On Kripke completeness of some modal predicate logics

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In this note we present some completeness results for modal predicate logics in the standard Kripke semantics. The proof is based on the technique developed by S.Ghilardi, G.Corsi and D. Skvorstov, but now we arrange it in a game-theoretic style [1].

1 Modal logics and Kripke frames

Let us recall some basic definitions and notation; most of them are the same as in the book [3].

Atomic formulas are constructed from predicate letters P_k^n (countably many for each arity $n \geq 0$) and a countable set of individual variables Var, without constants and function letters. *Modal (predicate) formulas* are obtained from atomic formulas by applying classical propositional connectives, quantifiers and the modal operator $\Box$.

In *modal propositional formulas* only the proposition letters (P_k^0) are used as atoms.

A *modal propositional logic* is a set of modal propositional formulas containing classical propositional tautologies, the axiom of **K** ($\Box(p \supset q) \supset (\Box p \supset \Box q)$, where p, q are proposition letters) and closed under the basic inference rules: Modus Ponens, $\Box$-introduction, and (propositional) Substitution.

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As usual $\mathbf{K}$ denotes the minimal propositional modal logic, $\mathbf{\Lambda} + A$ is the smallest logic containing a logic $\mathbf{\Lambda}$ and a formula A, and $\mathbf{K4} := \mathbf{K} + \Box p \supset \Box\Box p$.

Recall that Kripke semantics for propositional modal logics is given by *(propositional) Kripke frames* of the form (W, R), where $W \neq \varnothing$, $R \subseteq W \times W$. The set of all propositional formulas valid in a frame F (the *modal logic of* F) is denoted by $\mathbf{ML}(F)$. The class of all frames validating a propositional logic $\mathbf{\Lambda}$ ($\mathbf{\Lambda}$-*frames*) is denoted by $\mathbf{V}(\mathbf{\Lambda})$.

A *p-morphism* from (W, R) onto (W', R') is a surjective map $f : W \longrightarrow W'$ such that for any $x \in W$ $f[R(x)] = R'(f(x))$. In this case $\mathbf{ML}(W, R) \subseteq \mathbf{ML}(W', R')$ (the *p-morphism lemma*).

A *cone in* $F = (W, R)$ *with root* u (denoted by $F{\uparrow}u$) is the restriction of F to the smallest subset V containing u and such that $R(V) \subseteq V$; obviously, $V = R(u) \cup \{u\}$ if R is transitive.

A *modal predicate logic* is a set of modal predicate formulas containing classical predicate axioms, the axiom of $\mathbf{K}$ and closed under Modus Ponens, Generalization, $\Box$-introduction, and (predicate) Substitution.

$\mathbf{Q\Lambda}$ denotes the smallest predicate logic containing the propositional logic $\mathbf{\Lambda}$ (*the predicate version* of $\mathbf{\Lambda}$).

For predicate formulas we use the standard Kripke semantics. Recall that a *predicate Kripke frame* over a propositional Kripke frame $F = (W, R)$ is a pair $\mathbf{F} = (F, D)$, in which $D = (D_u)_{u \in W}$, $D_u \neq \varnothing$ and such that $D_u \subseteq D_v$ whenever uRv.

For a class of propositional frames $\mathcal{C}$, the class of all predicate frames (F, D) with $F \in \mathcal{C}$ is denoted by $\mathcal{KC}$.

A *valuation* ξ in $\mathbf{F}$ is a function sending every predicate letter P^n_k to a family of n-ary relations on the domains:

$$\xi(P^n_k) = (\xi_u(P^n_k))_{u \in W},$$

where $\xi_u(P^n_k) \subseteq D^n_u$ (D^0_u is a fixed two-element set $\{0, 1\}$).

