Ergodic Theorems

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Classical Ergodic Theorems

Boltzmann's ergodic hypothesis, 1880:

A system (ideal gas) develops freely.
Then it achieves all possible states with equal probability, or:

It visits every region in the phase space proportionally to its volume.

"time mean = space mean"

Mathematical model:

- (X, μ) probability space
- $T : X \to X$ μ-preserving.

Assumption: T is ergodic, i.e.,

$$A \subset X \text{ is } T\text{-invariant} \iff \mu(A) \in \{0, 1\}$$

Linearization:

- $A \rightsquigarrow \mathbf{1}_A$, $H := L^2(X, \mu)$,
- $T : H \to H$ linear: $(Tf)(x) := f(Tx)$.

$$T \text{ ergodic} \iff \mathrm{Fix}\, T = \{\mathbf{1}\}$$

Mean Ergodic Theorem, Von Neumann, 1931

For every $f \in L^2(X, \mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f = \int_X f d\mu \cdot \mathbf{1}$$

in $L^2(X, \mu)$.

<u>Idea</u>: Use

$$H = \langle \mathbf{1} \rangle \oplus \overline{\mathrm{rg}(I - T)}.$$

For $f = g - Tg$,

$$\frac{1}{N} \sum_{n=1}^{N} T^n f = \frac{1}{N} \sum_{n=1}^{N} (T^n g - T^{n+1} g) = \frac{Tg - T^{N+1} g}{N} \to 0.$$

<u>Remark</u>: A.e. convergence on $\langle \mathbf{1} \rangle \oplus (I - T)(L^\infty(X, \mu))$.

Pointwise Ergodic Theorem, Birkhoff, 1931

For every $f \in L^1(X, \mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f(x) = \int_X f d\mu$$

for a.e. $x \in X$.

Remark: $f = \mathbf{1}_A$, $(T^n f)(x) = \mathbf{1}_A(T^n x)$:

$$\lim_{N\to\infty} \frac{|\{n \in \{1, \ldots, N\} : \ T^n x \in A\}|}{N} = \mu(A)$$

"time mean = space mean"

Multiple Ergodic Theorems

Theorem (Furstenberg, ..., Host-Kra, Ziegler, 1977-2002)

Let (X, μ, T), $k \in \mathbb{N}$, $f_1, \ldots, f_k \in L^\infty$ *be given.*

- *(Multiple convergence)*

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{kn} f_k$$

  converge in L^2.

- *(Multiple recurrence) If* $f := f_1 = \ldots f_k > 0$, *then*

$$\langle f, \lim \ldots \rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int f \cdot T^n f \cdot \ldots \cdot T^{kn} f > 0.$$

Example: $f_1 = \ldots = f_k = 1_A$ - multiple recurrence of sets

$$\mu(A \cap T^{-n}A \cap \ldots \cap T^{-nk}A) > 0 \quad \text{for many } n$$

Motivation:

Theorem (Szemerédi, 1975)

Let $E \subset \mathbb{N}$ with

$$\limsup \frac{|E \cap \{1, \ldots, N\}|}{N} > 0.$$

Then $E \supset$ arithm. progressions of arbitrary length.

Furstenberg 1977: Multiple recurrence $\Rightarrow$ Szemerédi

Green-Tao 2004:

$\mathbb{P} \supset$ arithm. progressions of arbitrary length.

Ex.: $7, 37, 67, 97, 127, 157$

$$224.584.605.939.537.911 + 81.292.139 \cdot 223.092.870 \cdot j$$
$$\text{für } j = 0, \ldots, 26.$$

We want to understand

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot \ldots \cdot T^{kn} f_k$$

Idea: Decompose $H = \{k\text{-structured fcts}\} \oplus \{k\text{-random fcts}\}$

$k = 1$: $H = \langle \mathbf{1} \rangle \oplus \overline{\mathrm{rg}(I - T)}$ $\qquad \frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle| \to 0$

$k = 2$: $H = \overline{\mathrm{lin}}\{\text{eigenfcts of } T\} \oplus \{\text{weakly mixing fcts}\}$

- $Tf_1 = \lambda_1 f_1$, $Tf_2 = \lambda_2 f_2$:

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 = \frac{1}{N}\sum_{n=1}^{N} (\lambda_1 \lambda_2^2)^n \cdot f_1 f_2$$

- f_1 or f_2 weakly mixing: "van der Corput trick" $\implies \lim = 0$

We want to understand

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot \ldots \cdot T^{kn} f_k$$

Idea: Decompose $H = \{k\text{-structured fcts}\} \oplus \{k\text{-random fcts}\}$

$k \geq 3$: f is k-random $\Longleftrightarrow \|f\|_{U^k} = 0$

Gowers-Host-Kra (uniformity) seminorms:

$$\|f\|_{U^1} := \left|\int f\right|$$

$$\|f\|_{U^{k+1}}^{2^{k+1}} := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \|T^n f \cdot \overline{f}\|_{U^k}^{2^k}$$

Ex.: $\|f\|_{U^2}^4 = \lim \frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle|^2$

Further directions:

- polynomial powers: Bergelson-Leibman, '96 (polynomial Szemerédi), Leibman '05, Tao-Ziegler '08, '18 (primes)
- commuting transformations: Furstenberg, Katznelson, '78 (Szemerédi in $\mathbb{Z}^d$), Tao, Austin, Walsh, '08-12, Austin '15
- better understanding of $\|\cdot\|_{U^k}$: Green-Tao-Ziegler '10, T.E.-Tao '12, Tao-Ziegler '16, Neuman '20
- noncommutative systems ($L^\infty(X,\mu) \rightsquigarrow$ von Neumann algebra): Niculescu-Ströh-Zsidó '03, Austin-T.E.-Tao '11, Duvenhage, King '19, '21
- group actions
- ...

