Algebraic and Topological Methods in Lambda Calculus

Antonino Salibra

Dipartimento di Informatica, Università Ca'Foscari di Venezia
Via Torino 155, 30172 Venezia, Italy
salibra@dsi.unive.it

Dedicated to Istvan Németi
The author wishes to thank Istvan for his teachings over these years

Abstract. The untyped lambda calculus was introduced around 1930 by Church as part of an investigation in the formal foundations of mathematics and logic. Although lambda calculus is a very basic language, it is sufficient to express all the computable functions. The process of application and evaluation reflects the computational behaviour of many modern functional programming languages, which explains the interest in the lambda calculus among computer scientists.
In this paper we give an outline of the theory of lambda abstraction algebras (LAA's). These algebras constitute an equational class and were introduced by Pigozzi and Salibra to provide an algebraic version of the untyped lambda calculus, in the same way cylindric and polyadic algebras algebraize the first-order predicate logic. Questions related to the functional representation of various subclasses of lambda abstraction algebras were investigated by Pigozzi and Salibra in a series of papers. The problem of representability for LAA's was solved by Goldblatt and Salibra by showing that every LAA is isomorphic to a representable one. Algebraic methods were also applied by Lusin and Salibra to study the structure of the lattice of lambda theories (i.e., extensions of lambda calculus that are closed under derivation) and by Salibra to study the structure of the lattice of the subvarieties of LAA. For every variety of LAA's there exists a lambda theory whose term algebra generates the variety. In particular, the variety LAA is generated by the term algebra of the minimal lambda theory. Finally, algebraic and topological methods were applied by Salibra to study the incompleteness of the semantics of lambda calculus.

1 Introduction

The untyped lambda calculus was introduced by Church [5] as a foundation for logic. Although the appearance of paradoxes caused the program to fail, a consistent part of the theory turned out to be successful as a theory of "functions as rules" (formalized as terms of the lambda calculus) that stresses the computational process of going from the argument to the value. Every object in the lambda calculus is at the same time a function and an argument; in particular a function can be applied to itself. Although lambda calculus is a very basic language, it is sufficient to express all the computable functions.

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The two primitive notions of the lambda calculus are application, the operation of applying a function to an argument (expressed as juxtaposition of terms), and lambda (functional) abstraction, the process of forming a function from the "rule" that defines it. Then the set Λ_I of λ-terms of lambda calculus over an infinite set I of variables is constructed by induction as usual: every variable x is a λ-term, while, if M and N are λ-terms, then so are (MN) and $(\lambda x.M)$ for each variable $x \in I$.

The axioms of the untyped lambda calculus are defined by the schemes of (β)- and (α)-conversion. The first one expresses the way of calculating a function $(\lambda x.M)$ on an argument N, while the second one says that the name of bound variables does not matter. The variable-binding properties of lambda abstraction prevent variables in lambda calculus from operating as real algebraic variables. Indeed, the equations are not always preserved when arbitrary λ-terms are substituted for variables. The rules for deriving equations from instances of (α) and (β)-conversion are the usual ones from equational calculus asserting that equality is a congruence for application and abstraction. More precisely, consider the absolutely free algebra of λ-terms:

$$\mathbf{\Lambda}_I := (\Lambda_I, \cdot^{\mathbf{\Lambda}_I}, \lambda x^{\mathbf{\Lambda}_I}, x^{\mathbf{\Lambda}_I})_{x \in I}, \tag{1}$$

where, for all $s, t \in \Lambda_I$,

$$s \cdot^{\mathbf{\Lambda}_I} t = (st); \quad \lambda x^{\mathbf{\Lambda}_I}(t) = (\lambda x.t); \quad x^{\mathbf{\Lambda}_I} = x.$$

The lambda theory $\lambda\beta$ is the least congruence over $\mathbf{\Lambda}_I$ including (α) and (β)-conversion. The term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$ of $\lambda\beta$ is the quotient of $\mathbf{\Lambda}_I$ by the congruence $\lambda\beta$. The variety (= equational class) generated by the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$ is the starting point for studying the lambda calculus by universal algebraic methods.

In [30] Salibra has shown that the variety generated by the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$ is axiomatized by the finite schema of identities characterizing *lambda abstraction algebras* (LAA's). The equational theory of lambda abstraction algebras, introduced by Pigozzi and Salibra in [21] and [24], constitutes a purely algebraic theory of the untyped lambda calculus in the same spirit that Boolean algebras constitute an algebraic theory of classical propositional logic and, more to the point, cylindric and polyadic (Boolean) algebras of the first-order predicate logic. The equational theory of lambda abstraction algebras is intended as an alternative to combinatory logic (see Curry-Feys [9]) since it is a first-order algebraic description of lambda calculus, which allows to keep the lambda notation and hence all the functional intuitions. Among the seven identities characterizing LAAs, the first six constitute a recursive definition of the abstract substitution operator; they express precisely the metamathematical content of β-conversion. The last axiom is an algebraic translation of α-conversion.

The most natural LAA's are algebras of functions, called functional LAA's, which arise as "expansions" of suitable combinatory algebras (i.e., models of lambda calculus) by the variables of lambda calculus in a natural way. The situation in algebraic logic is analogous. The most natural cylindric (and polyadic) algebras are algebras of functions that are obtained by coordinatizing models of classical first-order logic. The smallest variety that includes all the functional algebras that are most closely connected with models of first-order logic constitutes the class of representable cylindric algebras. It is a proper subvariety of the class of all cylindric algebras, hence non-representable cylindric algebras exist. Much of the work in algebraic logic has been directed at finding wider classes of representable cylindric algebras with natural intrinsic characterizations. The main references for cilindric algebras are [14] [15]; for polyadic algebras it is [13]. We also mention here [20]. It contains an extensive survey of the various algebraic versions of quantifier logics. Questions related to the functional representation of various subclasses of lambda abstraction algebras were investigated by Pigozzi and Salibra in a series of papers [21], [22], [23], [24], [25]. Goldblatt [10] and Salibra-Goldblatt [31] have solved the problem of representability for LAA's, by showing that every LAA is isomorphic to a functional LAA and that the class of isomorphic images of functional lambda abstraction algebras constitutes a variety of algebras axiomatized by the finite schema of identities characterizing LAA's.

Algebraic methods were also applied in [29] [18] to study the structure of the lattice of lambda theories. We recall that lambda theories are extensions of the untyped lambda calculus (including (α) and (β)-conversion) that are closed under derivation. They correspond to different operational semantics of lambda calculus (see e.g. [1]) The set of all lambda theories is naturally equipped with a structure of complete lattice (see [1, Chapter 4]), with meet defined as set theoretical intersection. The bottom element of this lattice is the minimal lambda theory $\lambda\beta$, while the top element is the inconsistent lambda theory. The lattice of lambda theories, hereafter denoted by λT, has a continuum of elements and it is a very rich and complex structure. For example, every countable partially ordered set embeds into λT by an order-preserving map, and every interval of λT, whose bounds are recursively enumerable lambda theories, has a continuum of elements (see Visser [35]). The lattice of lambda theories is naturally isomorphic to the congruence lattice of the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$, the quotient of $\mathbf{\Lambda}_I$ by $\lambda\beta$. This is the starting point for studying the structure of the lattice of lambda theories by universal algebraic methods.

