{Cay}(\Gamma/N, S))$ if and only if $N$ is amenable.
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Let $\Gamma = \langle S \rangle$ and let $H$ be an IRS of $\Gamma$ of infinite index. Then $\rho(\text{Cay}(\Gamma, S)) = \rho(\text{Sch}(\Gamma/H, S))$ a.s. if and only if $H$ is amenable a.s.
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Exercise: free groups do not admit nontrivial amenable IRS’s. So, if $\Gamma$ is free and the IRS $H \neq 1$, we have $\rho(\text{Sch}(\Gamma / H, S)) > \rho(\text{Cay}(\Gamma, S))$. 
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**Corollary (A-Glasner-Virag)**

Let $G_n$ be finite $d$-regular graphs with $|G_n| \to \infty$. If $\lim \lambda_{G_n}$ is supported on $[-\rho(T_d), \rho(T_d)]$ then

$$\lim_{n \to \infty} \frac{\#\text{L-cycles in } G_n}{|G_n|} = 0 \quad (L > 0).$$
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$d_k(G)$: number of primitive, cyclically reduced cycles of length $k$ in $G$.

**Theorem (Serre)**

Let $(G_n)$ be finite $d$-regular graphs, such that $\gamma_k = \lim_{n \to \infty} d_k(G_n)/|G_n|$ exists ($k \geq 1$). Then $\lambda_{G_n}$ weakly converges. If $\sum_{k=1}^{\infty} \gamma_k (d-1)^{-k/2}$ converges then $\lim \lambda_{G_n}$ is absolutely continuous on $[-\rho(T_d), \rho(T_d)]$. 
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If Serre’s condition holds, then $\gamma_k = 0$ for all $k$ and $\lim_{n} \lambda_{G_n} = \lambda_{T_d}$. 

The Nevo-Stück-Zimmer theorem in IRS form

Let $G$ be a Lie group. For a lattice $\Gamma$ let $\mu_\Gamma$ denote the Haar-random conjugate of $\Gamma$. Let $\mu_1 = 1$ and $\mu_G = G$ a.s.
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**Theorem ([Stück-(Zimmer]-Nevo))**

Let \( G \) be a higher rank simple real Lie group and let \( H \) be an ergodic IRS in \( G \). Then \( H = \mu_G, \mu_1 \) or \( \mu_\Gamma \) for some lattice \( \Gamma \) of \( G \).
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The Nevo-Stück-Zimmer theorem in IRS form

Let $G$ be a Lie group. For a lattice $\Gamma$ let $\mu_{\Gamma}$ denote the Haar-random conjugate of $\Gamma$. Let $\mu_1 = 1$ and $\mu_G = G$ a.s.

**Theorem ([Stück-(Zimmer]-Nevo))**

Let $G$ be a higher rank simple real Lie group and let $H$ be an ergodic IRS in $G$. Then $H = \mu_G, \mu_1$ or $\mu_{\Gamma}$ for some lattice $\Gamma$ of $G$.

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Does not work for rank 1 in general ($\text{SL}_2(\mathbb{R})$).

Works for semisimple Lie groups. New proofs and extensions are in the works.
Other classifications and boundaries

Vershik: Classification of IRS’s for $\text{FSym}(\mathbb{IN})$

Bowen + Grigorchuk + Kravchenko: Zoos and shape of the simplex of IRS’s for large groups, analysis of IRS’s that are invariant under automorphisms (lamplighter group, $\text{Aut}(F_n)$).

[7Samurai] Let $K$ be any discrete subgroup in $G$ and let $H$ be a nontrivial IRS in $K$. Then the limit set of $H$ equals the limit set of $K$ a.s. In particular, any IRS in $G$ has full limit set.

[Cannizzo-Kaimanovich] Let $H$ be a stationary random subgroup of a free group $F$. Then the action of $H$ on the boundary of $F$ is conservative a.s.

The 7 Samurai

Tsachik Gelander
Jan Biringer
Miklos Abert
Nik Nikolov
Nicolas Bergeron
Jean Raimbault
Iddo Samet
For a Lie group $G$ let $X = G/K$ be its symmetric space. If $Y$ is connected, complete, locally-$X$, then $Y = \Gamma \backslash X$ where $\Gamma \leq G$ is discrete. Let
\[(Y)_< r = \{ x \in Y \mid \text{injrad}(x) < r \}\]be the $r$-thin part of $Y$. 

