Introduction to the world of CSPs

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Constraint Satisfaction Problems

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- 3-colourability; $\mathfrak{B} = K_3$
- 3-SAT (8 ternary relations!)
- linear equations over finite fields ...
- acyclic graphs; $\mathfrak{B} = (\mathbb{Q}; <)$

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- INPUT: A primitive positive (pp) sentence φ over $\mathfrak{B}$ ($\varphi \equiv \exists\exists\ldots\exists(\bigwedge(\text{atomic}))$)
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Expansion by pp-definable relations does not change the complexity of the CSP.

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$$f : \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ \vdots \\ a_{n1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ \vdots \\ a_{n2} \end{matrix}} & \begin{matrix} \dots \\ \\ \dots \end{matrix} & \boxed{\begin{matrix} a_{1k} \\ \vdots \\ a_{nk} \end{matrix}} \end{pmatrix} \mapsto \begin{pmatrix} \boxed{\begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix}} \end{pmatrix}$$

$$\begin{matrix} \pitchfork & \pitchfork & & \pitchfork & & \pitchfork \\ R & R & \dots & R & \Rightarrow & R \end{matrix}$$

Notation: $\mathrm{Pol}(\mathfrak{B})$.

Polymorphisms

Pol($\mathfrak{B}$) forms a *clone*.

Definition

$\mathcal{C} \subseteq \bigcup_{k=1}^{\infty} X^{X^k}$ is a clone if

1. $\mathcal{C}$ contains all projections $(\pi_i \colon (x_1, \ldots, x_k) \mapsto x_i)$
2. $f, g_1, \ldots, g_k \in \mathcal{C} \Rightarrow f \circ (g_1, \ldots, g_k) \in \mathcal{C}$.

$\mathscr{P}$: the clone of projections (on a 2-element set).

Polymorphisms

Theorem (Geiger; Bodnarčuk, Kalužnin, Kotov, Romov '69)

If $\mathfrak{B}$ *is finite then a relation* R *is pp-definable in* $\mathfrak{B}$ *iff* $\mathrm{Pol}(\mathfrak{B})$ *preserves* R*.*

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Theorem (Geiger; Bodnarčuk, Kalužnin, Kotov, Romov '69)

If $\mathfrak{B}$ is finite then a relation R is pp-definable in $\mathfrak{B}$ iff $\mathrm{Pol}(\mathfrak{B})$ preserves R.

Corollary (Jeavons '98)

If $\mathfrak{B}$ is finite then the complexity of $\mathrm{CSP}(\mathfrak{B})$ is uniquely determined by $\mathrm{Pol}(\mathfrak{B})$.

pp-interpretations

Definition

$\mathfrak{A}$ pp-interprets $\mathfrak{B}$ if
$\exists I \colon A^d \to B$ surjective partial map such that for all relations R of $\mathfrak{B}$

$$\{(a_1^1, \ldots, a_d^1, \ldots, a_1^k, \ldots, a_d^k) : (I(a_1), \ldots, I(a_k)) \in R\}$$

is pp-definable in $\mathfrak{A}$.

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$\mathrm{I}_{pp}(\mathfrak{A})$: structures pp-interpretable in $\mathfrak{A}$.

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Fact

If $\mathfrak{A}$ *pp-interprets* $\mathfrak{B}$ *then* CSP($\mathfrak{B}$) LOGSPACE *reduces to* CSP($\mathfrak{A}$).

pp-constructions

Definition

$\mathfrak{A}$ and $\mathfrak{B}$ are homomorphically equivalent iff there are homomorphisms $\mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{B} \to \mathfrak{A}$.
Notation: $\mathfrak{B} \in \mathsf{H}(\mathfrak{A})$.

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Reductions between CSPs

Clone and minion homomorphisms

Definition

$\mathcal{C}, \mathcal{D}$: clones.