In [18] Lusin and Salibra have shown that the lattice λT of lambda theories satisfies nontrivial quasi-identities in the language of lattices. Other quasi-identities, such as the ET-condition and the Zipper condition (see [17]), are satisfied by λT as a consequence of the dual isomorphism, described in [30], between λT and the lattice of subvarieties of LAA. An identity in the language of lattices (a lattice identity, for short) is trivial if it holds in every lattice and nontrivial otherwise. We conjecture that the lattice λT does not satisfy any nontrivial lattice identity. However, for every nontrivial lattice identity e, there exists a natural number n such that e fails in the lattice of lambda theories in a language of λ-terms with n constants (see [18]). In a more general result, it was shown that the variety LAA satisfies a lattice identity e (i.e., the congruence lattices of all LAA's satisfy e) if, and only if, e is trivial. In other words, for every nontrivial lattice identity e, there exists a lambda abstraction algebra (not necessarily equal to the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$) whose congruence lattice does not satisfy e. However, there exists a sublattice of λT satisfying nice lattice properties. In fact, in [18] it was defined a lambda theory $\mathcal{J}$ such that the lattice of all lambda theories extending $\mathcal{J}$ is shown to satisfy a restricted form of distributivity, called meet semidistributivity, and a nontrivial congruence identity (i.e., an identity in the language of lattices enriched by the relative product of binary relations).

Applications of topological and algebraic methods to the semantics of lambda calculus were obtained by the author in [26] [27] [28]. After Scott, mathematical models of the untyped lambda calculus are defined by order theoretic methods and classified into semantics according to the nature of their representable functions. Continuous semantics is given in the category of complete partial orders and Scott continuous functions, while stable and strongly stable semantics are strengthenings of the continuous semantics (see e.g. [1], [4]). A model of the untyped lambda calculus univocally induces a lambda theory through the kernel congruence relation of the interpretation function. A semantics of lambda calculus is called (equationally) incomplete if there exists a lambda theory which is not induced by any model in the semantics. The first incompleteness result was obtained by Honsell and Ronchi della Rocca [16] for the continuous semantics, while Gouy [12] proved the incompleteness of the stable semantics. The author has introduced a new technique, based on topological algebras, to prove in a uniform way the incompleteness of all denotational semantics of lambda calculus which have been proposed so far, including the strongly stable one, whose incompleteness had been conjectured. This technique was applied to prove the incompleteness of any semantics of lambda calculus given in terms of either partially ordered models with finitely many connected components (= minimal upward and downward closed sets), or topological models whose topology satisfies a suitable property of connectedness. In particular, it follows that any semantics of lambda calculus given in terms of partially ordered models which are semilattices, lattices, complete partial orderings, or which have a top or a bottom element, is incomplete.

2 Lambda Abstraction Algebras: Basic Notions and Notation

To keep this article self-contained, we summarize some definitions and results that we need in the subsequent part of the paper. Our main references will be [30] and [25] for lambda abstraction algebras and Barendregt's book [1] for lambda calculus.

2.1 Lambda calculus

The two primitive notions of the lambda calculus are *application*, the operation of applying a function to an argument (expressed as juxtaposition of terms), and *lambda* (*functional*) *abstraction*, the process of forming a function from the "rule" that defines it.

The set $\Lambda_I(C)$ of ordinary terms of lambda calculus over an infinite set I of variables and a set C of constants is constructed as usual [1]:

1. every variable $x \in I$ and every constant $c \in C$ is a λ-term;
2. if M and N are λ-terms, then so are (MN) and $(\lambda x.M)$ for each variable $x \in I$.

We will write Λ_I for $\Lambda_I(\emptyset)$, the set of λ-terms without constants.

The symbol $\equiv$ denotes syntactic equality.

The following are some well-known λ-terms:

$$\mathbf{i} \equiv \lambda x.x; \quad \mathbf{s} \equiv \lambda xyz.xz(yz); \quad \mathbf{k} \equiv \lambda xy.x; \quad \mathbf{1} \equiv \lambda xy.xy; \quad \Omega \equiv (\lambda x.xx)(\lambda x.xx).$$

An occurrence of a variable x in a λ-term is *bound* if it lies within the scope of a lambda abstraction λx; otherwise it is *free*. $FV(M)$ is the set of free variables of a λ-term M. A λ-term without free variables is said to be *closed*. $\Lambda_I^0(C)$ is the set of closed λ-terms of $\Lambda_I(C)$. A λ-term N is *free for* x in M if no free occurrence of x in M lies within the scope of a lambda abstraction with respect to a variable that occurs free in N. $M[N/x]$ is the result of substituting N for all free occurrences of x in M subject to the usual provisos about renaming bound variables in M to avoid capture of free variables in N. The above proviso is empty if N is free for x in M.

The axioms of the $\lambda\beta$-calculus are as follows: M and N are arbitrary λ-terms and x, y variables.

(α) $\lambda x.M = \lambda y.M[y/x]$ for any variable y that does not occur free in M;

(β) $(\lambda x.M)N = M[N/x]$ for any N free for x in M.

(β)-conversion expresses the way of calculating a function $(\lambda x.M)$ on an argument N, while (α)-conversion says that the name of bound variables does not matter. The rules for deriving equations from instances of (α) and (β) are the usual ones from equational calculus asserting that equality is a congruence for application and abstraction.

A λ-term $M \in \Lambda_I^0(C)$ is *solvable* if there exist an integer n and $N_1, \ldots, N_n \in \Lambda_I(C)$ such that $MN_1 \ldots N_n = \mathbf{i}$. $M \in \Lambda_I(C)$ is *solvable* if the closure of M, that is $\lambda x_1 \ldots x_n.M$ with $\{x_1 \ldots x_n\} = FV(M)$, is solvable. $M \in \Lambda_I(C)$ is *unsolvable* if it is not solvable. Unsolvable λ-terms represent "indefinite" computational processes.

A *compatible λ-relation* $\mathcal{T}$ is any set of equations between λ-terms that is closed under the following rules, for all λ-terms M, N and P:

- $M = N \in \mathcal{T} \Longrightarrow \lambda x.M = \lambda x.N \in \mathcal{T}$;
- $M = N \in \mathcal{T} \Longrightarrow MP = NP \in \mathcal{T}$;
- $M = N \in \mathcal{T} \Longrightarrow PM = PN \in \mathcal{T}$.

We will write on occasion $\mathcal{T} \vdash M = N$ (or $M =_{\mathcal{T}} N$) for $M = N \in \mathcal{T}$.