The pair $M = (\mathbf{F}, \xi)$ is a *Kripke model* over $\mathbf{F}$. The definition of truth in a Kripke model is standard. So at every point $u \in W$ we evaluate *modal* D_u*-sentences*, i.e., modal formulas, in which all parameters are replaced with elements of D_u; $M, u \vDash A$ means that A is true at u in M. Then

$$M, u \vDash P^n_k(a_1, \ldots, a_n) \text{ iff } (a_1, \ldots, a_n) \in \xi_u(P^n_k),$$
$$M, u \vDash P^0_k \text{ iff } \xi_u(P^0_k) = 1,$$
$$M, u \vDash A \supset B \text{ iff } (M, u \nvDash A \text{ or } M, u \vDash B),$$
$$M, u \nvDash \bot,$$
$$M, u \vDash \forall x A(x) \text{ iff } \forall a \in D_u \; M, u \vDash A(a),$$
$$M, u \vDash \Box A \text{ iff } \forall v \in R(u) \; M, v \vDash A.$$

A modal formula $A(x_1, \ldots, x_n)$ is called *true in* M (in symbols, $M \vDash A(x_1, \ldots, x_n)$) if $M, u \vDash A(\mathbf{a})$ for every $u \in W$ and $\mathbf{a} \in D^n_u$.

A modal formula A is *valid* in a frame $\mathbf{F}$ (in symbols, $\mathbf{F} \vDash A$) if it is true in every Kripke model over $\mathbf{F}$. $\mathbf{ML}(\mathbf{F}) := \{A \mid \mathbf{F} \vDash A\}$ is the *modal logic of* $\mathbf{F}$.

The *modal logic of a class of frames* $\mathcal{C}$ (or the logic *determined by* $\mathcal{C}$) is $\mathbf{ML}(\mathcal{C}) := \bigcap\{\mathbf{ML}(\mathbf{F}) \mid \mathbf{F} \in \mathcal{C}\}$. Logics of this form are called *Kripke complete.*

There is also the notion of *strong Kripke completeness*; a modal predicate logic L is strongly Kripke complete if every L-consistent theory is satisfied at a point of some Kripke model over a frame validating L.

Similar definitions are given for modal propositional logics. Also recall that a modal propositional logic *has the finite model property (fmp)* if it is determined by some class of finite frames.

From the definitions it follows that for a predicate frame (F, D) and a propositional formula A,

$$(F, D) \vDash A \text{ iff } F \vDash A.$$

So for a propositional logic $\mathbf{\Lambda}$ and a predicate frame $\mathbf{F}$

$$\mathbf{F} \vDash \mathbf{\Lambda} \text{ iff } \mathbf{F} \in \mathcal{K}\mathbf{V}(\mathbf{\Lambda}).$$

One can easily see that $\mathbf{Q\Lambda}$ is complete iff

$$\mathbf{Q\Lambda} = \mathbf{ML}(\mathcal{K}\mathbf{V}(\mathbf{\Lambda})).$$

2 Completeness and incompleteness in modal predicate logic

In modal predicate logic there are too many examples of incompleteness, and proofs of completeness can be rather nontrivial. For instance, for a propositional modal logic $\mathbf{\Lambda} \supseteq \mathbf{S4}$, $\mathbf{Q\Lambda}$ is complete only if $\mathbf{S5} \subseteq \mathbf{\Lambda}$ or $\mathbf{\Lambda} \subseteq \mathbf{S4.3}$ (cf. [5]). Still some logics $\mathbf{Q\Lambda}$ are complete, in particular, for the well-known modal logics $\mathbf{\Lambda} = \mathbf{K}$, $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{S5}$, $\mathbf{S4.2}$, $\mathbf{S4.3}$ (cf. [3], theorems 6.1.29, 6.6.7, 6.7.12). These results were obtained by different authors — S. Kripke, D. Gabbay, S. Ghilardi, G. Corsi and others.

In this paper we are mainly interested in the logic $\mathbf{K4}Ad := \mathbf{K4} + Ad$, where

$$Ad := \Box\Box p \supset \Box p$$

is the axiom of density; $(W, R) \models Ad$ iff R is dense, i.e., $R \subseteq R \circ R$.

An extension of $\mathbf{K4}Ad$ is $\mathbf{D4.3}Ad$ obtained by adding the axiom of non-branching (.3) and seriality ($\Diamond\top$). It is well-known that $\mathbf{D4.3}Ad = \mathbf{ML}(\mathbb{Q}, <)$, where $\mathbb{Q}$ denotes the set of rationals. Moreover, completeness transfers to the predicate version [1]:

$$\mathbf{Q}(\mathbf{D4.3}Ad) = \mathbf{ML}(\mathcal{K}(\mathbb{Q}, <)).$$

3 Unravelling and bulldozing

Let us first discuss Kripke semantics for the propositional logic $\mathbf{K4}Ad$.