$\xi : \mathcal{C} \to \mathcal{D}$ is a clone homomorphism if

1. ξ preserves arities,
2. $\xi(\pi_i) = \pi_i$,
3. $\xi(f \circ (g_1, \ldots, g_k)) = \xi(f) \circ (\xi(g_1), \ldots, \xi(g_k))$.

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Clone and minion homomorphisms

Theorem

$\mathfrak{A}, \mathfrak{B}$: *finite.*

1. $\exists : \mathrm{Pol}(\mathfrak{A}) \to \mathrm{Pol}(\mathfrak{B})$ *clone homomorphism iff* $\mathfrak{B} \in \mathrm{I}(\mathfrak{A})$.
   (Birkhoff, 1935)
2. $\exists : \mathrm{Pol}(\mathfrak{A}) \to \mathrm{Pol}(\mathfrak{B})$ *minion homomorphism iff* $\mathfrak{B} \in \mathrm{HI}(\mathfrak{A})$.
   (Barto, Opršal, Pinsker '18)

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Corollary

If $\mathfrak{A}$ *and* $\mathfrak{B}$ *are finite and* $\exists\, \mathrm{Pol}(\mathfrak{A}) \to \mathrm{Pol}(\mathfrak{B})$ *minion homomorphism then* $\mathrm{CSP}(\mathfrak{B})$ LOGSPACE *reduces to* $\mathrm{CSP}(\mathfrak{A})$

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Corollary

If $\mathfrak{A}$ *is finite and* $\exists\, \mathrm{Pol}(\mathfrak{A}) \to \mathscr{P}$ *minion homomorphism then* $\mathrm{CSP}(\mathfrak{A})$ *is* **NP**-*complete.*

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Every finite structure is homomorphically equivalent to a core.

Fact (Barto, Opršal, Pinsker '18)

If $\mathfrak{B}$ is a core and $c \in B$ then $(\mathfrak{B}; c) \in \mathrm{HI}(\mathfrak{B})$.

CSP dichotomy

Finite case

Theorem (Barto, Opršal, Pinsker, Kozik '12, '18)

$\mathfrak{A}$ *is a finite core. Then TFAE.*

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If 1 - 4 hold then $\mathrm{CSP}(\mathfrak{A})$ *is in* **P**.

Therefore if $\mathfrak{A}$ is finite then $\mathrm{CSP}(\mathfrak{A})$ is in **P** or it is **NP**-complete.

Generalizations to infinite structure

ω-categoricity

Disclaimer: every structure is assumed to be countable!

Definition (the useful one)

$\mathfrak{A}$ is ω-categorical if $\mathrm{Aut}(\mathfrak{A})$ has finitely many n-orbits for all $n \in \omega$.

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Corollary

If $\mathfrak{A}$ is ω-categorical, then the complexity of $\mathrm{CSP}(\mathfrak{A})$ is uniquely determined by $\mathrm{Pol}(\mathfrak{A})$.

$\mathscr{P}$: the clone of projections on a 2-element set.
$\mathfrak{A}$: ω-categorical, $\mathfrak{B}$ finite.
Facts:

- $\exists\,\mathrm{Pol}(\mathfrak{A}) \to \mathrm{Pol}(\mathfrak{B})$ uniformly continuous minion homomorphism, then $\mathrm{CSP}(\mathfrak{A})$ is at least as hard as $\mathrm{CSP}(\mathfrak{B})$.
- If $\mathrm{Pol}(\mathfrak{B}) = \mathscr{P}$, then $\mathrm{CSP}(\mathfrak{B})$ is **NP**-complete.
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Model-complete cores

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Theorem (Bodirsky '05)

Every ω-categorical structure is homomorphically equivalent to a model-complete core.
This is a unique up to isomorphism, and again ω-categorical.

$\mathfrak{A}$ is an ω-categorical model-complete core. Then TFAE.