A *lambda theory* $\mathcal{T}$ is any compatible λ-relation which is an equivalence relation and includes (α) and (β) conversion. The set of all lambda theories is naturally equipped with a structure of complete lattice with meet defined as set theoretical intersection. The join of two lambda theories $\mathcal{T}$ and $\mathcal{S}$ is the least equivalence relation including $\mathcal{T} \cup \mathcal{S}$. The least lambda theory including a set $\mathcal{W}$ of equations will be denoted by $\mathcal{W}^+$.

$\lambda\beta$ is the least lambda theory, while $\lambda\eta$ is the least extensional lambda theory (axiomatized by $\mathbf{i} = \mathbf{1}$). $\mathcal{H}$ is the lambda theory generated by equating all the unsolvable λ-term (i.e., $\mathcal{H} = \mathcal{H}_0^+$ where $\mathcal{H}_0 = \{M = N \mid M, N$ closed and unsolvable$\}$), while $\mathcal{H}^*$ is the unique maximal consistent extension of $\mathcal{H}$ (see [1]).

A lambda theory $\mathcal{T}$ is *sensible* if $\mathcal{H} \subseteq \mathcal{T}$. $\mathcal{T}$ is *semisensible* if $\mathcal{T}$ does not equate a solvable and an unsolvable.

2.2 Lambda abstraction algebras

Let I be a nonempty set. The similarity type of *lambda abstraction algebras of dimension* I is constituted by a binary operation symbol "$\cdot$" formalizing application, a unary operation symbol "λx" for every $x \in I$, and a constant symbol (i.e., nullary operation symbol) "x" for every $x \in I$. The elements of I are the variables of lambda calculus although in their algebraic transformation they no longer play the role of variables in the usual sense. In the remaining part of the paper we will refer to them as λ-*variables*. The actual variables of the similarity type of lambda abstraction algebras are referred to as *context variables* and denoted by the Greek letters ξ, ν, and μ possibly with subscripts.

The terms of the similarity type of lambda abstraction algebras are called λ-*contexts*. They are constructed in the usual way: every λ-variable x and context variable ξ is a λ-context; if t and s are λ-contexts, then so are $t \cdot s$ and $\lambda x(t)$.

Because of their similarity to the terms of the lambda calculus we use the standard notational conventions of the latter. The application operation symbol "$\cdot$" is normally omitted, and the application of t and s is written as juxtaposition ts. When parentheses are omitted, association to the left is assumed. The left parenthesis delimiting the scope of a lambda abstraction is replaced with a period and the right parenthesis is omitted. For example, $\lambda x(ts)$ is written $\lambda x.ts$. Successive λ-abstractions $\lambda x \lambda y \lambda z \dots$ are written $\lambda xyz \dots$.

An occurrence of a λ-variable x in a λ-context is *bound* if it falls within the scope of the operation symbol λx; otherwise it is *free*. The *free* λ-*variables* of a λ-context are the λ-variables that have at least one free occurrence. A λ-context without free λ-variables is said to be *closed*. Note that λ-contexts without any context variables coincide with ordinary terms of the lambda calculus without constants.

Our notion of a λ-context coincides with the notion of *context* defined in Barendregt ([1], Def.14.4.1); our context variables correspond to Barendregt's notion of a 'hole'. The main difference between Barendregt's notation and our's is that 'holes' are denoted here by Greek letters $\xi, \mu, \dots$, while in Barendregt's book by $[\,], [\,]_1, \dots$. The essential feature of a λ-context is that a free λ-variable in a λ-term may become bound when we substitute it for a 'hole' within the context. For example, if $t(\xi) = \lambda x.x(\lambda y.\xi)$ is a λ-context, in Barendregt's notation: $t([\,]) = \lambda x.x(\lambda y.[\,])$, and $M = xy$ is a λ-term, then $t(M) = \lambda x.x(\lambda y.xy)$.

A lambda theory has a natural algebraic interpretation. Let $\mathcal{T}$ be a lambda theory over the language $\Lambda_I(C)$ and let $\mathbf{\Lambda}_I(C)$ be the absolutely free algebra in the similarity type of lambda abstraction algebras (of dimension I) over the set C of generators:

$$\mathbf{\Lambda}_I(C) := \langle \Lambda_I(C), \cdot^{\mathbf{\Lambda}_I(C)}, \lambda x^{\mathbf{\Lambda}_I(C)}, x^{\mathbf{\Lambda}_I(C)} \rangle_{x \in I} \tag{2}$$

where for $M, N \in \Lambda_I(C)$:

$$M \cdot^{\mathbf{\Lambda}_I(C)} N = (MN); \quad \lambda x^{\mathbf{\Lambda}_I(C)}(M) = (\lambda x.M); \quad x^{\mathbf{\Lambda}_I(C)} = x.$$

We will write $\mathbf{\Lambda}_I$ for $\mathbf{\Lambda}_I(\emptyset)$. The lambda theory $\mathcal{T}$ is a congruence (i.e. a compatible equivalence relation) on $\mathbf{\Lambda}_I(C)$. We denote by $\mathbf{\Lambda}_I^{\mathcal{T}}(C)$ the quotient of $\mathbf{\Lambda}_I(C)$ by $\mathcal{T}$ and call it the *term algebra* of the lambda theory $\mathcal{T}$.

We say that $\mathcal{T}$ satisfies an identity between contexts $t(\xi_1, \ldots, \xi_n) = u(\xi_1, \ldots, \xi_n)$ if the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}(C)$ of $\mathcal{T}$ satisfies it; i.e., if all the instances of the above identity, obtained by substituting λ-terms for context variables in it, fall within the lambda theory: $\mathcal{T} \vdash t(M_1, \cdots, M_n) = u(M_1, \cdots, M_n)$, for all λ-terms $M_1, \cdots, M_n \in \Lambda_I(C)$. For example, every lambda theory satisfies the identity $(\lambda x.x)\xi = \xi$ because $\lambda\beta \vdash (\lambda x.x)M = M$ for every λ-term M.

Lambda abstraction algebras are meant to axiomatize those identities between contexts that are valid for the lambda calculus.

We now give the formal definition of a lambda abstraction algebra (see [22], [23], [25], [29], [30]).

