A SIMPLIFIED FINITE AXIOMATIZATION FOR THE SAIN TYPE ALGEBRAIZATION OF THE FIRST ORDER LOGIC

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Abstract. The problem of finding a finite axiomatization to the algebraic counterpart of the first order logic was solved by Ildikó Sain. She gave a finite scheme of axioms, however she stated as an open problem the task of giving amore simple one. In this work we give a small elegant system of axioms, solving the above mentioned problem. I would like to thank Ildikó Sain and István Németi to inspire this work.

This paper is a continuation of the work was initiated earlier together with Ildikó Sain and announced in [I.Sain 87] I. Thm 1. (p.3.).

The problem of finding a "finite scheme" algebraization of the first order logic goes back to [J.D.Monk 70] and [L.Henkin,J.D.Monk 74] and was recalled by [I.Sain 87]:

"*Devise an algebraic version of predicate logic in which the class of representable algebras forms a finitely based equational class*"

An equational class is finitely based if it is axiomatizable by a finite amount of equations. An algebra is called representable if it is isomorphic to a subdirect product of set algebras. According to [A.Tarski 66] we add the requirement that the operations should be logical. An operation is called logical iff it is invariant under permutation of the base of the algebra. By [I.Sain 93] (p.2.) the logical counterpart of this requirement is that isomorphic models satisfy the same formulas. Sain found the following solution to the above stated problem. We shall use the notation of [HMT I] and [HMT II].

$[i,j] \in \omega^\omega$ is the permutation of i and j, $[i/j] \in \omega^\omega$ is the replacement of i by j (i.e. $[i/j](i) = j$ and $[i/j](k) = k$ if $k \neq i$). Let f $\in \omega^\omega$ and $i, j \in \omega$, then let $f(i/j) \in \omega^\omega$ according to the definition $f(i/j)(i) = f(j)$ and $f(i/j)(n) = f(n)$ if $n \neq i$. $Sb(X)$ is the class of all subsets of X.

Definition 1.:([Sain 87.] I. Def.1. p.3.)

By a Sgws we understand a subalgebra:

$\mathfrak{A} \subseteq \langle Sb({}^\omega U), \cup, \cap, \widetilde{{}_V}, \emptyset, V, \mathbf{S}_{suc}, \mathbf{S}_{pred}, \mathbf{S}^i_j, \mathbf{c}_i\rangle_{i,j\in\omega}$

where: $\mathbf{S}_\tau(x) = \mathbf{S}^V_\tau(x) = \{q \in V \mid q \circ \tau \in x\}$ for every $\tau \in \omega^\omega$ and $x \subseteq V$,

$\mathbf{S}^i_j = \mathbf{S}_{[i/j]}$, and $suc \in \omega^\omega$ is the usual successor, and $pred \in \omega^\omega$ is its inverse with $pred(0) = 0$,

$V \subseteq {}^\omega U$ let $\mathrm{Gws}_\omega-$unit in the sense of [HMT II.] Def.3.1.1. (5.o.) definition and let $\mathbf{S}_{suc}(V) = \mathbf{S}_{pred}(V) = \mathbf{S}^i_j(V) = \mathbf{c}_i(V) = V$

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Remark 2.: It is clear from the definition that Sgws algebras do not contain the constants of the cylindric algebras $d_{ij}, i,j < \omega$. The logical counterpart of this is that Sgws algebras are related to the first order logic without equality.

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Theorem 3.:([Sain 87.] I. Thm.1. (Sain & student), p.3.)

ISgws is a variety axiomatizable by a finite scheme of equations.

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A detailed introduction to the history of the problem solved above and an overview of the results can be found in [I.Németi 91]. During the last years many results were published in this area.
The negative results are: in [B.Biró 87] the non-finitizability of logic with equality, the strengthening of Biró's result in [I.Németi 91] and [H.Andréka 91]. In [I.Sain, R.J.Thompson 91] there is a non-finitizability result for quasi-polyadic algebras. In [Sági 95] and [Sági,Németi 95] there are a non-finitizability results for polyadic algebras.
The positive results are: in [I.Sain 93] there are results on adding $d_{i,j}, i, j < \omega$ constants to Sgws. In [I.Sain, V.Gyuris 95] there are finitizability results for first order logic with equality too. Also in [Sági 95] it was shown that Sgws can be replaced by Cs-style algebras i.e. by such Sgws's the geatest elements of which are (full) Cartesian spaces. Strong positive results in non-well-founded set theories are in [Németi Tit1],[Németi Tit2] and in [Simon-Németi 95].

Our main theorem originated in [I.Sain 87] II. p.38. Claim (under Remark 10.1.) as an open problem. The [I.Sain 87] preprint introduced the class of Sgws-like algebras. She gave a finite axiom scheme [I.Sain 87] II. p.37. (Remark 10.1) axiomatizing the class of Sgws algebras. However she left the task of finding simpler and more elegant axiomatization(s) as an open problem. We list the original system of axioms and we will give a simplified version of it as stated in [S.Csizmazia 92], [S.Csizmazia 95]. The presently reported research was conducted with guidance from I.Sain and R.J.Thompson during the period 1987-1992 (though some corrections were made later). During 1993-1995 strongly related and sometimes slightly overlapping results were found by I.Sain and V.Gyuris of [I.Sain, V.Gyuris 95].

We will denote the variable symbols of the language of Sgws algebras with x, y.

The function symbols of the type t_{Sgws} are the following ones:

$0 :\ <: d>$, the "least element" constant symbol
$1 :\ <: d>$, the "greatest element" constant symbol
$- :\ < d : d >$, the symbol of the complementation
$+ :\ < d, d : d >$, the symbol of the join
$\cdot :\ < d, d : d >$, the symbol of the meet
$\mathbf{c}_i :\ < d : d >$, the symbols of the cilindrifications $(i < \omega)$
$\mathbf{S}^i_j :\ < d : d >$, the symbols of the replacements $(i, j < \omega)$
$\mathbf{S}_{suc} :\ < d : d >$, the symbol of the successor
$\mathbf{S}_{pred} :\ < d : d >$, the symbol of the predecessor

Let us denote with L_{Sgws} the t_{Sgws} type first order language.

We will introduce the following defined symbols:

$\mathbf{P}^i_j :\ < d : d >$, the symbols of the interchange (where $i, j < \omega$)

$$\mathbf{P}^i_j \stackrel{\text{def}}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}$$

First we will detail the **AX** system of axioms according to [I.Sain 87] II. (in the [I.Sain 87] II. Proof of Theorem 1. 34-38.o.), after this we will give its simplified version as **PAX**.

The **AX** system of axioms contains the following schemes:

With arbitrary $i, j, k, l < \omega$:

(**S**0) The usual axioms of the Boolean algebras

($\mathbf{S}1$) The $\mathbf{S}_{pred}, \mathbf{S}_{suc}, \mathbf{S}^i_j$ are Boolean endomorphisms

With arbitrary $\mathrm{S} \in \{\mathbf{S}_{pred}, \mathbf{S}_{suc}, \mathbf{S}^i_j\}$:

S(x + y) = Sx + Sy

S(x · y) = Sx · Sy

S0 = 0

S1 = 1

($\mathbf{S}2$) The connections of $\mathbf{S}_{pred}, \mathbf{S}_{suc}$ with each other and with $\mathbf{S}^i_j$:

(1.1) $\mathbf{S}_{pred}\mathbf{S}_{suc}x = x$

(1.2) $\mathbf{S}_{suc}\mathbf{S}_{pred}x = \mathbf{S}^0_1 x$

(1.3) $\mathbf{S}_{suc}\mathbf{S}^i_j x = \mathbf{S}^{i+1}_{j+1}\mathbf{S}_{suc}x$ if $i \neq j$

(1.4) $\mathbf{S}^i_j\mathbf{S}_{pred}x = \mathbf{S}_{pred}\mathbf{S}^{i+1}_{j+1}x$ if $i \neq 0$ and $i \neq j$

(1.5) $\mathbf{S}^0_j\mathbf{S}_{pred}x = \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^1_{j+1}x$ if $j \neq 0$

(1.6) Jónsson's seven schemes about the connections of the $\mathbf{S}^i_j$-s and the $\mathbf{P}^i_j$-s (see [HMT II.] (p.68.)):

(1.6.1) $\mathbf{P}^i_j x = \mathbf{P}^j_i x$

(1.6.2) $\mathbf{P}^i_j\mathbf{P}^j_i x = x$

(1.6.3) $\mathbf{P}^i_j\mathbf{P}^i_k x = \mathbf{P}^j_k\mathbf{P}^i_j x$ if $j \neq k, i \neq k, i \neq j$

(1.6.4) $\mathbf{P}^i_j\mathbf{S}^k_i x = \mathbf{S}^k_j\mathbf{P}^i_j x$ if $j \neq k, i \neq k$

(1.6.5) $\mathbf{P}^i_j\mathbf{S}^j_i x = \mathbf{S}^i_j x$ if $i \neq j$

(1.6.6) $\mathbf{S}^i_j\mathbf{S}^k_l x = \mathbf{S}^k_l\mathbf{S}^i_j x$ if $k \neq j, i \neq k, l$