1. $\nexists\, \mathrm{Pol}(\mathfrak{A}) \to \mathscr{P}$ *minion homomorphism.*
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3. $\mathrm{Pol}(\mathfrak{A})$ *contains a Siggers operation:*
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Infinite-domain CSP dichotomy

Algebraic formulation

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$\mathfrak{A}$ *is an* ω*-categorical model-complete core. Then* ❶ $\Rightarrow$ ❷ $\Leftrightarrow$ ❸.

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Remark

❶ $\Leftrightarrow$ ❷ does not hold in general, but it does hold for "reasonable" structures.
(Barto, Kompatscher, Olšák, Pham, Pinsker '17).

Conjecture (Bodirsky, Pinsker '11)

If $\mathfrak{A}$ is a first-order reduct of a finitely bounded homogeneous structure (FOROFBHS) then CSP($\mathfrak{A}$) is in **P** or it is **NP**-complete,

Infinite-domain CSP dichotomy

The conjecture

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Homogeneous: every finite partial isomorphism extends to an automorphism of $\mathfrak{A}$

Infinite-domain CSP dichotomy

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Finitely bounded: Age($\mathfrak{A}$) can be described by finitely many forbidden substructures

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Infinite-domain CSP dichotomy

Known CSP dichotomies

Solved for

- reducts of $(\mathbb{N}; =)$ (Bodirsky, Kára '08)
- reducts of $(\mathbb{Q}; <)$ (Bodirsky, Kára '09)
- reducts of the homogeneous binary branching C-structure (Bodirsky, Jonsson, Pham '16)
- reducts of the random poset (Kompatscher, Pham '18)
- reducts of unary ω-categorical structures (Bodirsky, Mottet '18)
- MMSNPs (Bodirsky, Madelaine, Mottet '18)
- reducts of homogeneous graphs (Bodirsky, Martin, Pinsker, Pongrácz '19)
- reducts of the random tournament (Mottet, Pinsker '21)
- first-order expansions of the homogeneous RCC5 structure (Bodirsky, B. '21)
- hereditarily cellular structures (B. '22)
- first-order expansions of powers of $(\mathbb{Q}; <)$ (Bodirsky, Jonsson, Martin, Mottet, Semanišinová '22)
- reducts of random uniform hypergraphs (Mottet, Nagy, Pinsker '23)
- reducts of Johnson graphs (Bodirsky, B. '25)

More restrictive classes of ω-categorical structures to consider:

1. Stability, NIP, NSOP, etc.

Infinite-domain CSP dichotomy

Tame ω-categoricity

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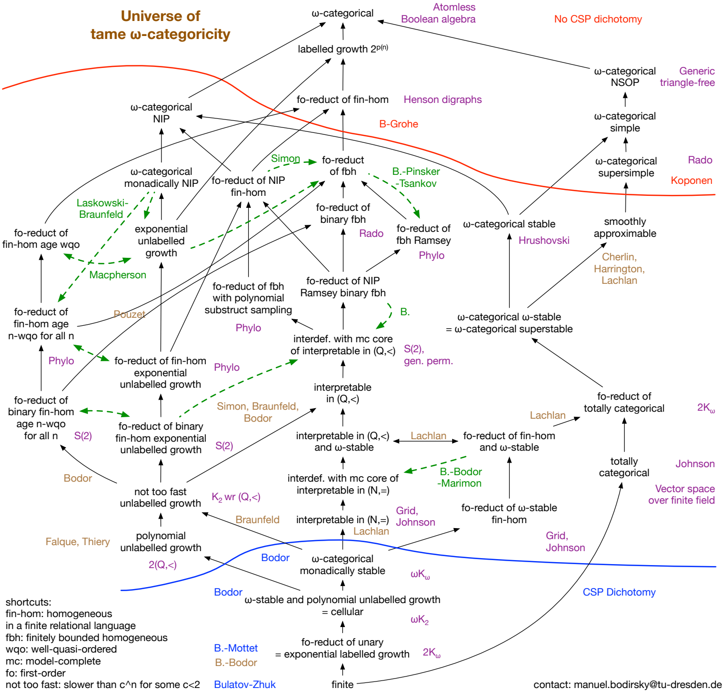
3. First-order interpretability in certain structures
4. From second-order logic: MMSNP, GMSNP