A *(transitive) tree* is a strictly ordered set $(W, <)$ with the least element such that every subset $\{y \mid y < x\}$ is linearly ordered and finite. Recall that a transitive frame (W, R) is *rooted* with root u if $W = R(u)$, or equivalently, if it has the first cluster.

Lemma 3.1 *Every rooted transitive frame is a p-morphic image of a tree.*

A well-known proof is by unravelling: for a rooted frame $F = (W, R)$ with root u we construct a tree $F^\sharp = (W^\sharp, <)$, where $W^\sharp$ is the set of all finite paths from u to points of W (i.e., finite sequences $x_0 x_1 \dots x_n$ such that $x_0 = u$ and $x_i R x_{i+1}$ for any $i < n$), and $\alpha < \beta$ iff β prolongs α. The required p-morphism sends every path to its last point.

Hence we have

Proposition 3.2 **K4** *is determined by the class of all (at most) countable trees.*

This follows from lemma 3.1, the p-morphism lemma and the fmp of **K4**; note that unravelling of a finite frame is finite or countable.

Definition 3.3 *Let $(W, <)$ be a tree, and consider a frame $(W, <')$, in which $<'$ is obtained from $<$ by making some points reflexive. Then $(W, <')$ is called a* semireflexive tree.

One can easily check that a semireflexive tree $(W, <')$ validates Ad iff its irreflexive points can have only reflexive successors. Such a semireflexive tree is called *dense*.

Proposition 3.4 **K4**Ad *is determined by the class of all (at most) countable dense semireflexive trees.*

Proof A standard filtration argument shows that **K4**Ad has the fmp, so it is determined by finite rooted **K4**Ad-frames (cf. [6]). Finite **K4**-frames consist of clusters, some of which can be degenerate (i.e., irreflexive singletons), while in finite **K4**Ad-frames successors of degenerate clusters are non-degenerate.

Now let us unravel a finite **K4**Ad-frame $F = (W, R)$ with root u more carefully than in lemma 3.1. Call a path $x_0 \dots x_n$ *long* if

$$\forall i < n \, \forall y \in F(x_i R y R x_{i+1} \Rightarrow y R x_i \vee x_{i+1} R y).$$

Consider the set W_1 of all long paths from u to points in F and take the restriction $F_1 := F^\sharp | W_1$. This frame is a tree, and the map f sending a path to its last point is still a p-morphism $F_1 \longrightarrow F$. This is because every two R-related points can be connected by a long path.

Now we extend the relation in F_1 by making reflexive every point a such that $f(a)$ is reflexive. We obtain a semireflexive tree F_2 and again f is a p-morphism $F_2 \longrightarrow F$.

F_2 is a dense semireflexive tree. In fact, if in F_2 we have an irreflexive a and its successor b, then a is a long path in F ending at an irreflexive point $f(a)$, and the cluster of $f(b)$ is a successor of $f(a)$. So $f(b)$ is reflexive, and thus b is reflexive in F_2. ∎

To obtain a class of irreflexive transitive frames determining **K4**Ad we can use Segerberg's bulldozing method (cf. [6]). Viz., given a dense semireflexive tree F_2, we can replace each its reflexive point with a strict dense linear order without the last element (e.g., the non-negative rationals $\mathbf{Q}_+$). Then we obtain **K4**Ad-frame F_3, and there is a p-morphism from F_3 sending every irreflexive point from F_2 to itself and every copy of $\mathbf{Q}_+$ to the corresponding reflexive point in F_2. We call such a frame F_3 a *sprouting tree.* So we have

Proposition 3.5 **K4**Ad *is determined by the class of sprouting trees.*

Remark 3.6 It is not clear if predicate frames over sprouting trees determine the predicate logic **QK4**Ad. The completeness proof proposed below yields more complicated frames.

4 Completeness of QK4Ad

To prove completeness for **QK4**Ad we use a method originating from G. Cosri's paper [1] and further developed by D. Skvortsov [9]; also cf. [3], sec. 6.4.