(1.6.7) $\mathbf{S}^i_j\mathbf{S}^i_k x = \mathbf{S}^i_k x$ if $i \neq k$

($\mathbf{S}3$) The $q_1 - q_9$ axiom schemes of Pinter [73]:

(q_1) $\mathbf{S}^i_j(-x) = -\mathbf{S}^i_j x$

(q_2) $\mathbf{S}^i_j(x + y) = \mathbf{S}^i_j x + \mathbf{S}^i_j y$

(q_3) $\mathbf{S}^i_i x = x$

(q_4) $\mathbf{S}^i_j\mathbf{S}^k_i x = \mathbf{S}^i_j\mathbf{S}^k_j x$

(q_5) $\mathbf{c}_i(x + y) = \mathbf{c}_i x + \mathbf{c}_i y$

(q_6) $x \leq \mathbf{c}_i x$

(q_7) $\mathbf{S}^i_j\mathbf{c}_i x = \mathbf{c}_i x$

(q_8) $\mathbf{c}_i\mathbf{S}^i_j x = \mathbf{S}^i_j x$ if $i \neq j$

(q_9) $\mathbf{S}^i_j\mathbf{c}_k x = \mathbf{c}_k\mathbf{S}^i_j x$ if $k \neq i, j$

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We give the simplified version as **PAX**:

Let **PAX** be the following finite amount of axiom schemes with arbitrary $i, j, k, l < \omega$:

(**S**0) The usual axioms of the Boolean algebras

(**S**1) The $\mathbf{S}_{pred}, \mathbf{S}_{suc}, \mathbf{S}^i_j$ are Boolean endomorphisms:

with arbitrary S $\in \{\mathbf{S}_{pred}, \mathbf{S}_{suc}, \mathbf{S}^i_j\}$:

(0.1) S(x + y) = Sx + Sy

(0.2) S(-x) = - Sx

(**S**2) The connections of $\mathbf{S}_{pred}$ and $\mathbf{S}_{suc}$ with each other and with $\mathbf{S}^i_j$:

(1.1) $\mathbf{S}_{pred}\mathbf{S}_{suc}x = x$

(1.2) $\mathbf{S}_{suc}\mathbf{S}_{pred}x = \mathbf{S}^0_1 x$

(1.3) $\mathbf{S}_{suc}\mathbf{S}^i_j x = \mathbf{S}^{i+1}_{j+1}\mathbf{S}_{suc}x$ if $i \neq j$

(**S**3) The connections of $\mathbf{c}_i$-s and $\mathbf{S}^i_j$-s:

(p_1) $\mathbf{S}^i_j\mathbf{c}_i x = \mathbf{c}_i x$

(p_2) $\mathbf{c}_i\mathbf{S}^i_j x = \mathbf{S}^i_j x$ if $i \neq j$

(p_3) $\mathbf{S}^i_j\mathbf{c}_k x = \mathbf{c}_k\mathbf{S}^i_j x$ if $k \neq i, j$

The properties of the $\mathbf{S}^i_j$-s:

(p_4) $\mathbf{S}^i_j\mathbf{S}^k_i x = \mathbf{S}^i_j\mathbf{S}^k_j x$

(p_5) $\mathbf{S}^i_i x = x$

The properties of the $\mathbf{c}_i$-s:

(p_6) $\mathbf{c}_i x \geq x$

(p_7) $\mathbf{c}_i(x + y) = \mathbf{c}_i x + \mathbf{c}_i y$

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PAX fully describes Sgws:

Theorem 4.: **I**Sgws = Mod(**PAX**)

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The proof uses significantly [I.Sain 87] I. Lemma 1. (p.6.): **I**Sgws = Mod(**AX**). As it will be proved in Lemma 5.: **AX** and **PAX** are equivalent, thus Mod(**PAX**) = Mod(**AX**). Out of this we conclude **I**Sgws = Mod(**PAX**).

Lemma 5.: **AX** $\models$ **PAX** and **PAX** $\models$ **AX**

Proof 5.:
1. The part **AX** $\models$ **PAX** is obvious, because every axiom and axiom scheme of **PAX** except of (**S**1)(0.2) are part of **AX**. The axiom (**S**1)(0.2) is a consequence of **AX**(**S**0), (**S**1):
$1 = S(1) = S(x + -x) = S(x) + S(-x)$

$0 = S(0) = S(x \cdot -x) = S(x) \cdot S(-x)$

and because of the existence of the unique complementer in the Boolean algebras: $-S(x) = S(-x)$.

2. We have to prepare some lemmas as evidence to the other part: $\mathbf{PAX} \models \mathbf{AX}$

As a proof we will use the results of [C.C.Pinter 73]. Because **PAX** contains every axiom and scheme of [C.C.Pinter 73] we will not repeat the proofs of these results. The results are the following ones, where $i, j, k, l < \omega$ are arbitrary:

(i) [C.C.Pinter 73] Lemma 2.2 (i) p.363.:

$\mathbf{PAX} \models \mathbf{c}_i 0 = 0$

(ii) [C.C.Pinter 73] Lemma 2.2 (ii) p.363.:

$\mathbf{PAX} \models \mathbf{c}_i(x \cdot \mathbf{c}_i y) = \mathbf{c}_i x \cdot \mathbf{c}_i y$

(iii) [C.C.Pinter 73] Lemma 2.2 (iii) p.363.:

$\mathbf{PAX} \models \mathbf{c}_i \mathbf{c}_j x = \mathbf{c}_j \mathbf{c}_i x$

(iv) [C.C.Pinter 73] Lemma 2.2 (iv) p.363.:

$\mathbf{PAX} \models \mathbf{S}^i_j \mathbf{S}^i_k x = \mathbf{S}^i_k x$ if $i \neq k$

(v) [C.C.Pinter 73] Lemma 2.2 (v) p.363.:

$\mathbf{PAX} \models \mathbf{S}^i_j \mathbf{S}^k_l x = \mathbf{S}^k_l \mathbf{S}^i_j x$ if $i \neq k, l$ and $k \neq j$

(vi) [C.C.Pinter 73] Lemma 5 Proof (4) p.363.:

$\mathbf{PAX} \models \mathbf{c}_i x = min\{y \mid y \geq x$ and $y = S^i_j z$ for some z $\}$ if $i \neq j$

<u>Lemma 6.</u>: $\mathbf{PAX} \models \mathbf{S}^i_j \mathbf{S}^k_i x = \mathbf{S}^k_j x$, if $\mathbf{c}_i x = x$

<u>Proof 6.</u>:

Let us assume that $\mathbf{c}_i x = x$

If i = j then

$$\mathbf{S}^i_i \mathbf{S}^k_i x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}3)/(p_5)}}{=} \mathbf{S}^k_i x$$

If i = k then

$$\mathbf{S}^i_j \mathbf{S}^k_k x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}3)/(p_5)}}{=} \mathbf{S}^i_j x$$

We can assume that $i \neq j, k$

(3) $$x = \mathbf{c}_i x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}3)/(p_1)}}{=} \mathbf{S}^i_j \mathbf{c}_i x \underset{\substack{\uparrow \\ \mathbf{c}_i x = x}}{=} \mathbf{S}^i_j x$$

Then if $j \neq k$

$$\mathbf{S}^i_j\mathbf{S}^k_i x \underset{\uparrow}{=} \mathbf{S}^i_j\mathbf{S}^k_j x \qquad \underset{\uparrow}{=} \mathbf{S}^k_j\mathbf{S}^i_j x \qquad \underset{\uparrow}{=} \mathbf{S}^k_j x$$

$\mathbf{PAX}(\mathbf{S}3)/(p_4)$ $\quad$ (v) because $i \neq j, k$ and $j \neq k$ $\quad$ (3)

If $j = k$

$$\mathbf{S}^i_j\mathbf{S}^k_i x \underset{\uparrow}{=} \mathbf{S}^i_j\mathbf{S}^j_j x \qquad \underset{\uparrow}{=} \mathbf{S}^j_j x \qquad \underset{\uparrow}{=} x$$

$\mathbf{PAX}(\mathbf{S}3)/(p_4)$ $\quad$ $\mathbf{PAX}(\mathbf{S}3)/(p_5)$ $\quad$ (3)

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<u>Lemma 7.</u>:

Let S $\in \{\mathbf{S}_{pred}, \mathbf{S}_{suc}, \mathbf{S}^i_j\}$ be arbitrary. Then the following statements are consequences of **PAX**:

S(x · y) = S(x) · S(y)

S(0) = 0

S(1) = 1

<u>Proof 7.</u>:

$$-S(x\cdot y) \underset{\uparrow}{=} S(-(x\cdot y)) \qquad \underset{\uparrow}{=} S(-x+-y) \qquad \underset{\uparrow}{=} S(-x)+S(-y) \qquad \underset{\uparrow}{=} -S(x)+-S(y)$$