Universe of
tame ω-categoricity
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labelled growth $2^{p(n)}$
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fo-reduct of
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fo-reduct of fbh
with polynomial
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Ramsey binary fbh
B.
ω-categorical ω-stable
= ω-categorical superstable
Phylo
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K_2 wr (Q,<)
Vector space
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CSP Dichotomy
ω-stable and polynomial unlabelled growth
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shortcuts:
fin-hom: homogeneous
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fbh: finitely bounded homogeneous
wqo: well-quasi-ordered
mc: model-complete
fo: first-order
not too fast: slower than c^n for some c<2
B.-Mottet
B.-Bodor
Bulatov-Zhuk
fo-reduct of unary
= exponential labelled growth
$2K_\omega$
finite
contact: manuel.bodirsky@tu-dresden.de

Hereditarily cellular structures

Definition

A structure $\mathfrak{A}$ is hereditarily cellular if $\mathfrak{A}$ can be constructed from finite structure by taking

- finite disjoint unions
- infinite unlabelled copies
- first-order reducts

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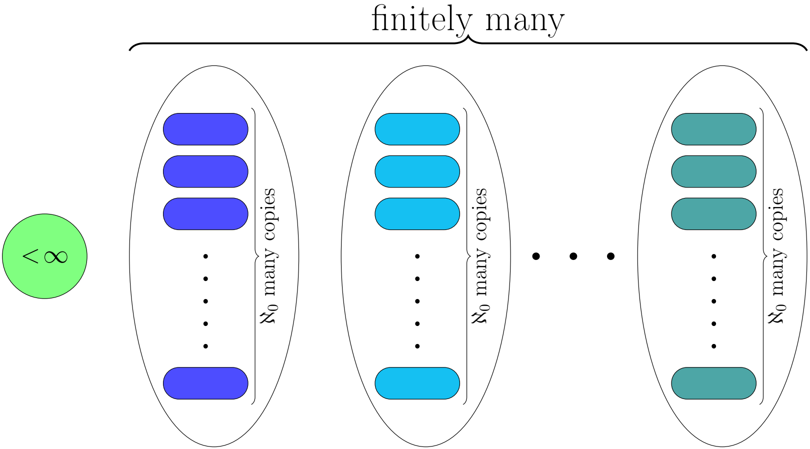
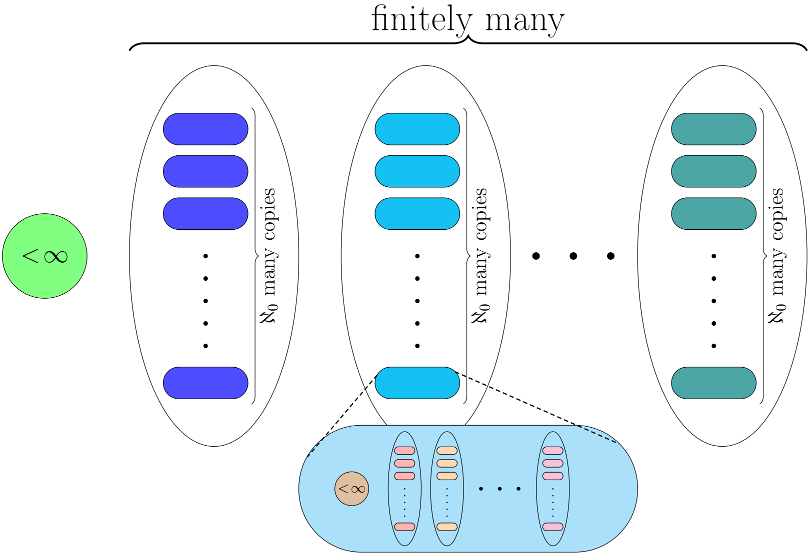
- Disjoint union: the domains of the original structures are named by unary predicates
- Infinite unlabelled copies: we add an equivalence relation E whose equivalence classes are the copies

Hereditarily cellular structures



Hereditarily cellular structures



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The infinite-domain CSP dichotomy conjecture holds for hereditarily cellular classes.