Definition 1. *By a* **lambda abstraction algebra of dimension** I *we mean an algebraic structure of the form:*

$$\mathbf{A} := \langle A,\, \cdot^{\mathbf{A}},\, \lambda x^{\mathbf{A}},\, x^{\mathbf{A}} \rangle_{x \in I}$$

satisfying the following identities between λ-contexts, for all $x, y, z \in I$:

(β_1) $(\lambda x.x)\xi = \xi;$

(β_2) $(\lambda x.y)\xi = y, \quad x \neq y;$

(β_3) $(\lambda x.\xi)x = \xi;$

(β_4) $(\lambda xx.\xi)\mu = \lambda x.\xi;$

(β_5) $(\lambda x.\xi\mu)\nu = (\lambda x.\xi)\nu((\lambda x.\mu)\nu);$

(β_6) $(\lambda xy.\mu)((\lambda y.\xi)z) = \lambda y.(\lambda x.\mu)((\lambda y.\xi)z), \quad x \neq y, z \neq y;$

(α) $\lambda x.(\lambda y.\xi)z = \lambda y.(\lambda x.(\lambda y.\xi)z)y, \quad z \neq y.$

I is called the **dimension set** *of $\mathbf{A}$. $\cdot^{\mathbf{A}}$ is called* **application** *and $\lambda x^{\mathbf{A}}$ is called* **λ-abstraction** *with respect to x.*

The class of lambda abstraction algebras of dimension I is denoted by LAA_I and the class of all lambda abstraction algebras of any dimension by LAA. We also use LAA_I as shorthand for the phrase "lambda abstraction algebra of dimension I", and similar for LAA. An LAA_I is *infinite dimensional* if I is infinite.

LAA_I is a variety (= equational class) for every dimension set I, and therefore it is closed under the formation of subalgebras, homomorphic (in particular isomorphic) images, and Cartesian products.

In [25] it is shown the following result.

Proposition 1. ([25]) *Let $\mathcal{T}$ be a lambda theory over the language $\Lambda_I(C)$. Then the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}$ of $\mathcal{T}$ is an LAA_I.*

We note here one very useful immediate consequence of the axioms $(\beta_1) - (\beta_6)$ and (α): in any LAA_I $\mathbf{A}$ the functions λx are always one-one, i.e., for all $x \in I$,

$$\lambda x.a = \lambda x.b \quad \text{iff} \quad a = b, \quad \text{for all} \quad a, b \in A.$$

In fact, if $\lambda x.a = \lambda x.b$, then by (β_3), $a = (\lambda x.a)x = (\lambda x.b)x = b$.

An LAA with only one element is said to be *trivial*. It is interesting that any nontrivial LAA_I $\mathbf{A}$ of positive dimension is infinite, since the one-one map λx is not onto. To see this, assume by way of contradiction that x is in the range of λx; then $x = \lambda x.b$ for some element $b \in A$. Since $\mathbf{A}$ is nontrivial, there exists an element $a \in A$ such that $a \neq x$. Then a contradiction results from (β_1) and (β_4):

$$a = (\lambda x.x)a = (\lambda x.\lambda x.b)a = \lambda x.b = x.$$

Definition 2. ([24]; Def. 1.3) *Let* $\mathbf{A}$ *be an* LAA_I. *Let* $a \in A$ *and* $x \in I$. a *is said to be* **algebraically dependent on** x (**over** $\mathbf{A}$) *if* $(\lambda x.a)z \neq a$ *for some* $z \in I$*; otherwise* a *is* **algebraically independent of** x (**over** $\mathbf{A}$). *The set of all* $x \in I$ *such that* a *is algebraically dependent on* x *over* $\mathbf{A}$ *is called the* **dimension set** *of* a *and is denoted by* $\mathbf{\Delta}a$*; thus:*

$$\mathbf{\Delta}a = \{x \in I : (\lambda x.a)z \neq a \quad \text{for some} \quad z \in I\}.$$

a *is* **finite** (**infinite**) **dimensional** *if* $\mathbf{\Delta}a$ *is finite (infinite). An element* a *is called* **zero-dimensional** *if* $\mathbf{\Delta}a = \emptyset$. *We denote the set of zero-dimensional elements by* Zd $\mathbf{A}$.

For example, if $a = xy$ then a is algebraically dependent on x because $(\lambda x.xy)z = zy \neq xy$ for every $z \in I \setminus \{x, y\}$.

Proposition 2. ([24]; Prop. 1.7) *Let* $\mathbf{A} \in \mathsf{LAA}_I$, $a, b \in A$, *and* $x \in I$.

1. $\mathbf{\Delta}(ab) \subseteq \mathbf{\Delta}a \cup \mathbf{\Delta}b$.
2. $\mathbf{\Delta}(\lambda x.a) = \mathbf{\Delta}a \setminus \{x\}$.
3. $\mathbf{\Delta}x \subseteq \{x\}$, *with equality holding if* $\mathbf{A}$ *is nontrivial.*

If M is a λ-term without constants and $\mathbf{A}$ is an LAA_I, then $M^{\mathbf{A}}$ will denote the value of M in $\mathbf{A}$ when each λ-variable x occurring in M is interpreted as $x^{\mathbf{A}}$. By Prop. 2 the dimension set of $M^{\mathbf{A}}$ is a subset of the set of free λ-variables of M.

Suitable reducts of arbitrary LAA's turn out to be combinatory algebras. Recall that a combinatory algebra is an algebra $\mathbf{C} = \langle C, \cdot^{\mathbf{C}}, \mathbf{k}^{\mathbf{C}}, \mathbf{s}^{\mathbf{C}} \rangle$, where $\cdot^{\mathbf{C}}$ is a binary operation and $\mathbf{k}^{\mathbf{C}}, \mathbf{s}^{\mathbf{C}}$ are constants, satisfying the following identities: (as usual the symbol $\cdot$ and the superscript $^{\mathbf{C}}$ are omitted, and association is to the left)

$$\mathbf{k}xy = x; \qquad \mathbf{s}xyz = xz(yz).$$

Let $\mathbf{A}$ be an LAA_I. By the *combinatory reduct* of $\mathbf{A}$ we mean the algebra

$$\mathrm{Cr}\,\mathbf{A} = \langle A, \cdot^{\mathbf{A}}, \mathbf{k}^{\mathbf{A}}, \mathbf{s}^{\mathbf{A}} \rangle$$

where

$$\mathbf{k}^{\mathbf{A}} = (\lambda xy.x)^{\mathbf{A}} \quad \text{and} \quad \mathbf{s}^{\mathbf{A}} = (\lambda xyz.xz(yz))^{\mathbf{A}}.$$

$\mathrm{Cr}\,\mathbf{A}$ is a combinatory algebra [31]. A subalgebra of the combinatory reduct of an LAA_I $\mathbf{A}$ (i.e., a subset of $\mathbf{A}$ containing $\mathbf{k}^{\mathbf{A}}$ and $\mathrm{s}^{\mathbf{A}}$ and closed under $\cdot^{\mathbf{A}}$) is called a *combinatory subreduct* of $\mathbf{A}$. The *zero-dimensional subreduct* of $\mathbf{A}$ is the combinatory subreduct

$$\mathbf{Zd}\,\mathbf{A} = \langle \mathrm{Zd}\,\mathbf{A}, \cdot^{\mathbf{A}}, \mathbf{k}^{\mathbf{A}}, \mathrm{s}^{\mathbf{A}} \rangle,$$

where $\mathrm{Zd}\,\mathbf{A} = \{\, a \in A : \mathbf{\Delta} a = \emptyset \,\}$, the set of zero-dimensional elements of $\mathbf{A}$.