$\mathbf{PAX}(\mathbf{S}1)(0.2)$ $\quad$ $\mathbf{PAX}(\mathbf{S}0)$ $\quad$ $\mathbf{PAX}(\mathbf{S}1)(0.1)$ $\quad$ $\mathbf{PAX}(\mathbf{S}1)(0.2)$

$$\underset{\uparrow}{=} -(S(x)\cdot S(y))$$

$\mathbf{PAX}(\mathbf{S}0)$

Then according to $(\mathbf{S}0)$:
$S(x\cdot y) = S(x)\cdot S(y)$ follows.

$$(4) \quad S(1) = S(x+-x) \qquad \underset{\uparrow}{=} S(x)+S(-x) \qquad \underset{\uparrow}{=} S(x)+-S(x) \qquad \underset{\uparrow}{=} 1$$

$\mathbf{PAX}(\mathbf{S}1)(0.1)$ $\quad$ $\mathbf{PAX}(\mathbf{S}1)(0.2)$ $\quad$ $\mathbf{PAX}(\mathbf{S}0)$

$$S(0) \underset{\uparrow}{=} S(-1) \qquad \underset{\uparrow}{=} -S(1) \qquad \underset{\uparrow}{=} -1 \qquad \underset{\uparrow}{=} 0$$

$\mathbf{PAX}(\mathbf{S}0)$ $\quad$ $\mathbf{PAX}(\mathbf{S}1)(0.2)$ $\quad$ (4) $\quad$ $\mathbf{PAX}(\mathbf{S}0)$

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<u>Lemma 8.</u>:

$\mathbf{PAX} \models \mathbf{c}_i\mathbf{S}^j_i x = \mathbf{c}_j\mathbf{S}^i_j x$

<u>Proof 8.</u>:

	$\mathbf{S}^i_j x$	$\leq \mathbf{c}_i x$	$\mathbf{PAX}(\mathbf{S}3)/(p_6), (\mathbf{S}3)/(p_1), (\mathbf{S}1)(0.1)$
(5)	$\mathbf{c}_j\mathbf{S}^i_j x$	$\leq \mathbf{c}_j\mathbf{c}_i x$	$\mathbf{PAX}(\mathbf{S}3)/(p_7)$

a. $(i \neq j)$ case:

$$\mathbf{c}_i\mathbf{S}_i^j x \underset{\uparrow}{=} \mathbf{c}_i\mathbf{c}_j\mathbf{S}_i^j x \underset{\uparrow}{=} \mathbf{c}_j\mathbf{c}_i\mathbf{S}_i^j x \underset{\uparrow}{\geq} \mathbf{c}_j\mathbf{S}_j^i\mathbf{S}_i^j x \underset{\uparrow}{=} \mathbf{c}_j\mathbf{S}_j^i x$$

$\mathbf{PAX}(\mathbf{S}3)/p_2$, $i \neq j$; (iii), $i \neq j$; (5)-ben with $x \doteq \mathbf{S}_i^j x$; $\mathbf{PAX}(\mathbf{S}3)/(p_4), (p_5)$

The direction "$\leq$" is verifiable interchanging the role of the i and j.

b. $(i = j)$ case: obvious.

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Lemma 9.:

$\mathbf{PAX} \models \mathbf{c}_0\mathbf{S}_{suc}x = \mathbf{S}_{suc}x$

Proof 9.:

$$\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_1^0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{c}_0\mathbf{S}_1^0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{c}_0\mathbf{S}_{suc}x$$

$\mathbf{PAX}(\mathbf{S}2)(1.1)$; $\mathbf{PAX}(\mathbf{S}2)/(1.2)$; $\mathbf{PAX}(\mathbf{S}3)/(p_2)$; $\mathbf{PAX}(\mathbf{S}2)(1.2), (1.1)$

■

Lemma 10.:

$\mathbf{PAX} \models$ if $x = \mathbf{S}_j^i y$ for some y and $i \neq j$, then $\mathbf{S}_j^i x = x$

Proof 10.:

$$\mathbf{S}_j^i x = \mathbf{S}_j^i\mathbf{S}_j^i y \underset{\uparrow}{=} \mathbf{S}_j^i y \underset{\uparrow}{=} x$$

(iv) (because $i \neq j$); (because $x = \mathbf{S}_j^i y$)

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The following lemma have been already published in [I. Sain, V. Gyuris 94] (p.17.) from the author.

Lemma 11.:

$\mathbf{PAX} \models \mathbf{c}_1\mathbf{S}_{suc}x = \mathbf{S}_{suc}\mathbf{c}_0 x$

Proof 11.:

We remark that the minimums exist in this proof is due to the celebrated Jónsson-Tarski theorem: every Mod($\mathbf{PAX}$) algebra is embeddable into a complete algebra ([HMT I.] Thm. 10.5.(i) (p.412.)), where the positive equations (our system of axioms essentially looks like) are preserved. Because of the completness the minimums exist in the extended algebra, as well as in it's subalgebra, in Mod($\mathbf{PAX}$).

because of (vi): $\mathbf{c}_1\mathbf{S}_{suc}x = min\underbrace{\{y \mid y \geq \mathbf{S}_{suc}x \text{ and } y = \mathbf{S}_2^1 z \text{ for some } z\ \}}_{B_1}$ (1)

because of (vi): $\mathbf{S}_{suc}\mathbf{c}_0 x = \mathbf{S}_{suc} min \underbrace{\{y \mid y \geq x \text{ and } y = \mathbf{S}_1^0 z \text{ for some } z\ \}}_{B_2}$ (2)

In the next step we will verify that (1) = (2). Starting with (1) we will get (2) applying some transformations.

a.) The following identity is valid:

$$min \underbrace{\{y \mid y \geq \mathbf{S}_{suc}x \text{ and } y = \mathbf{S}_2^1 z \text{ for some } z\ \}}_{B_1}$$

$$= min \underbrace{\{y \mid y \geq \mathbf{S}_{suc}x \text{ and } \overbrace{y = \mathbf{S}_2^1 z_1 \text{ for some } z_1}^{(3)} \text{ and } y = \mathbf{S}_1^0 z_2 \text{ for some } z_2\ \}}_{A_1}$$

The identity above is true because of $A_1 \subseteq B_1$ and for every $y \in B_1$ there is less $y^{'} \in A_1$, especially $y^{'} \stackrel{\text{def}}{=} -\mathbf{c}_0(-y)$ will be available as we shall see later.
Let $y \in B_1$.
a.1.)
Because of $\mathbf{PAX}(\mathbf{S}3)/(p_6)$: $\mathbf{c}_0(-y) \geq -y \Rightarrow -\mathbf{c}_0(-y) \leq y \Rightarrow y^{'} \leq y$
a.2.)
$y^{'} \geq \mathbf{S}_{suc}x$ because if $y \geq \mathbf{S}_{suc}x$, then

$$y' = -\mathbf{c}_0(-y) \underset{\substack{\uparrow \\ \mathbf{P}AX(\mathbf{S}3)/(p_7)}}{\geq} -\mathbf{c}_0(-\mathbf{S}_{suc}x) \underset{\substack{\uparrow \\ \text{Lemma 9.} \\ \mathbf{P}AX(\mathbf{S}1)(0.2)}}{=} -\mathbf{S}_{suc}(-x) \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}1)(0.2)}}{=} \mathbf{S}_{suc}x$$

a.3.)

$$\mathbf{S}_2^1 y^{'} = \mathbf{S}_2^1(-\mathbf{c}_0(-y)) \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}3)/(p_3),(\mathbf{S}1)(0.2)}}{=} -\mathbf{c}_0(-\mathbf{S}_2^1 y) \underset{\substack{\uparrow \\ \text{(3) and Lemma 10.}}}{=} -\mathbf{c}_0(-y) = y^{'}$$

and

$$\mathbf{S}_1^0 y^{'} = \mathbf{S}_1^0(-\mathbf{c}_0(-y)) \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}1)(0.2),(\mathbf{S}3)/(p_1)}}{=} -\mathbf{c}_0(-y) = y^{'}$$

Thus because of **a**.2.) and **a**.3.) it is true that $y^{'} \in A_1$
b.) It is valid that

$$min \underbrace{\{y \mid y \geq \mathbf{S}_{suc}x \text{ and } y = \mathbf{S}_2^1 z_1 \text{ for some } z_1 \text{ and } \overbrace{y = \mathbf{S}_1^0 z_2 \text{ for some } z_2}^{(4)}\}}_{A_1} =$$

$$min \underbrace{\{\mathbf{S}_{suc}y \mid y \geq x \text{ and } \overbrace{y = \mathbf{S}_1^0 z \text{ for some } z}^{(5)}\}}_{A_2} \text{ because } A_1 = A_2.$$

b.1.) In the identity above first we shall proof the direction $A_1 \subseteq A_2$:
Let $y \in A_1$.