Very much not true in rank 1 in general (lattices with cyclic quotients). When $\Gamma$ is a fixed arithmetic lattice and $\Gamma_n \Gamma$ is a sequence of congruence subgroups, we have explicit bounds on the size of the thin part and the typical injrad.
Big higher rank locally symmetric spaces are also fat

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**Theorem (7Samurai)**

Let $G$ be a higher rank simple Lie group with symmetric space $X$. Let $\Gamma_n \leq G$ be lattices and let $X_n = \Gamma_n \backslash X$ with $\text{vol}(X_n) \to \infty$. Then for all $r > 0$ we have

$$\lim_{n \to \infty} \frac{\text{vol}( (X_n)_< )}{\text{vol}(X_n)} = 0.$$
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The IRS behind

**Theorem (7Samurai)**

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$m(\pi, \Gamma)$: multiplicity of $\pi \in \hat{G}$ in $L^2(\Gamma \backslash G)$. $d(\pi)$: multiplicity in $L^2(G)$.

**Theorem (7Samurai Limit Multiplicity)**

Let $(\Gamma_n)$ be a uniformly discrete sequence of lattices in $G$ such that $\lim_{n \to \infty} \mu_{\Gamma_n} = \mu_1$. Then for all $\pi \in \hat{G}$, we have

$$\frac{m(\pi, \Gamma_n)}{\text{vol}(\Gamma_n \backslash G)} \to d(\pi).$$

Also implies weak convergence of Plancherel measures. For chains, these are due to DeGeorge-Wallach and Delorme. Lots of deep papers. For the non-uniform case, recent work of Finis, Lapid and Müller.
Character rigidity

A character of $\Gamma$ is a conjugacy invariant, positive definite complex function on $\Gamma$ with value 1 at the identity. (Thoma, Kirillov).
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Let $H$ be an IRS of $\Gamma$. Then $f(g) = \mathcal{P}(g \in H)$ is a character of $\Gamma$.
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**Theorem (Peterson-Thom)**

No nontrivial characters (and hence IRS’s) for $\text{SL}_n(K)$ where $K$ is an infinite field or the localization of an order in a number field.

Much more on semisimple lattices: Creutz, Creutz-Peterson.
Open problems: rank 1 simple Lie groups

In rank 1, not every sequence of lattices approximate $G$.

- Question [7Sam + Sarnak] Let $G$ be a rank 1 simple Lie group and let $\Gamma_n$ be a sequence of congruence lattices in $G$. Then $\mu_{\Gamma_n} \to \mu_1$. 

Theorem (Raimbault) True for the Bianchi groups $\Gamma_D = \text{SL}_2(\mathbb{Z}[\sqrt{D}])$ (and more).

A lattice $\Gamma G$ is Ramanujan, if $\lambda_1(\Gamma G) = \lambda_0(G)$. Selberg $1/4$.

Theorem (A-Virag) Let $G$ be a simple Lie group and let $\Gamma_n$ be a sequence of Ramanujan lattices in $G$. Then $\mu_{\Gamma_n} \to \mu_1$.

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Question [Weinberger] Assume $G$ has finitely many non-conjugate lattices below any given volume. Do random lattices converge to $\mu_1$?
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Covering towers (chains) admit a stronger limit: graphing, profinite action, foliated space with transversal measure. Let the *rank* of a measured groupoid be the infimum of measures of its generating subsets.

Theorem (A-Nikolov)
The rank is continuous for towers of measured groupoids. In particular, the rank gradient of an approximating chain equals the cost of the limiting profinite action.

Is there a stronger limit notion for arbitrary sequences? (Local-global limit?)

Let $S^\Gamma$ be finite and $k > 0$. Fix a map $A: k^\Gamma \to k^\Gamma$. For every subgroup $H\Gamma$, $A$ induces a map $A_H: k^\Gamma/\Gamma \to k^\Gamma/\Gamma$ (look at $S$-neighbors).

[A-Szegedy] The normalized entropy $h(A, \Gamma, H) = H(A_H(k\text{-i.i.d.}))/\mu(\Gamma)$ is continuous in IRS convergence. Would imply Lück Approx. mod $p$. 
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