The *open term model* of a lambda theory $\mathcal{T}$, as defined in Barendregt's book [1], is the combinatory reduct $\mathrm{Cr}\,\mathbf{\Lambda}_I^{\mathcal{T}}(C)$ of the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}(C)$, while the *closed term model* of $\mathcal{T}$ is its zero-dimensional subreduct $\mathbf{Zd}\,\mathbf{\Lambda}_I^{\mathcal{T}}(C)$.

2.3 Locally finite LAA's

There is a strong connection between the lambda theories and the subclass of LAA's whose elements are finite dimensional.

Definition 3. ([24]; Def. 2.1) *A lambda abstraction algebra* $\mathbf{A}$ *is* **locally finite** *if it is of infinite dimension (i.e., I is infinite) and every $a \in A$ is of finite dimension (i.e., $|\mathbf{\Delta} a| < \aleph_0$).*

The class of locally finite LAA_I's is denoted by LFA_I, which is also used as shorthand for the phrase "locally finite lambda abstraction algebra of dimension I".

For every infinite I the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}$ of a lambda theory $\mathcal{T}$ is locally finite. This is a direct consequence of the trivial fact that every λ-term is a finite string of symbols and hence contains only finitely many λ-variables.

The following result characterizes those congruences on the algebra $\mathbf{\Lambda}_I(C)$ (defined in Section 2.2) that are lambda theories.

Lemma 1. ([30]; Lemma. 8) *Let I be an infinite set. A congruence θ on $\mathbf{\Lambda}_I(C)$ is a lambda theory over the language $\mathbf{\Lambda}_I(C)$ if, and only if, the following two conditions are satisfied:*

1. *The quotient algebra $\mathbf{\Lambda}_I(C)/\theta$ is an LAA_I;*
2. *$(\lambda x.c)y\ \theta\ c$ for all $c \in C$ and all $x, y \in I$, i.e., the equivalence class c/θ of every element $c \in C$ is a zero-dimensional element of $\mathbf{\Lambda}_I(C)/\theta$.*

Proposition 3. ([25]; Prop. 2.4) *Let I be an infinite set. An algebra $\mathbf{A}$ in the similarity type of lambda abstraction algebras of dimension I is (isomorphic to) the term algebra of a lambda theory if, and only if, it is an LFA_I.*

Recall that the LAA_I-free algebra over an empty set of generators is the quotient of the absolutely free algebra $\mathbf{\Lambda_I}$ of λ-terms by the smallest congruence θ making $\mathbf{\Lambda}_I/\theta$ an LAA_I.

The following proposition provides an algebraic characterization of the term algebra of $\lambda\beta$.

Proposition 4. ([30]; Prop. 10) *Let I be an infinite set. The term algebra $\mathbf{\Lambda_I^{\lambda\beta}}$ of the minimal lambda theory $\lambda\beta$ is the LAA_I-free algebra over an empty set of generators.*

Proposition 5. ([30]; Prop. 11) *Let I be an infinite set. For all λ-terms $t, u \in \Lambda_I$, $\mathsf{LAA}_I \models t = u$ iff $\lambda\beta \vdash t = u$.*

3 Subvarieties of LAA's and Lambda Theories

In this Section we briefly survey the main results connecting lambda abstraction algebras, lambda theories and functional (i.e., representable) lambda abstraction algebras.

In [30] it was shown that the complete lattice of the subvarieties of LAA_I is isomorphic to the complete lattice of the lambda theories over the language Λ_I. It follows that every variety of lambda abstraction algebras is generated by the term algebra of a suitable lambda theory over the language Λ_I (with an empty set of constants). In particular, the term algebra of the least lambda theory $\lambda\beta$ generates the variety LAA_I. Hence the explicit finite equational axiomatization for the variety of lambda abstraction algebras provides also an explicit axiomatization of the identities between contexts satisfied by the term algebra of $\lambda\beta$. Note that the lattice of all lambda theories is naturally isomorphic to the lattice of all congruences of the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$. We would like to explicitly mention at this point that the equational theory of lambda abstraction algebras (axiomatized by $(\beta_1) - (\beta_6)$ and (α)) is a conservative extension of lambda beta-calculus: for any two λ-terms M and N, the identity $M = N$ between λ-terms is a logical consequence of $(\beta_1) - (\beta_6)$ and (α) (in symbols, $LAA_I \models M = N$) if, and only if, $M = N$ is derivable in the lambda beta-calculus. This can be immediately inferred from the fact that LAA_I is generated as a variety by the term algebra of the lambda theory $\lambda\beta$ (see Thm. 2 below and Prop. 5 above).

In Thm. 1 below it is shown that the satisfiability of an identity between contexts is equivalent to the satisfiability of a suitable identity between λ-terms. This result is applied in the proof of Thm. 2 below (see [30]).

Let $t(\xi_1, \ldots, \xi_n)$ be a λ-context over $\overline{x}$ (i.e., $\overline{x} = x_1 \ldots x_k$ is the finite sequence of λ-variables which contains all the λ-variables occurring in t either as constants x_i or as λ-abstractions λx_i). Let $\overline{y} = y_1 \ldots y_n$ be an n-tuple of λ-variables such that $\overline{y} \cap \overline{x} = \emptyset$. Define

$$t(y_1 x_1 \ldots x_k, \ldots, y_n x_1 \ldots x_k)$$

as the λ-term obtained from the λ-context t by substituting the λ-term $y_i x_1 \ldots x_k$ for all the occurrences of the context variable ξ_i in t ($i = 1, \ldots, n$). (Recall that $y_i x_1 \ldots x_k$ means $(\ldots((y_i x_1)x_2)\ldots)x_k)$.)

If $\overline{y} = y_1 \ldots y_n$ is a sequence of λ-variables and $\overline{\xi} = \xi_1 \ldots \xi_n$ is a sequence of context variables, we will write $\lambda\overline{y}$ for $\lambda y_1 \ldots y_n$; $t(\overline{\xi})$ for $t(\xi_1, \ldots, \xi_n)$; and $t(y_1\overline{x}, \ldots, y_n\overline{x})$ for $t(y_1 x_1 \ldots x_k, \ldots, y_n x_1 \ldots x_k)$. We always assume that $\overline{\xi}$ and $\overline{y}$ have the same length.

Theorem 1. ([30]) *Let $\mathbf{A}$ be an infinite dimensional LAA_I. Let $t(\overline{\xi})$, $u(\overline{\xi})$ be λ-contexts over $\overline{x} = x_1 \ldots x_k$ and let $\overline{y} = y_1 \ldots y_n$ such that $\overline{y} \cap \overline{x} = \emptyset$. Then,*

$$\mathbf{A} \models t(\overline{\xi}) = u(\overline{\xi}) \text{ if and only if } \mathbf{A} \models t(y_1\overline{x}, \ldots, y_n\overline{x}) = u(y_1\overline{x}, \ldots, y_n\overline{x}).$$

Let $\mathcal{V}$ be an arbitrary variety of algebras and $\mathbf{A} \in \mathcal{V}$. Then $\mathbf{A}$ is said to be *generic* in $\mathcal{V}$ if an identity holds in $\mathbf{A}$ iff it holds in $\mathcal{V}$; equivalently, $\mathbf{A}$ is generic iff it generates $\mathcal{V}$ as a variety.