$$y \underset{\substack{\uparrow \\ \text{Lemma 10.}}}{=} \mathbf{S}_1^0 y \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.2)}}{=} \mathbf{S}_{suc}\mathbf{S}_{pred}y$$

$$y \geq \mathbf{S}_{suc}x \Rightarrow \mathbf{S}_{pred}y \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}1)(0.1)}}{\geq} \mathbf{S}_{pred}\mathbf{S}_{suc}x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} x$$

$$y \underset{\substack{\uparrow \\ \text{Lemma 10}}}{=} \mathbf{S}_1^0 y \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.2)}}{=} \mathbf{S}_{suc}\mathbf{S}_{pred}y \text{ and } y \geq \mathbf{S}_{suc}x \Rightarrow \mathbf{S}_{pred}y \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}1)(0.1)}}{\geq} \mathbf{S}_{pred}\mathbf{S}_{suc}x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} x$$

and

$$\mathbf{S}_1^0\mathbf{S}_{pred}y \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} \mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{S}_1^0\mathbf{S}_{pred}y \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.3)}}{=} \mathbf{S}_{pred}\mathbf{S}_2^1\mathbf{S}_{suc}\mathbf{S}_{pred}x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.2)}}{=} \mathbf{S}_{pred}\mathbf{S}_2^1\mathbf{S}_1^0 y \underset{\substack{\uparrow \\ y \in A_1}}{=} \mathbf{S}_{pred}y$$

Thus it is true that $y \in A_2$.
b.2.) After this we shall proof the direction $A_2 \subseteq A_1$:
Let $y \in A_2$. Then $y = \mathbf{S}_{suc}y^{'}$ for some $y^{'} \geq x$, and for which (5) is valid too.
Because of $\mathbf{PAX}(\mathbf{S}1)$ if $y^{'} \geq x \Rightarrow y = \mathbf{S}_{suc}y^{'} \geq \mathbf{S}_{suc}x$.

$$\mathbf{S}_1^0 y = \mathbf{S}_1^0\mathbf{S}_{suc}y^{'} \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.2)}}{=} \mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{S}_{suc}y^{'} \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} \mathbf{S}_{suc}y^{'} = y$$

and

$$\mathbf{S}_2^1 y = \mathbf{S}_2^1\mathbf{S}_{suc}y^{'} \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.3)}}{=} \mathbf{S}_{suc}\mathbf{S}_1^0 y^{'} \underset{\substack{\uparrow \\ \text{(5) and Lemma 10}}}{=} \mathbf{S}_{suc}y^{'} = y$$

verifies that $y \in A_1$.
c.) The following identity is valid:

$$min\,\underbrace{\{\mathbf{S}_{suc}y \mid y \geq x \text{ and } y = \mathbf{S}_1^0 z \text{ for some } z\ \}}_{A_2} = \mathbf{S}_{suc}min\,\underbrace{\{y \mid y \geq x \text{ and } y = \mathbf{S}_1^0 z \text{ for some } z\ \}}_{B_2}$$

Because $y \in B_2 \leftrightarrow \mathbf{S}_{suc}y \in A_2$ thus according to $\mathbf{PAX}(\mathbf{S}1)(0.1)$ the identity above is true and this verifies the lemma.

■

<u>Remark 12.</u>:

Similarly with arbitrary $i < \omega$: $\mathbf{PAX} \models \mathbf{c}_{i+1}\mathbf{S}_{suc}x = \mathbf{S}_{suc}\mathbf{c}_i x$.

<u>Lemma 13.</u>:

$\mathbf{PAX} \models \mathbf{S}_{suc}\mathbf{c}_0 x = \mathbf{c}_0\mathbf{S}_0^1\mathbf{S}_{suc}x$

<u>Proof 13.</u>:

(1)
$$\mathbf{S}_{pred}\mathbf{c}_0\mathbf{S}_0^1\mathbf{S}_{suc}x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}3)/(p_1)}}{=} \mathbf{S}_{pred}\mathbf{S}_1^0\mathbf{c}_0\mathbf{S}_0^1\mathbf{S}_{suc}x \underset{\substack{\uparrow \\ \text{Lemma 8}}}{=} \mathbf{S}_{pred}\mathbf{S}_1^0\mathbf{c}_1\mathbf{S}_1^0\mathbf{S}_{suc} \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.2)}}{=}$$
$$\mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{c}_1\mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{S}_{suc}x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} \mathbf{S}_{pred}\mathbf{c}_1\mathbf{S}_{suc}x \underset{\substack{\uparrow \\ \text{Lemma 10.}}}{=} \mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{c}_0 x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} \mathbf{c}_0 x$$

from (1):

$\mathbf{S}_{pred}\mathbf{c}_0\mathbf{S}_0^1\mathbf{S}_{suc}x = \mathbf{c}_0 x$	applying	$\mathbf{S}_{suc}$
$\mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{c}_0\mathbf{S}_0^1\mathbf{S}_{suc}x = \mathbf{S}_{suc}\mathbf{c}_0 x$	because of	$\mathbf{PAX}(\mathbf{S}2)/(1.2)$ and $\mathbf{PAX}(\mathbf{S}3)/(p1)$
$\mathbf{c}_0\mathbf{S}_0^1\mathbf{S}_{suc}x = \mathbf{S}_{suc}\mathbf{c}_0 x$		

■

We will give the deduction of the system of axioms **AX** from the system of axioms **PAX**

AX(**S**0) exactly **PAX**(**S**0).

AX(**S**1) is provable from **PAX**(**S**1) because of the Lemma 7.

AX(**S**2)(1.1)-(1.3) exactly **PAX**(**S**2)(1.1)-(1.3).

AX(**S**2)(1.4): **PAX** $\models \mathbf{S}^i_j \mathbf{S}_{pred} x = \mathbf{S}_{pred} \mathbf{S}^{i+1}_{j+1} x$ if $i \neq 0$ and $i \neq j$ proof:

Let $i \neq j$ and $i \neq 0$, then because of **PAX**(**S**2)(1.3):

$$\begin{aligned}
\mathbf{S}_{suc}\mathbf{S}^i_j\mathbf{S}_{pred}x &= \mathbf{S}^{i+1}_{j+1}\mathbf{S}_{suc}\mathbf{S}_{pred}x && \mathbf{PAX}(\mathbf{S}2)(1.2)\\
\mathbf{S}_{suc}\mathbf{S}^i_j\mathbf{S}_{pred}x &= \mathbf{S}^{i+1}_{j+1}\mathbf{S}^0_1x && \\
\mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{S}^i_j\mathbf{S}_{pred}x &= \mathbf{S}_{pred}\mathbf{S}^{i+1}_{j+1}\mathbf{S}^0_1x && \mathbf{PAX}(\mathbf{S}2)(1.1)\\
\mathbf{S}^i_j\mathbf{S}_{pred}x &= \mathbf{S}_{pred}\mathbf{S}^{i+1}_{j+1}\mathbf{S}^0_1x && \text{(v)(because } i \neq 0)\\
\mathbf{S}^i_j\mathbf{S}_{pred}x &= \mathbf{S}_{pred}\mathbf{S}^0_1\mathbf{S}^{i+1}_{j+1}x && \mathbf{PAX}(\mathbf{S}2)(1.2)\\
\mathbf{S}^i_j\mathbf{S}_{pred}x &= \mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{S}^{i+1}_{j+1}x && \mathbf{PAX}(\mathbf{S}2)(1.1)\\
\mathbf{S}^i_j\mathbf{S}_{pred}x &= \mathbf{S}_{pred}\mathbf{S}^{i+1}_{j+1}x &&
\end{aligned}$$

AX(**S**2)(1.5): **PAX** $\models \mathbf{S}^0_j \mathbf{S}_{pred} x = \mathbf{S}_{pred} \mathbf{S}^0_{j+1} \mathbf{S}^1_{j+1} x$ if $j \neq 0$ proof:

Let $j \neq 0$. Because of **PAX**(**S**2)(1.1):

$$\begin{aligned}
\mathbf{S}^0_jx &= \mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{S}^0_jx && \mathbf{PAX}(\mathbf{S}2)(1.3)\ \text{(because } j \neq 0)\\
\mathbf{S}^0_jx &= \mathbf{S}_{pred}\mathbf{S}^1_{j+1}\mathbf{S}_{suc}x && \text{Lemma 6 (by Lemma 9.: } 0 \notin \Delta(\mathbf{S}_{suc}x))\\
\mathbf{S}^0_jx &= \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^1_0\mathbf{S}_{suc}x && \mathbf{PAX}(\mathbf{S}2)(1.2)\\
\mathbf{S}^0_j\mathbf{S}_{pred}x &= \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^1_0\mathbf{S}^0_1x && \mathbf{PAX}(\mathbf{S}3)/(p_4)\\
\mathbf{S}^0_j\mathbf{S}_{pred}x &= \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^1_0\mathbf{S}^0_0x && \mathbf{PAX}(\mathbf{S}3)/(p_5)\\
\mathbf{S}^0_j\mathbf{S}_{pred}x &= \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^1_0x && \mathbf{PAX}(\mathbf{S}3)/(p_4)\\
\mathbf{S}^0_j\mathbf{S}_{pred}x &= \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^1_{j+1}x &&
\end{aligned}$$

We will not proof Jónsson schemes in their original sequence from **PAX**, because some of the schemes will be used later to prove the other ones.