Remark: Hereditary cellularity can also be described in terms of orbit growths.

Interpretation of structures

Definition (reminder)

$\mathfrak{A}$ (first-order) interprets $\mathfrak{B}$ if
$\exists I \colon A^d \to B$ surjective partial map such that for all relations R of $\mathfrak{B}$

$$\{(a_1^1, \ldots, a_d^1, \ldots, a_1^k, \ldots, a_d^k) : (I(a_1), \ldots, I(a_k)) \in R\}$$

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Fact

Herederitily cellular structure are in $\mathrm{I}_{fo}((\mathbb{N}; =))$.

Example: Johnson graphs

Definition

$\mathcal{J}_k := (\binom{\mathbb{N}}{k}; S_0, S_1, \ldots, S_{k-1})$ where $S_i = \{(a,b) : |a \cap b| = i\}$.

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- These are all the primitive actions of $\mathrm{Sym}(\mathbb{N})$ on countable sets.

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I think $1 < \ell < k$ is not possible in the theorem above.

Problem with interpretations

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Theorem (Lachlan's class, Lachlan '87+easy)

TFAE.

- $\mathfrak{A} \in \mathsf{I}_{fo}((\mathbb{Q};<))$ *and* $\mathfrak{A}$ *is stable.*
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Corollary (using a Ramsey transfer result by Mottet, Pinsker '21)

Lachlan's class is closed under taking model-complete cores.

Lachlan's class

Primitive structures

Theorem (Cherlin, Lachlan, Harrington '85+Bodirsky, B., Marimon '25+)

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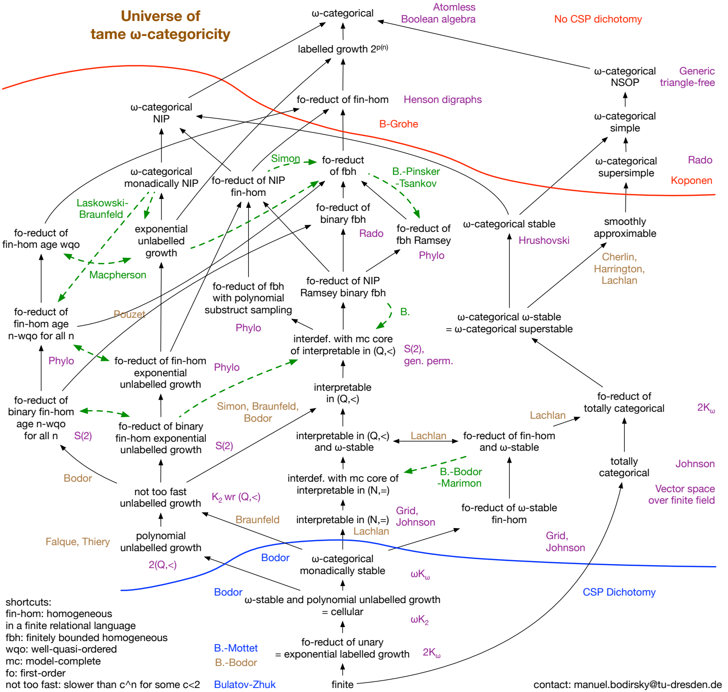
Theorem (Bodirsky, B., Marimon '25+)

Every model-complete core as in item ?? has a hard CSP unless $n = k = 1$.

Universe of tame ω-categoricity



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