Recall from Section 1 that, if $\mathcal{T}$ is a lambda theory we denote by $\mathbf{\Lambda}_I^{\mathcal{T}}$ the term algebra of the lambda theory $\mathcal{T}$. So, $\mathbf{\Lambda}_I^{\lambda\beta}$ is the term algebra of the minimal lambda theory $\lambda\beta$.

Theorem 2. ([30]) *For any infinite set I, the variety generated by the term algebra $\mathbf{\Lambda}_I^{\lambda\beta}$ of the minimal lambda theory $\lambda\beta$ is the variety of* LAA_I*'s, in symbols,*

$$\mathsf{LAA}_I = \mathbf{H}\ \mathbf{S}\ \mathbf{P}(\mathbf{\Lambda}_I^{\lambda\beta}).$$

It follows that the set of identities between λ-contexts true in the term algebra of $\lambda\beta$ is axiomatized by the identities (β_1)-(β_6) and (α) characterizing the variety of lambda abstraction algebras.

The class FLA of (isomorphic images of) functional LAA's is defined, for example, in [24]. There is a strong relationship between the class FLA and the class of models of lambda calculus (see [24] [25]). We recall that, although lambda calculus has been the subject of research by logicians since the early 1930's, its model theory developed only much later, following the pioneering model construction made by Dana Scott. There exists an intrinsic characterization of what might be meant by mathematical model of the untyped lambda calculus, as an elementary class of combinatory algebras called λ-*models* ([1, Def. 5.2.7]). They were first axiomatized by Meyer [19] and independently by Scott [32]; the axiomatization, while elegant, is not equational. The functional representation theorem (Thm. 3 below) connects lambda abstraction algebras and models of lambda calculus. Indeed, every lambda abstraction algebra is isomorphic to a functional lambda abstraction algebra arising from a suitable λ-model.

The proof that FLA is a variety was given by Goldblatt [10], while the proof that $\mathsf{FLA} = \mathsf{LAA}$ was obtained in Salibra-Goldblatt [31] with a very technical and difficult proof. One of the consequences of Thm. 2 is a simplification of the proof of the general representation theorem for lambda abstraction algebras (see [30]).

Theorem 3. ([10] [31]) *For any infinite set I,*

$$\mathsf{LAA}_I = \mathsf{FLA}_I.$$

There exists a one-to-one correspondence between the set of lambda theories over the set Λ_I of λ-terms (without constants) and the set of congruences over the term algebra of the minimal lambda theory $\lambda\beta$. So, the set of lambda theories over Λ_I constitutes a complete lattice.

We now characterize the lattice of subvarieties of the variety LAA_I. The variety generated by the term algebra of a lambda theory $\mathcal{T}$ will be denoted by $\mathsf{LAA}_I^{\mathcal{T}}$.

Theorem 4. ([30]) *Let $\mathcal{V}$ be a subvariety of the variety LAA_I. Then there exists exactly one lambda theory $\mathcal{T}$ over Λ_I such that the term algebra $\mathbf{\Lambda}_I^{\mathcal{T}}$ is generic in $\mathcal{V}$:*

$$\mathcal{V} = \mathbf{H}\ \mathbf{S}\ \mathbf{P}(\mathbf{\Lambda}_I^{\mathcal{T}}) = \mathsf{LAA}_I^{\mathcal{T}}.$$

The following theorem is now immediate.

Theorem 5. ([30]) *There is a complete lattice dual isomorphism between the lattice of subvarieties of LAA_I and the lattice of lambda theories over Λ_I (or the lattice of congruences over the term algebra of the minimal lambda theory $\lambda\beta$).*

The above theorem has important consequences for the structure of the lattice of lambda theories as it will be explained in the next Section.

4 The Structure of the Lattice of Lambda Theories

Techniques of universal algebra were applied in [18] to study the properties of the lattice λT of lambda theories. As a consequence of the dual isomorphism between λT and the lattice of subvarieties of LAA (see Thm. 5), some nontrivial quasi-identities, such as the ET condition and the Zipper condition (see [17]), hold in λT. Moreover, the following other results hold:

(i) A lattice identity e holds in LAA if, and only if, it is trivial;
(ii) For every nontrivial lattice identity e, there exists a natural number n such that e fails in the lattice of lambda theories in a language of λ-terms with n constants.
(iii) There exists a sublattice of λT satisfying good lattice properties.

The language of bounded lattices is constituted by two binary operators, "$\cdot$" (meet) and "$+$" (join), and two constants "0" (bottom) and "1" (top). If q is an identity or a quasi-identity in the language of bounded lattices and λT is the lattice of lambda theories, we write $\lambda T \models q$ if the (quasi-)identity q holds in the lattice λT.

The lattice λT satisfies nontrivial quasi-identities.

Theorem 6. ([18]) *Let $\mathcal{T}$, $\mathcal{G}$ and $\mathcal{S}_k$ ($k \in K$) be lambda theories. Then*

$$\lambda T \models\ \sqcup_{k\in K}\mathcal{S}_k = 1,\ \ \mathcal{T} \geq \mathcal{G}(\mathcal{S}_k + \mathcal{T}\mathcal{G})\ (k \in K)\ \Rightarrow\ \mathcal{G} \leq \mathcal{T}.$$

The ET condition and the Zipper condition defined below hold in every lattice dually isomorphic to the lattice of subvarieties of a variety (see [17]). Then the following two corollaries follows from Thm. 5 above.

Corollary 1. (ET Condition) *Let $(\mathcal{S}_k : k \in K)$ be a family of lambda theories. If*

$$\sqcup_{k\in K}\mathcal{S}_k = 1_{\lambda T}$$

then there is a finite sequence $\mathcal{S}_0, \dots, \mathcal{S}_n$ with $\mathcal{S}_j \in \{\mathcal{S}_k : k \in K\}$ for $j \leq n$ such that:

$$(((\dots(((\mathcal{S}_0\mathcal{G}) + \mathcal{S}_1)\mathcal{G}) + \mathcal{S}_2)\mathcal{G}) + \dots + \mathcal{S}_n)\mathcal{G} = \mathcal{G} \tag{3}$$

for every lambda theory $\mathcal{G}$.

Corollary 2. (Zipper Condition) *Let $\mathcal{T}$, $\mathcal{G}$ and $(\mathcal{S}_k : k \in K)$ be lambda theories. Then,*

$$\lambda T \models \sqcup_{k\in K}\mathcal{S}_k = 1,\ \ \mathcal{S}_k\mathcal{G} = \mathcal{T}\ (k \in K)\ \Rightarrow\ \mathcal{G} = \mathcal{T}.$$

Corollary 3. *Let $\mathcal{S}$ and $\mathcal{T}$ be two lambda theories. If $\mathcal{S} + \mathcal{T} = 1_{\lambda T}$ and $\mathcal{S}\mathcal{G} = \mathcal{T}\mathcal{G}$ then $\mathcal{G} \leq \mathcal{S}$ and $\mathcal{G} \leq \mathcal{T}$.*

Two well-known lattices cannot be sublattices of λT.