AX(**S**2)(1.6.6): **PAX** $\models \mathbf{S}^i_j \mathbf{S}^k_l x = \mathbf{S}^k_l \mathbf{S}^i_j x$ if $(k \neq j, i \neq k, l)$ proof:

Similar as (v).

AX(**S**2)(1.6.7): **PAX** $\models \mathbf{S}^i_j \mathbf{S}^i_k x = \mathbf{S}^i_k x$ if $(i \neq k)$ proof:

Similar as (iv).

$\mathbf{AX}(\mathbf{S}2)(1.6.5)$: $\mathbf{PAX} \models \mathbf{P}^i_j\mathbf{S}^j_i x = \mathbf{S}^i_j x$ if $(i \neq j)$ proof:

Let $i \neq j$. Then:

$$\mathbf{P}^i_j\mathbf{S}^j_i x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}\mathbf{S}^j_i x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^{j+1}_{i+1}\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

$\mathbf{P}^i_j$ definition — $\mathbf{PAX}(\mathbf{S}2)(1.3.)$ (because $i \neq j$) — $\mathbf{PAX}$ $(\mathbf{S}3)/(p_4)$

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^{j+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{j+1}_0\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

(v) (because $i \neq j$) — $\mathbf{PAX}(\mathbf{S}3)/(p_4)$

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{j+1}_0\mathbf{S}^{i+1}_{j+1}\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{j+1}_0\mathbf{S}_{suc}\mathbf{S}^i_j x \underset{\uparrow}{=}$$

$\mathbf{PAX}(\mathbf{S}2)(1.3)$ (because $i \neq j$) — (iv)

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_0\mathbf{S}_{suc}\mathbf{S}^i_j x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^{j+1}_{j+1}\mathbf{S}_{suc}\mathbf{S}^i_j x \underset{\uparrow}{=}$$

Lemma 6. (because $0 \notin \Delta(\mathbf{S}_{suc}\mathbf{S}^i_j x)$:Lemma 9.) — $\mathbf{PAX}(\mathbf{S}3)/p_5$

$$\mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{S}^i_j x \underset{\uparrow}{=} \mathbf{S}^i_j x$$

$\mathbf{PAX}(\mathbf{S}2)(1.1)$

$\blacksquare$

$\mathbf{AX}(\mathbf{S}2)(1.6.4)$: $\mathbf{PAX} \models \mathbf{P}^i_j\mathbf{S}^k_i x = \mathbf{S}^k_j\mathbf{P}^i_j x$ if $(j \neq k, i \neq k)$ proof:
a. Let $k \neq 0$ and $k \neq i, j$

$$\mathbf{P}^i_j\mathbf{S}^k_i x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}\mathbf{S}^k_i x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^{k+1}_{i+1}\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

$\mathbf{P}^i_j$definition — $\mathbf{PAX}(\mathbf{S}2)(1.3.)$ (because $i \neq k$) — $\mathbf{PAX}$ $(\mathbf{S}3)/(p_4)$

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^{k+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{k+1}_0\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

(v) (because $i \neq k$) — (v) (because $j, i \neq k$)

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{k+1}_0\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{k+1}_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

$\mathbf{PAX}(\mathbf{S}3)/(p_4)$ — (v) (because $k \neq j$)

$$\mathbf{S}_{pred}\mathbf{S}^{k+1}_{j+1}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}^k_j\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}^k_j\mathbf{P}^i_j$$

$\mathbf{AX}(\mathbf{S}2)(1.4.)$ (because $k \neq j, k \neq 0$) — $\mathbf{P}^i_j$ definition

$\blacksquare$

b. Let $k = 0$ and $k \neq i, j$

Because $k \neq j, i$, then $j, i \neq 0$

$$\mathbf{P}^i_j\mathbf{S}^0_i x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}\mathbf{S}^0_i x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^1_{i+1}\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

$\mathbf{P}^i_j$definition — **PAX**(**S**2)(1.3.) (because $i \neq 0$) — **PAX** (**S**3)/(p_4)

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^1_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^1_0\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

(v) (because $i \neq 0$) — (v) (because $j, i \neq 0$)

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^1_0\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^1_{j+1}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

(iv), **PAX**(**S**3)/(p_4), (v) (because $j \neq 0$) — **AX**(**S**2)(1.5) (because $0 \neq j$)

$$\mathbf{S}^0_j\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}^0_j\mathbf{P}^i_j$$

$\mathbf{P}^i_j$ definition

∎

AX(**S**2)(1.6.2): **PAX** $\models \mathbf{P}^i_j\mathbf{P}^j_i x = x$ proof:

a. Let $i \neq j$

$$\mathbf{P}^i_j\mathbf{P}^j_i x \underset{\uparrow}{=}$$

$\mathbf{P}^i_j$definition

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{j+1}\mathbf{S}^{j+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

PAX(**S**2)(1.2)

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^0_1\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{j+1}\mathbf{S}^{j+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

(iv)

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{j+1}\mathbf{S}^{j+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

PAX(**S**3)/(p_4)

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^0_0\mathbf{S}^{i+1}_{j+1}\mathbf{S}^{j+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

PAX(**S**3)/(p_5), (iv) (because $i \neq j$)

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_{j+1}\mathbf{S}^{j+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{j+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$

PAX(**S**3)/(p_4), (p_5) — (iv)

$$\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^{j+1}_{j+1}\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}_{suc}x \underset{\uparrow}{=} x$$

Lemma 6. (because $0 \notin \Delta(\mathbf{S}_{suc}x)$:Lemma 9.) — **PAX**(**S**3)/(p_5) — **PAX** (**S**2)(1.1)

b. Let $i = j$

$$\mathbf{P}^i_i\mathbf{P}^i_i x \underset{\uparrow}{=}$$
$\mathbf{P}^i_i$definition

$$\mathbf{S}_{pred}\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$
$\mathbf{PAX}(\mathbf{S}2)(1.2)$

$$\mathbf{S}_{pred}\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^0_1\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$
(iv)

$$\mathbf{S}_{pred}\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$
Lemma 6.
because $i+1 \notin \Delta(\mathbf{S}^{i+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x)$

$$\mathbf{S}_{pred}\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_{i+1}\mathbf{S}^0_0\mathbf{S}^{i+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=}$$
$\mathbf{PAX}(\mathbf{S}3)/(p_5)$

$$\mathbf{S}_{pred}\mathbf{S}^0_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^{i+1}_{i+1}\mathbf{S}_{suc}x \quad \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}_{suc}x \quad \underset{\uparrow}{=} x$$
Lemma 6. (because $0 \notin \Delta(\mathbf{S}_{suc}x)$:Lemma 9.) $\qquad$ $\mathbf{PAX}(\mathbf{S}3)/(p_5)$ $\qquad$ $\mathbf{PAX}$ $(\mathbf{S}2)(1.1)$

<u>Lemma 14.</u>:[I.Sain,R.J.Thompson 88] (p.553.)

$\mathbf{PAX} \models \mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^j_k\mathbf{S}^k_j\mathbf{S}^j_i\mathbf{S}^i_k x = \mathbf{S}^k_i x$ if $i \neq j, k$

<u>Proof 14.</u>:

Let $i \neq j, k$. Then:

$$\mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^j_k\mathbf{S}^k_j\mathbf{S}^j_i\mathbf{S}^i_k x \underset{\uparrow}{=} \mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^j_k\mathbf{S}^j_i\mathbf{S}^i_k x \quad \underset{\uparrow}{=} \mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^j_i\mathbf{S}^i_k x \quad \underset{\uparrow}{=}$$
$\mathbf{PAX}(\mathbf{S}3)/(p_4),(p_5)$ $\qquad$ (iv) (because $i \neq j$) $\qquad$ $\mathbf{PAX}(\mathbf{S}3)/(p_4),(p_5)$

$$\mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^i_k x \underset{\uparrow}{=} \mathbf{S}^k_i\mathbf{S}^i_k x \quad \underset{\uparrow}{=} \mathbf{S}^k_i x$$
(iv) (because $i \neq k$) $\qquad$ $\mathbf{PAX}(\mathbf{S}3)/(p_4),(p_5)$

■

Lemma 15.:

$\mathbf{PAX} \models \mathbf{c}_i\mathbf{c}_i x = \mathbf{c}_i x$ if $i \neq j, k$

Proof 15.:

Let $j \neq i$. Then:

$$\mathbf{c}_i\mathbf{c}_i x \underset{\uparrow}{=} \mathbf{c}_i\mathbf{S}^i_j\mathbf{c}_i x \qquad \underset{\uparrow}{=} \mathbf{S}^i_j\mathbf{c}_i x \qquad \underset{\uparrow}{=} \mathbf{c}_i x$$

$\mathbf{PAX}(\mathbf{S}3)/(p_1)$ $\mathbf{PAX}(\mathbf{S}3)/(p_2)$ (because $i \neq j$) $\mathbf{PAX}(\mathbf{S}3)/(p_1)$

