Discretization of uniform norm of polynomials on unbounded domains

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We consider the problem of finding discrete *norming sets* $\{x_1, ..., x_N\} \subset K \subset \mathbb{R}^d$ so that with some $c_K > 0$ depending only on the domain K for every algebraic polynomial $p \in P_n^d$ of d variables and degree $\leq n$

$$\max_{x \in K} |p(x)| \leq c_K \max_{1 \leq j \leq N} |p(x_j)|. \tag{1}$$

The main goal is to find discrete sets of possibly smallest cardinality N. Since $\dim P_n^d = \binom{n+d}{n} \sim n^d$ we must have $N > cn^d$ in order for (1) to be possible. This leads to the concept of ***optimal meshes*** which are defined as discrete sets of cardinality $N \sim n^d$ satisfying (1). Optimal meshes have been widely investigated in the recent years, their existence has been verified for various **compact** sets in $\mathbb{R}^d$, in particular **convex bodies** and **smooth star like domains**.

In this talk we will discuss a novel question of finding optimal meshes on **unbounded** domains. Naturally, this requires consideration of weighted norms. Here is a sample of what can be done: for every $\gamma > 0$ and $n \in \mathbb{N}$ there exist optimal meshes $\{x_1, ..., x_N\} \subset \mathbb{R}^d, N \sim n^d$ so that

$$\max_{x \in \mathbb{R}^d} e^{-|x|^\gamma} |p(x)| \leq c_{d,\gamma} \max_{1 \leq j \leq N} e^{-|x_j|^\gamma} |p(x_j)|, \quad \forall p \in P_n^d.$$

Optimal meshes with the above property can be constructed explicitly, moreover this can be done for more general weights and unbounded domains.