Corollary 4. *Let $\mathbf{M}_k$ be the lattice having k atoms, a zero and one, and no other elements. $\mathbf{M}_k$ cannot be a sublattice of λT, provided the top element is the inconsistent lambda theory $1_{\lambda T}$.*

In every lattice L the modularity law is equivalent to the requirement that L has no sublattice isomorphic to the "pentagon" $\mathbf{N}_5$. The pentagon $\mathbf{N}_5$ is constituted by five distinct elements $0, a, b, c, 1$ such that $a \leq c$, $1 = c + b = a + b$ and $0 = ab = cb$. In [29] it was shown the following result. Recall from Section 2.1 the definitions of the lambda terms $\mathbf{i}$, Ω and of the lambda theories $\mathcal{H}, \mathcal{H}^*$.

Theorem 7. ([29]) *Let $\mathcal{T}$ be the lambda theory generated by the equation $\Omega = \mathbf{i}$. The lattice λT is not modular because the pentagon $\mathbf{N}_5$, defined by*

$$0 \equiv \mathcal{T}\mathcal{H}^*; \quad 1 \equiv 1_{\lambda T}; \quad b \equiv \mathcal{T}; \quad a \equiv \mathcal{H} + (\mathcal{T}\mathcal{H}^*); \quad c \equiv \mathcal{H}^*,$$

is a sublattice of λT.

An identity in the binary symbols $\{\cdot, +, \circ\}$ is called a *congruence identity*, while an identity in the language $\{\cdot, +\}$ of lattices is called a *lattice identity*. We interpret the variables of a congruence (lattice) identity as congruence relations, and for arbitrary binary relations γ and δ we interpret $\gamma + \delta$ as the congruence relation generated by the union of the two relations, $\gamma \cdot \delta$ as the intersection and $\gamma \circ \delta$ as the composition of the two relations (as usual, we will write $\gamma\delta$ for $\gamma \cdot \delta$). We say that a variety $\mathcal{V}$ *satisfies a congruence (lattice) identity* if it holds in all congruence lattices of members of $\mathcal{V}$. A congruence identity is *trivial* if it holds in the congruence lattice of any algebra, while a lattice identity is *trivial* if it holds in every lattice. Varieties are often characterized in terms of congruence identities.

The following theorem is a consequence of the sequentiality theorem of lambda calculus (see [1, Thm. 14.4.8]). It has no immediate consequence for the lattice of lambda theories.

Theorem 8. ([18]) *Let $\mathcal{T}$ be a semisensible lambda theory. Then the variety $\mathsf{LAA}_I^{\mathcal{T}}$ generated by the term algebra of $\mathcal{T}$ satisfies a congruence identity e if, and only if, e is trivial.*

Notoce that there exists a continuum of semisensible lambda theories and that all the most important lambda theories, such as $\lambda\beta$, $\lambda\eta$, $\mathcal{H}$, $\mathcal{H}^*$ etc., are semisensible.

As a consequence of the above theorem, for every nontrivial lattice identity e, there exists an LAA_I $\mathbf{A}$ whose congruence lattice does not satisfy e. In the following theorem we show that $\mathbf{A}$ can be chosen a term algebra if we modify the language of lambda calculus with a finite number of constants.

As a matter of notation, $\Lambda_I(n)$ is the set of lambda terms constructed from an infinite set I of λ-variables and a finite set of constants of cardinality n.

Theorem 9. ([18]) *Let e be a nontrivial lattice identity. Then there exists a natural number n such that the identity e fails in the lattice of the lambda theories over the language $\Lambda_I(n)$.*

There exists a sublattice of λT satisfying good lattice properties. First we introduce a lambda theory $\mathcal{J}$, whose consistency is obtained in [18] by using intersection types for defining a filter model for it (see [2], [8]). Then the equations defining $\mathcal{J}$ are used to define a semilattice term operation on the term algebra $\mathbf{\Lambda}_I^{\mathcal{J}}$. It follows from this result that the lattice of all lambda theories including $\mathcal{J}$ has the lattice properties described in Thm. 12 below.

Theorem 10. ([18]) *The lambda theory $\mathcal{J}$, axiomatized by*

$$\Omega xx = x; \quad \Omega xy = \Omega yx; \quad \Omega x(\Omega yz) = \Omega(\Omega xy)z, \tag{4}$$

is consistent.

Theorem 11. ([18]) *The variety $\mathsf{LAA}_I^{\mathcal{J}}$ generated by the term algebra of the lambda theory $\mathcal{J}$ satisfies a nontrivial congruence identity.*

Theorem 12. ([18]) *The interval sublattice $[\mathcal{J}) = \{\mathcal{T} : \mathcal{J} \subseteq \mathcal{T}\}$ of the lattice of lambda theories satisfies the following properties:*

(i) The finite lattice $\mathbf{M}_3$ is not a sublattice of $[\mathcal{J})$.
(ii) $[\mathcal{J})$ satisfies a nontrivial congruence identity.
(iii) $[\mathcal{J})$ is congruence meet semidistributive, i.e. the following implication holds for all lambda theories $\mathcal{S}, \mathcal{T}, \mathcal{G} \in [\mathcal{J})$.

$$\mathcal{S}\mathcal{T} = \mathcal{S}\mathcal{G} \;\Rightarrow\; \mathcal{S}\mathcal{T} = \mathcal{S}(\mathcal{T} + \mathcal{G}).$$

5 Order incompleteness and topological incompleteness

In this Section we survey some results of incompleteness for the semantics of lambda calculus (see [26] [27] [28]).

A model of the untyped lambda calculus univocally induces a lambda theory (i.e., a congruence relation on λ-terms closed under α- and β-conversion) through the kernel congruence relation of the interpretation function. A semantics of lambda calculus is called *(equationally) incomplete* if there exists a lambda theory which is not induced by any model in the semantics.

One of the most interesting open problems of lambda calculus is whether every lambda theory arises as the equational theory of a non-trivially ordered model (in other words, whether the semantics of lambda calculus given in terms of non-trivially ordered models is complete). Selinger [33] gave a syntactical characterization, in terms of so-called generalized Mal'cev operators, of the order-incomplete lambda theories (i.e., the theories not induced by any non-trivially ordered model). A lambda theory $\mathcal{T}$ is order-incomplete if, and only if, there exist a natural number $n \geq 1$ and a sequence $M_1, \dots, M_n$ of closed λ-terms such that the following Mal'cev conditions are satisfied:

$$x =_{\mathcal{T}} M_1 xyy; \; M_i xxy =_{\mathcal{T}} M_{i+1} xyy; \; M_n xxy =_{\mathcal{T}} y \; (1 \leq i < n).$$

In other words, $\mathcal{T}$ is order-incomplete if, and only if, the variety of LAA's generated by the term algebra of $\mathcal{T}$ is $(n+1)$-permutable for some $n \geq 1$. Plotkin and Simpson (see [33]) have shown that the above Mal'cev conditions are inconsistent with lambda calculus for $n = 1$, while Plotkin and Selinger (see [33]) obtained the same result for $n = 2$. It is an open problem whether n can be greater than or equal to 3.