∎

Lemma 16.:

$\mathbf{PAX} \models \mathbf{S}^k_i\mathbf{c}_i x = \mathbf{P}^i_k\mathbf{c}_i x$ if $k \neq i$

Proof 16.:

a. Let $k \neq 0$ and $k \neq i$

$$\mathbf{S}^k_i\mathbf{c}_i x \underset{\uparrow}{=} \mathbf{S}^k_i\mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{c}_i x \qquad \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^{k+1}_{i+1}\mathbf{c}_0\mathbf{S}_{suc}\mathbf{c}_i x \qquad \underset{\uparrow}{=}$$

$\mathbf{PAX}(\mathbf{S}2)(1.1)$ Lemma 9. $\mathbf{PAX}$
$\mathbf{AX}(\mathbf{S}2)(1.4)$(because $k \neq 0, k \neq i$)$(\mathbf{S}3)/(p_3)$

$$\mathbf{S}_{pred}\mathbf{c}_0\mathbf{S}^{k+1}_{i+1}\mathbf{S}_{suc}\mathbf{c}_i x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{k+1}\mathbf{c}_0\mathbf{S}^{k+1}_{i+1}\mathbf{S}_{suc}\mathbf{c}_i x \qquad \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{k+1}\mathbf{S}^{k+1}_{i+1}\mathbf{S}_{suc}\mathbf{c}_i x \qquad \underset{\uparrow}{=}$$

$\mathbf{PAX}(\mathbf{S}3)/(p_1)$ Lemma 9. $\mathbf{PAX}(\mathbf{S}3)/(p_3)$ Remark 11.

$$\mathbf{S}_{pred}\mathbf{S}^0_{k+1}\mathbf{S}^{k+1}_{i+1}\mathbf{c}_{i+1}\mathbf{S}_{suc} x \underset{\uparrow}{=} \mathbf{S}_{pred}\mathbf{S}^0_{k+1}\mathbf{S}^{k+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{c}_{i+1}\mathbf{S}_{suc} x \qquad \underset{\uparrow}{=}$$

$\mathbf{PAX}(\mathbf{S}3)/(p_1)$ Remark 12.

$$\mathbf{S}_{pred}\mathbf{S}^0_{k+1}\mathbf{S}^{k+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}\mathbf{c}_i x \underset{\uparrow}{=} \mathbf{P}^i_k\mathbf{c}_i x$$

$\mathbf{P}^i_k$definition

∎

b. Let $k = 0$ and $k \neq i$

Because $k \neq i$, then $i \neq 0$

$$\mathbf{S}_i^0\mathbf{c}_i x \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} \mathbf{S}_i^0\mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{c}_i x \underset{\substack{\uparrow\\ \text{Lemma 9.}\\ \mathbf{AX}(\mathbf{S}2)(1.5)(\text{because } 0\neq i)}}{=} \mathbf{S}_{pred}\mathbf{S}^0_{i+1}\mathbf{S}^1_{i+1}\mathbf{c}_0\mathbf{S}_{suc}\mathbf{c}_i x \underset{\substack{\uparrow\\ (v)\\ (\text{because } i\neq 0)}}{=}$$

$$\mathbf{S}_{pred}\mathbf{S}^1_{i+1}\mathbf{S}^0_{i+1}\mathbf{c}_0\mathbf{S}_{suc}\mathbf{c}_i x \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}3)/(p_1)}}{=} \mathbf{S}_{pred}\mathbf{S}^1_{i+1}\mathbf{c}_0\mathbf{S}_{suc}\mathbf{c}_i x \underset{\substack{\uparrow\\ \text{Lemma 9.}, 15.\\ \text{Remark 12.}}}{=} \mathbf{S}_{pred}\mathbf{S}^1_{i+1}\mathbf{c}_{i+1}\mathbf{S}_{suc}\mathbf{c}_i x \underset{\substack{\uparrow\\ \mathbf{PAX}\\ (\mathbf{S}3)/(p_1)}}{=}$$

$$\mathbf{S}_{pred}\mathbf{S}^1_{i+1}\mathbf{S}^{i+1}_0\mathbf{c}_{i+1}\mathbf{S}_{suc}\mathbf{c}_i x \underset{\substack{\uparrow\\ \text{Remark 12,15}}}{=} \mathbf{S}_{pred}\mathbf{S}^1_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}\mathbf{c}_i x \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=}$$

$$\mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{S}^1_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}\mathbf{c}_i x \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}2)(1.2)}}{=} \mathbf{S}_{pred}\mathbf{S}^0_1\mathbf{S}^1_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc}\mathbf{c}_i x \underset{\substack{\uparrow\\ \mathbf{P}^i_0\text{definition}}}{=} \mathbf{P}^i_0\mathbf{c}_i x$$

$\blacksquare$

<u>Lemma 17.</u>:[I.Sain,R.J.Thompson 88] (p.553.) :

$\mathbf{PAX} \models \mathbf{S}^k_j\mathbf{P}^i_j x = \mathbf{S}^k_j\mathbf{S}^j_i\mathbf{S}^i_k x$ if i, j, k are distinct.

<u>Proof 17.</u>:

Let i, j, k be distinct. Then:

$$\mathbf{S}^k_j\mathbf{P}^i_j(x\cdot -\mathbf{S}^i_k x) \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}3)/(p_6)}}{\leq} \mathbf{S}^k_j\mathbf{P}^i_j\mathbf{c}_j(x\cdot -\mathbf{S}^i_k x) \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}3)/(p_1)}}{=}$$

$$\mathbf{S}^k_j\mathbf{P}^i_j\mathbf{S}^j_i\mathbf{c}_j(x\cdot -\mathbf{S}^i_k x) \underset{\substack{\uparrow\\ \mathbf{AX}(1.6.5.)\\ (\text{because } i\neq j)}}{=} \mathbf{S}^k_j\mathbf{S}^i_j\mathbf{c}_j(x\cdot -\mathbf{S}^i_k x) \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}3)/(p_4)}}{=} \mathbf{S}^k_j\mathbf{S}^i_k\mathbf{c}_j(x\cdot -\mathbf{S}^i_k x) \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}3)/(p_3)\\ (\text{because } j\neq i,k)}}{=}$$

$$\mathbf{S}^k_j\mathbf{c}_j\mathbf{S}^i_k(x\cdot -\mathbf{S}^i_k x) \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}1)}}{=} \mathbf{S}^k_j\mathbf{c}_j(\mathbf{S}^i_k x\cdot \mathbf{S}^i_k(-\mathbf{S}^i_k x)) \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}1)}}{=} \mathbf{S}^k_j\mathbf{c}_j(\mathbf{S}^i_k x\cdot -\mathbf{S}^i_k\mathbf{S}^i_k x) \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}3)/(p_1),(p_2)\\ (\text{because } i\neq k)}}{=}$$

$$\mathbf{S}^k_j\mathbf{c}_j(\mathbf{S}^i_k x\cdot -\mathbf{S}^i_k x) \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}0)}}{=} \mathbf{S}^k_j\mathbf{c}_j(0) \underset{\substack{\uparrow\\ (i)}}{=} \mathbf{S}^k_j(0) \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}1)}}{=} 0$$

Thus $\mathbf{S}^k_j\mathbf{P}^i_j x \cdot -\mathbf{S}^k_j\mathbf{P}^i_j\mathbf{S}^i_k x = 0$ because of $\mathbf{PAX}(\mathbf{S}1)$.

Then:

$$(1)\ \mathbf{S}^k_j\mathbf{P}^i_j x \underset{\uparrow}{\leq} \mathbf{S}^k_j\mathbf{P}^i_j\mathbf{S}^i_k x \underset{\uparrow}{=} \mathbf{S}^k_j\mathbf{P}^i_j\mathbf{c}_i\mathbf{S}^i_k x \underset{\uparrow}{=} \mathbf{S}^k_j\mathbf{S}^j_i\mathbf{c}_i\mathbf{S}^i_k x \underset{\uparrow}{=} \mathbf{S}^k_j\mathbf{S}^j_i\mathbf{S}^i_k x$$

$\mathbf{PAX}(\mathbf{S}0)$; $\mathbf{PAX}(\mathbf{S}3)/(p_2)$ (because $i \neq k$); Lemma 16. (because $i \neq j$); $\mathbf{PAX}(\mathbf{S}3)/(p_2)$ (because $i \neq k$)

$$(2)\ \mathbf{S}^k_i x \underset{\uparrow}{=} \mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^j_k\mathbf{S}^k_j\mathbf{S}^j_i\mathbf{S}^i_k x \underset{\uparrow}{\geq} \mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^j_k\mathbf{S}^k_j\mathbf{P}^i_j x \underset{\uparrow}{=} \mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^j_k\mathbf{P}^i_j x$$

Lemma 14. (because $i \neq k, j$); (1) (because $i \neq k, j$) and $\mathbf{PAX}(\mathbf{S}1)(0.1)$; $\mathbf{PAX}(\mathbf{S}3)/(p_4), (p_5)$

Thus:

$$\mathbf{S}^k_i\mathbf{P}^j_i x \underset{\uparrow}{\geq} \mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^j_k\mathbf{P}^i_j\mathbf{P}^j_i x \underset{\uparrow}{=} \mathbf{S}^k_i\mathbf{S}^i_j\mathbf{S}^j_k x$$

(2) – ből; $\mathbf{AX}(1.6.2.)$

Interchanging the role of i and j we will get: (3) $\mathbf{S}^k_j\mathbf{P}^i_j x \geq \mathbf{S}^k_j\mathbf{S}^j_i\mathbf{S}^i_k$

The statement follows from (1) and (3).