Open: a.e. convergence for $k \geq 3$ ($k = 2$: Bourgain)

Subsequential Ergodic Theorems

Find "good" powers $(k_n) \subset \mathbb{N}$:

$$\frac{1}{N}\sum_{n=1}^{N} T^{k_n} f$$

converge for $\forall (X, \mu, T)$, $\forall f \in L^p$

Mean conv.:

$$(k_n) \text{ is good } \iff \frac{1}{N}\sum_{n=1}^{N} \lambda^{k_n} \text{ conv. } \forall \lambda \in \mathbb{T}$$

Pointwise conv.: no characterization

Examples (pointwise):

$p > 1$

- 2^n - bad (Bellow, '83)
- $p(n)$, p polynomial - good (Bourgain '88)
- $\mathbb{P}$ - good (Bourgain, Wierdl '88)
- $p(\mathbb{P})$ - good (Wierdl '89, Nair '93)

$p = 1$

- n^2 - bad (Buczolich, Mauldin '10)
- $\mathbb{P}$, n^d - bad (LaVictoire, '11)
- sparse good sequence (Buczolich '07)

Open: a.e. convergence of

$$\frac{1}{N}\sum_{n=1}^{N} T^n f \cdot T^{n^2} g$$

Weighted Ergodic Theorems

Find "good" (bounded) weights $(a_n) \subset \mathbb{C}$:

$$\frac{1}{N}\sum_{n=1}^{N} a_n T^n f$$

conv. $\forall (X, \mu, T)$, $\forall\ f \in L^\infty$.

Mean conv.:

$$(a_n) \text{ good } \iff \frac{1}{N}\sum_{n=1}^{N} a_n \lambda^n \text{ conv. } \forall \lambda \in \mathbb{T}$$

Pointwise conv.: no characterization

Motivation: Wiener-Wintner Theorem

Theorem (Wiener-Wintner, 1941)

Let (X, μ, T), $f \in L^1$ *be given. Then* $\exists X' \subset X$ *with* $\mu(X') = 1$*:*

$$\frac{1}{N}\sum_{n=1}^{N} \lambda^n T^n f(x)$$

conv. $\forall x \in X'$ *and* $\forall \lambda \in \mathbb{T}$*.*

Idea: Use $H = \overline{\mathrm{lin}}\{\text{eigenfcts of } T\} \oplus \text{Rest}$

In particular: (λ^n), $\lambda \in \mathbb{T}$, are good weights (pointwise).

Further examples (pointwise):

- $(\lambda^{p(n)})$, p polynomial (Lesigne '93, Frantzikinakis '06)
- return time seq.: (Y, ν, S), $g \in L^1$, then for a.e. y,

$$a_n := g(S^n y)$$

  is good (Bourgain, '89)
- for nilsystems (coming from rotation on nilpotent groups) - $\forall g \in C(Y)$, $\forall y \in Y$, independently of (a_n) (Host-Kra '09, T.E.–Zorin-Kranich '13)
- "linear" seq. $a_n := \langle L^n z, z' \rangle$, $L : E \to E$ lin., bdd, "good" (T.E., '13)
- certain log-exp fcts (T.E.-Krause '16)
- automatic sequences (T.E.-Konieczny '18)

- Möbius fct $\mu(n) = \begin{cases} 1, & n = p_1 \dots p_{2k},\ p_j \text{ distinct}, \\ -1, & n = p_1 \dots p_{2k+1},\ p_j \text{ distinct}, \\ 0, & n \text{ not square-free.} \end{cases}$

  (Sarnak '11, El Abdalaoui, Kulaga, Lemańczyk, de la Rue '14, T.E. '15)

Sarnak's conjecture '11:

$\forall$ "deterministic" (X, T) $\forall f \in C(X)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)(T^n f)(x) = 0$$

$\forall x \in X$. Or:

$\mu \perp$ deterministic sequences

Sarnak's Conjecture

holds for many systems: rotations (Davenport), nilsystems (Green-Tao '05), horocycle flow (Bourgain-Sarnak-Ziegler '11), ...

Connection to Chowla's conjecture

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n+a_1) \cdot \ldots \cdot \mu(n+a_k) = 0$$

for all k and $a_1 < \ldots < a_k$.

- Chowla $\implies$ Sarnak (Sarnak '11)
- Sarnak $\implies$ Chowla for some (N_j) (Gomilko, Kwietniak, Lemańczyk '18, Tao '17)
- Logarithmic versions equivalent (Tao '17)
- Log. Chowla holds for $k = 2$ and k odd (Tao '16, Tao-Teräväinen '18)

Liouville fct λ: analogously.

Book manuscript

"Ergodic Theorems"

(with Bálint Farkas)

KÖSZÖNÖM

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