We may relax the order-incompleteness problem by considering semantics of lambda calculus given in terms of partially ordered models satisfying suitable conditions. For example, we have shown in [26] [27] the incompleteness of any semantics of lambda calculus given in terms of partially ordered models which are semilattices, lattices, complete partial orderings, or which have a top or a bottom element. Moreover, we have obtained the incompleteness of any semantics of lambda calculus, given in terms of topological models satisfying a suitable property of connectedness called *closed-open-connectedness*. The proof is based on a general theorem of separation for topological algebras.

We begin the technical part of this Section by introducing the notion of a semisubtractive algebra. These algebras satisfy a very weak form of subtractivity. We recall that the notion of subtractivity in Universal Algebra was introduced by Ursini [34]: an algebra is *subtractive* if it satisfies the identities

$$s(x,x) = 0; \quad s(x,0) = x \tag{5}$$

for some binary term s and constant 0. Subtractive algebras abound in classical algebras and in algebraic logic.

Definition 4. *An algebra* $\mathbf{A}$ *is* **semisubtractive** *if there exist a binary term* $s(x,y)$ *and a constant* 0 *in the similarity type of* $\mathbf{A}$ *such that*

$$s(x,x) = 0.$$

A *partially ordered algebra* is a pair $(\mathbf{A}, \leq)$, where $\mathbf{A}$ is an algebra and $\leq$ is a compatible (i.e., the basic operations are monotone) partial order on A.

A semisubtractive algebra is *trivial* if $s(x,y) = 0$ for all x and y. A semisubtractive algebra is trivial if it admits a compatible partial order with a bottom element and a top element.

As a matter of notation, ω denotes the first infinite ordinal, i.e., the set of natural numbers.

Definition 5. *Let* $\mathbf{A}$ *be a semisubtractive algebra. The* **subtraction sequence** (c_n) *of a pair* $(a,b) \in A^2$ *is defined by induction as follows:*

$$c_1 = s(a,b); \quad c_{n+1} = s(c_n, 0). \tag{6}$$

We say that (a,b) **has order** $k \in \omega$ *if* $c_k \neq 0$*, while* (a,b) **has order** ω *if* $c_k \neq 0$ *for all* $k \in \omega$*.*

Notice that, if a pair (a,b) has order k, then $a \neq b$ and (a,b) has order n for all $n \leq k$.

The *inequality graph* of a partially ordered algebra $(\mathbf{A}, \leq)$ has the elements of A as nodes, while an edge connects two distinct nodes a and b if either $a < b$ or $b < a$. Two nodes are in the same connected component if they are either not distinct or joined by a path. The equivalence classes of the relation "to be in the same connected component" define the partition of the inequality graph into *connected components*. A connected component can be also characterized as a minimal subset of A which is both upward closed and downward closed.

Lemma 2. *Let* $(\mathbf{A}, \leq)$ *be a semisubtractive partially ordered algebra. If a pair* $(a,b) \in A^2$ *has order* ω*, then* a *and* b *are in distinct connected components (in the inequality graph of* A*).*

Theorem 13. ([27], [28]) *Any semantics of the untyped lambda calculus, given in terms of partially ordered models which have only finitely many connected components (in the inequality graph), is incomplete.*

Corollary 5. *Any semantics of lambda calculus given in terms of partially ordered models which are semilattices, lattices, complete partial orderings, or which have a top or a bottom element, is incomplete.*

A topological algebra is a pair $(\mathbf{A}, \tau)$, where $\mathbf{A}$ is an algebra and τ is a topology on A making the basic operations of $\mathbf{A}$ continuous.

We recall that separation axioms in topology stipulate the degree to which distinct points may be separated by open sets or by closed neighbourhoods of open sets. In Thm. 14 below we prove that in every semisubtractive T_0-topological algebra every pair of elements of order 3 is $T_{2_{1/2}}$-separable. We recall that a and b are $T_{2_{1/2}}$-separable if there exists two open sets U and V such that $a \in U$, $b \in V$ and the closures of U and V have empty intersection.

We were inspired with Bentz [3] and Coleman [6] [7] for the idea of this theorem and for the techniques used in its proof.

Theorem 14. ([26], [28]) *If $(\mathbf{A}, \tau)$ is a semisubtractive T_0-topological algebra, then every pair $(a, b) \in A^2$ of order 3 is $T_{2_{1/2}}$-separable.*

Connectedness axioms in topology examine the structure of a topological space in an orthogonal way with respect to separation axioms. They deny the existence of certain subsets of a topological space with properties of separation. We introduce a strong property of connectedness, called closed-open-connectedness, which is orthogonal to the property of $T_{2_{1/2}}$-separability, and it is satisfied by a topological space if there exist no $T_{2_{1/2}}$-separable elements.

Definition 6. *We say that a space is* **closed-open-connected** *if it has no disjoint closures of open sets. In other words, if, for all open sets U and V, we have that the closures of U and V have empty intersection.*

The following proposition provides a wide class of topological spaces whose topology is closed-open-connected.

Proposition 6. *Let (X, τ) be a T_0-topological space, whose specialization order $\leq_\tau$, defined by*

$$a \leq_\tau b \text{ iff } a \text{ belongs to the closure of set } \{b\},$$

satisfies the following property: every pair of nodes of the inequality graph of $(X, \leq_\tau)$ is joined by a path of length less or equal to 3. Then (X, τ) is closed-open-connected.

Corollary 6. *Every T_0-topological space (X, τ), whose specialization order either admits a bottom (top) element or makes X an upward (downward) direct set, is closed-open-connected.*

In particular complete partial orderings with the Scott topology (see [1]) are closed-open-connected.

As a consequence of Thm. 14 of separation, we get the topological incompleteness theorem.

A topological model of the lambda calculus is a topological algebra $(\mathbf{C}, \tau)$, where $\mathbf{C}$ is a model of lambda calculus.

Theorem 15. ([26], [28]) *Any semantics of the lambda calculus given in terms of closed-open-connected T_0-topological models is incomplete.*

A topological model, whose topology is T_0 but not T_1, has a non-trivial specialization order. This means that there exists at least two elements a and b such that a belongs to the closure of set $\{b\}$. Then from Thm. 13 and from Thm. 15 it follows the incompleteness of any semantics of lambda calculus given in terms of T_0-topological models, whose topology is either closed-open-connected or admits a specialization order with a finite number of connected components.

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