■

<u>Lemma 18.</u>:

$\mathbf{PAX} \models \mathbf{S}_{suc}\mathbf{P}^i_j x = \mathbf{P}^{i+1}_{j+1}\mathbf{S}_{suc} x$ if $i \neq j$

<u>Proof 18.</u>:

Let $i \neq j$

$$\mathbf{S}_{suc}\mathbf{P}^i_j x \underset{\uparrow}{=} \mathbf{S}_{suc}\mathbf{S}_{pred}\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc} x \underset{\uparrow}{=} \mathbf{S}^0_1\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc} x \underset{\uparrow}{=}$$

$\mathbf{P}^i_j$definition; $\mathbf{PAX}(\mathbf{S}2)/(1.2)$; (iv)

$$\mathbf{S}^0_{j+1}\mathbf{S}^{j+1}_{i+1}\mathbf{S}^{i+1}_0\mathbf{S}_{suc} x \underset{\uparrow}{=} \mathbf{S}^0_{j+1}\mathbf{P}^{i+1}_{j+1}\mathbf{S}_{suc} x \underset{\uparrow}{=} \mathbf{P}^{i+1}_{j+1}\mathbf{S}^0_{i+1}\mathbf{S}_{suc} x \underset{\uparrow}{=}$$

Lemma 17. (because $i \neq j$); $\mathbf{AX}(1.6.4)$; Lemma 9.

$$\mathbf{P}^{i+1}_{j+1}\mathbf{S}^0_{i+1}\mathbf{c}_0\mathbf{S}_{suc} x \underset{\uparrow}{=} \mathbf{P}^{i+1}_{j+1}\mathbf{c}_0\mathbf{S}_{suc} x \underset{\uparrow}{=} \mathbf{P}^{i+1}_{j+1}\mathbf{S}_{suc} x$$

$\mathbf{PAX}(\mathbf{S}3)/(p_1)$; Lemma 9.

■

<u>Lemma 19.</u>:

$\mathbf{PAX} \models \mathbf{S}^k_i\mathbf{c}_i\mathbf{S}^i_0\mathbf{c}_0 x = \mathbf{P}^i_k\mathbf{P}^0_i\mathbf{c}_0 x$ if $i \neq 0, k \neq i$

Proof 19.:

Let $i \neq 0, k \neq i$. Then:

$$\mathbf{S}_i^k\mathbf{c}_i\mathbf{S}_0^i\mathbf{c}_0x \underset{\substack{\uparrow\\ \text{Lemma 16.}\\ (\text{because } k\neq i)}}{=} \mathbf{P}_k^i\mathbf{c}_i\mathbf{S}_0^i\mathbf{c}_0x \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}3)/(p_2)\\ (\text{because } 0\neq i)}}{=} \mathbf{P}_k^i\mathbf{S}_0^i\mathbf{c}_0x \underset{\substack{\uparrow\\ \text{Lemma 16.}\\ (\text{because } i\neq 0)}}{=} \mathbf{P}_k^i\mathbf{P}_i^0\mathbf{c}_0x$$

■

$\mathbf{AX}(\mathbf{S}2)$(1.6.3): $\mathbf{PAX} \models \mathbf{P}_j^i\mathbf{P}_k^i x = \mathbf{P}_k^j\mathbf{P}_j^i x$ if $j \neq k, i \neq k, i \neq j$ proof:

We will prove it in two steps. In the step a. we will prove the statement for "x" where $\mathbf{c}_0x = x$. In the step b. the case $\mathbf{c}_0x \neq x$ will be attributed to the application of step a.

Let i, j, k be distinct.

a. Let $\mathbf{c}_0x = x$

Starting from $\mathbf{PAX}(\mathbf{S}3)/(p_4)$ we can deduce:

$$\begin{array}{lrll}
 & \mathbf{S}_k^j\mathbf{S}_j^i x &= \mathbf{S}_k^j\mathbf{S}_k^i x & \\
 & \mathbf{P}_k^j\mathbf{S}_j^k\mathbf{P}_j^i\mathbf{S}_i^j x &= \mathbf{S}_k^j\mathbf{P}_k^i\mathbf{S}_i^k x & \mathbf{AX}(\mathbf{S}3)(1.6.5)\ (\text{because } i\neq j, j\neq k, i\neq k)\\
 & \mathbf{P}_k^j\mathbf{P}_j^i\mathbf{S}_i^k\mathbf{S}_i^j x &= \mathbf{P}_k^i\mathbf{S}_i^j\mathbf{S}_i^k x & \mathbf{AX}(\mathbf{S}3)(1.6.4)\ (\text{because } i\neq j, j\neq k, i\neq k)\\
(3) & \mathbf{P}_k^j\mathbf{P}_j^i\mathbf{S}_i^j\mathbf{S}_i^k x &= \mathbf{P}_k^i\mathbf{S}_i^j\mathbf{S}_i^k x & (\text{v})\ (\text{because } i\neq j, j\neq k, i\neq k)
\end{array}$$

Starting from (iv) we conclude the following, because $i \neq k$:

$$\begin{array}{lrll}
 & \mathbf{S}_j^i\mathbf{S}_k^i x &= \mathbf{S}_k^i x & \\
 & \mathbf{P}_j^i\mathbf{S}_i^j\mathbf{P}_k^i\mathbf{S}_i^k x &= \mathbf{P}_k^i\mathbf{S}_i^k x & \mathbf{A}X(\mathbf{S}3)(1.6.5)\ (\text{because } i\neq j, i\neq k)\\
 & \mathbf{P}_j^i\mathbf{S}_i^j\mathbf{P}_k^i\mathbf{S}_i^k\mathbf{S}_i^j x &= \mathbf{P}_k^i\mathbf{S}_i^k\mathbf{S}_i^j x & \text{replacing x by } \mathbf{S}_i^j(x)\\
(4) & \mathbf{P}_j^i\mathbf{S}_i^j\mathbf{P}_k^i\mathbf{S}_i^j\mathbf{S}_i^k x &= \mathbf{P}_k^i\mathbf{S}_i^j\mathbf{S}_i^k x & (\text{v})\ (\text{because } i\neq j, j\neq k, i\neq k)
\end{array}$$

According to (3) and (4) starting from the following identity:

$$\begin{array}{lrll}
 & \mathbf{P}_k^j\mathbf{P}_j^i\mathbf{S}_i^j\mathbf{S}_i^k x &= \mathbf{P}_j^i\mathbf{S}_i^j\mathbf{P}_k^i\mathbf{S}_i^j\mathbf{S}_i^k x & \\
 & \mathbf{P}_k^j\mathbf{P}_j^i\mathbf{S}_i^j\mathbf{S}_i^k x &= \mathbf{P}_j^i\mathbf{S}_i^j\mathbf{S}_k^j\mathbf{P}_k^i\mathbf{S}_i^k x & \mathbf{AX}(\mathbf{S}3)(1.6.4)\ (\text{because } i\neq j, i\neq k)\\
 & \mathbf{P}_k^j\mathbf{P}_j^i\mathbf{S}_i^j\mathbf{S}_i^k x &= \mathbf{P}_j^i\mathbf{S}_k^j\mathbf{P}_k^i\mathbf{S}_i^k x & (\text{iv})\ (\text{because } j\neq k)\\
(5) & \mathbf{P}_k^j\mathbf{P}_j^i\mathbf{S}_i^j\mathbf{S}_i^k x &= \mathbf{P}_j^i\mathbf{P}_k^i\mathbf{S}_i^j\mathbf{S}_i^k x & \mathbf{AX}(\mathbf{S}3)(1.6.4)\ (\text{because } i\neq j, j\neq k)\\
 & \mathbf{P}_k^j\mathbf{P}_j^i\mathbf{S}_i^j\mathbf{S}_i^k\mathbf{c}_i\mathbf{S}_0^i\mathbf{c}_0 x &= \mathbf{P}_j^i\mathbf{P}_k^i\mathbf{S}_i^j\mathbf{S}_i^k\mathbf{c}_i\mathbf{S}_0^i\mathbf{c}_0 x & \text{replacing } x \text{ by } \mathbf{c}_i\mathbf{S}_0^i\mathbf{c}_0 x
\end{array}$$

$\mathbf{P}_k^j\mathbf{P}_j^i\mathbf{S}_i^j\mathbf{P}_k^i\mathbf{P}_i^0\mathbf{c}_0x = \mathbf{P}_j^i\mathbf{P}_k^i\mathbf{S}_i^j\mathbf{P}_k^i\mathbf{P}_i^0\mathbf{c}_0x$ because of Lemma 19.

(6) $\mathbf{P}^j_k\mathbf{P}^i_j\mathbf{S}^j_i\mathbf{P}^i_k\mathbf{P}^0_i x = \mathbf{P}^i_j\mathbf{P}^i_k\mathbf{S}^j_i\mathbf{P}^i_k\mathbf{P}^0_i x$ because $\mathbf{c}_0 x = x$

We can transform the identity (6) with help of $\mathbf{AX}(\mathbf{S}2)(1.6.2)$ for the following form, when the "x" is replaced by $\mathbf{P}^i_0 x$ and $\mathbf{P}^k_i x$:

(7) $\mathbf{P}^j_k\mathbf{P}^i_j\mathbf{S}^j_i x = \mathbf{P}^i_j\mathbf{P}^i_k\mathbf{S}^j_i x$

The method eliminating the $\mathbf{S}^k_i$-s from (5) is called "Method of the elimination of replacements"

Let us repeat the "Method of the elemination of replacements" in the identity (7) for the $\mathbf{S}^j_i$. We shall get the following identity.

(8) $\mathbf{P}^j_k\mathbf{P}^i_j x = \mathbf{P}^i_j\mathbf{P}^i_k x$, thus we have proved (1.6.3), if $\mathbf{c}_0 x = x$.

b. Let $\mathbf{c}_0 x \neq x$

$$\mathbf{P}^j_k\mathbf{P}^i_j x \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} \mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{P}^j_k\mathbf{P}^i_j x \underset{\substack{\uparrow\\ \text{Lemma 18.}\\ (\text{because } j \neq k, i \neq j)}}{=} \mathbf{S}_{pred}\mathbf{P}^{j+1}_{k+1}\mathbf{P}^{i+1}_{j+1}\mathbf{S}_{suc} x \underset{\substack{\uparrow\\ \text{step a. of the deduction}\\ (\mathbf{c}_0\mathbf{S}_{sux}x = \mathbf{S}_{suc}x\text{:Lemma 9.)}}}{=}$$

$$\mathbf{S}_{pred}\mathbf{P}^{i+1}_{j+1}\mathbf{P}^{i+1}_{k+1}\mathbf{S}_{suc} x \underset{\substack{\uparrow\\ \text{Lemma 18.}\\ (\text{because } i \neq k, i \neq j}}{=} \mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{P}^i_j\mathbf{P}^i_k x \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} \mathbf{P}^i_j\mathbf{P}^i_k x$$

Thus we have completed the proof of the identity (1.6.3).

$\mathbf{AX}(\mathbf{S}2)(1.6.1)$: $\mathbf{PAX} \models \mathbf{P}^i_j x = \mathbf{P}^j_i x$ proof:

If $i = j$ then $P^i_i x = P^i_i x$ is obvious. Thus we can assume that $i \neq j$.

Let $i \neq j$.

a. Let $\mathbf{c}_0 x = x$

Let $k \neq i, j$. Then:

$$(1) \qquad \mathbf{S}^j_k\mathbf{S}^i_j x \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}3)/(p_4)}}{=} \mathbf{S}^j_k\mathbf{S}^i_k x \underset{\substack{\uparrow\\ (v)\\ (\text{because } j \neq i, k \text{ and } k \neq i)}}{=} \mathbf{S}^i_k\mathbf{S}^j_k x \underset{\substack{\uparrow\\ \mathbf{PAX}(\mathbf{S}3)/(p_4)}}{=} \mathbf{S}^i_k\mathbf{S}^j_i x$$

Let us start with the identity (1):

$$
\begin{aligned}
\mathbf{S}^j_k\mathbf{S}^i_j x &= \mathbf{S}^i_k\mathbf{S}^j_i x && \\
\mathbf{P}^j_k\mathbf{S}^k_j\mathbf{P}^i_j\mathbf{S}^j_i x &= \mathbf{P}^i_k\mathbf{S}^k_i\mathbf{P}^j_i\mathbf{S}^i_j x && \mathbf{AX}(\mathbf{S}2)(1.6.5)\ \text{(because } j \neq i, k \text{ and } k \neq i\text{)} \\
\mathbf{P}^j_k\mathbf{P}^i_j\mathbf{S}^k_i\mathbf{S}^j_i x &= \mathbf{P}^i_k\mathbf{P}^j_i\mathbf{S}^k_j\mathbf{S}^i_j x && \mathbf{AX}(\mathbf{S}2)(1.6.4)\ \text{(because } j \neq k \text{ and } k \neq i\text{)} \\
\mathbf{P}^i_j\mathbf{P}^j_k\mathbf{S}^k_i\mathbf{S}^j_i x &= \mathbf{P}^j_i\mathbf{P}^i_k\mathbf{S}^k_j\mathbf{S}^i_j x && \mathbf{AX}(\mathbf{S}2)(1.6.3)\ \text{(because } j \neq i, k \text{ and } k \neq i\text{)} \\
\mathbf{P}^i_j\mathbf{S}^i_k\mathbf{S}^j_i x &= \mathbf{P}^j_i\mathbf{S}^j_k\mathbf{S}^i_j x && \mathbf{AX}(\mathbf{S}2)(1.6.5)\ \text{(because } j \neq k \text{ and } k \neq i\text{)} \\
\mathbf{P}^i_j\mathbf{S}^i_k\mathbf{S}^j_k x &= \mathbf{P}^j_i\mathbf{S}^j_k\mathbf{S}^i_k x && \mathbf{PAX}(\mathbf{S}3)/(p_4) \\
\mathbf{P}^i_j\mathbf{S}^i_k\mathbf{S}^j_k x &= \mathbf{P}^j_i\mathbf{S}^i_k\mathbf{S}^j_k x && \text{(v) (because } j \neq i, k \text{ and } k \neq i\text{)}
\end{aligned}
$$

In the last identity we can apply the "Method of the elimination of replacements" to the $\mathbf{S}^j_k$ and $\mathbf{S}^i_k$. We will get the following identity:

$\mathbf{P}^i_j x = \mathbf{P}^j_i x$ (if $j \neq i$ and $c_0 x = x$) b. Let $\mathbf{c}_0 x \neq x$

$$
\mathbf{P}^j_i x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} \mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{P}^j_i x \underset{\substack{\uparrow \\ \text{Lemma 16.} \\ \text{(because } i \neq j\text{)}}}{=} \mathbf{S}_{pred}\mathbf{P}^{j+1}_{i+1}\mathbf{S}_{suc} x \underset{\substack{\uparrow \\ \text{step a. in the deduction} \\ (\mathbf{c}_0\mathbf{S}_{sux} x = \mathbf{S}_{suc} x\text{:Lemma 9.})}}{=}
$$

$$
\mathbf{S}_{pred}\mathbf{P}^{i+1}_{j+1}\mathbf{S}_{suc} x \underset{\substack{\uparrow \\ \text{Lemma 16.} \\ \text{(because } i \neq j\text{)}}}{=} \mathbf{S}_{pred}\mathbf{S}_{suc}\mathbf{P}^i_j x \underset{\substack{\uparrow \\ \mathbf{PAX}(\mathbf{S}2)(1.1)}}{=} \mathbf{P}^i_j x
$$

Thus we have completed the proof of the identity (1.6.1).

$\mathbf{AX}(\mathbf{S}3)/(q_1)$ exactly $\mathbf{PAX}(\mathbf{S}1)(0.2.)$ if we shall choose as S the $\mathbf{S}^i_j$.

$\mathbf{AX}(\mathbf{S}3)/(q_2)$ exactly $\mathbf{PAX}(\mathbf{S}1)(0.1.)$ if we shall choose as S the $\mathbf{S}^i_j$.

$\mathbf{AX}(\mathbf{S}3)/(q_3)$ exactly $\mathbf{PAX}(\mathbf{S}3)/(p_5)$.

$\mathbf{AX}(\mathbf{S}3)/(q_4)$ exactly $\mathbf{PAX}(\mathbf{S}3)/(p_4)$.

$\mathbf{AX}(\mathbf{S}3)/(q_5)$ exactly $\mathbf{PAX}(\mathbf{S}3)/(p_4)$.

$\mathbf{AX}(\mathbf{S}3)/(q_6)$ exactly $\mathbf{PAX}(\mathbf{S}3)/(p_5)$.

$\mathbf{AX}(\mathbf{S}3)/(q_7)$ exactly $\mathbf{PAX}(\mathbf{S}3)/(p_1)$.

$\mathbf{AX}(\mathbf{S}3)/(q_8)$ exactly $\mathbf{PAX}(\mathbf{S}3)/(p_2)$.

$\mathbf{AX}(\mathbf{S}3)/(q_9)$ exactly $\mathbf{PAX}(\mathbf{S}3)/(p_3)$.

Thus we have completed the proof of the Lemma 5. as well as the Theorem 4.

$\blacksquare$

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