The main idea is to extract an appropriate submodel from a canonical model of a given logic L and to make a sort of unravelling which leads to a frame validating L. More exactly, this frame is obtained as a direct limit of a sequence of finite trees. This sequence can be constructed by induction, or equivalently, by playing a game.

First we recall some definitions from [3], sections 6.1, 6.3, with little changes.

We fix a denumerable set of extra constants S^*. Its subset is called *small* if its complement to S^* is infinite.

Definition 4.1 *For a modal predicate logic L, an L-*place *is an L-consistent theory (i.e, a set of sentences) Γ in the basic language with extra constants from S^* and with the* Henkin property*: for any formula $\varphi(x)$ with at most one parameter x there exists a constant c such that $(\exists x\varphi(x) \supset \varphi(c)) \in \Gamma$. An L-place is* small *if the set of its constants is small.*

The canonical model VM_L *is (VP_L, R_L, D_L, ξ_L), where*

- *VP_L is the set of all small L-places,*
- *$\Gamma R_L \Delta$ iff $\Box^-\Gamma \subseteq \Delta$, where $\Box^-\Gamma := \{A \mid \Box A \in \Gamma\}$,*
- *$(D_L)_\Gamma$ (also denoted by D_Γ) is the set of constants occurring in Γ,*
- $(\xi_L)_\Gamma(P_k^m) := \{\mathbf{c} \in (D_\Gamma)^m \mid P_k^m(\mathbf{c}) \in \Gamma\}$
  for $m > 0$, and
  $(\xi_L)_\Gamma(P_k^0) := 1$ *iff* $P_k^0 \in \Gamma$.

Then for any A in the language of Γ

$$VM_L, \Gamma \vDash A \text{ iff } A \in \Gamma$$

(the *Canonical model theorem*).

Note that for arbitrary L-places an analogue of this theorem does not hold, but we still need them for further considerations. So put $VM_L^+ := (VP_L^+, R_L, D_L, \xi_L)$, where VP_L^+ is the set of all L-places, and R_L, D_L, ξ_L are the same as above.[2] This VM_L^+ is actually a submodel of a canonical model for some larger set of extra constants.

Henceforth we assume that L contains **QK4**, so L-frames are transitive.

Definition 4.2 *Let L be a predicate logic, $F = (W, R)$ a transitive propositional frame. An* L-network *over F is a monotonic map from F to (VP_L^+, R_L), i.e. a map $h : W \longrightarrow VP_L^+$ such that for any $u, v \in W$*

$$uRv \Rightarrow h(u) R_L h(v).$$

[2]More exactly, R_L extended to $VP_L^+ \times VP_L^+$, etc.

The frame F is denoted by $dom(h)$ and called the domain *of h. An L-network is* small *if every $h(u)$ is small.*

With every L-network h we associate a predicate Kripke frame $\mathbf{F}(h) := (F, D)$, where $D_u = (D_L)_{h(u)}$ for $u \in W$, and a Kripke model $M(h) := (\mathbf{F}(h), \xi(h))$, where

$$\xi(h)_u(P_k^m) := \{\mathbf{c} \in D_u^m \mid P_k^m(\mathbf{c}) \in h(u)\}$$

for $m > 0$ and

$$\xi(h)_u(P_k^0) := 1 \text{ iff } P_k^0 \in h(u).$$

We define the partial order on networks.

$$h \leq h' := dom(h) \text{ is a subframe of } dom(h') \text{ and } \forall u \in dom(h)\ h(u) \subseteq h'(u).$$

Definition 4.3 *A* defect *in a network h over a frame (W, R) is a pair (u, A) such that $u \in W$ and $\Diamond A \in h(u)$. A defect (u, A) is* eliminated *in h if there exists $v \in R(u)$ such that $A \in h(v)$.*

We will call an L-network h *finite* if it is small and $dom(h)$ is a finite tree.

Lemma 4.4 *(On elimination of defects) Let h be a finite L-network with a defect (u, A). Then there is a finite L-network $h' \geq h$ eliminating this defect.*

Proof If h eliminates (u, A), take $h' = h$. Otherwise extend $dom(h)$ by adding a new successor v of u (such that v has no successors). Since $\Diamond A \in h(u)$, by the properties of the canonical model VM_L, there exists an L-place Γ such that $A \in \Gamma$ and $h(u) R_L \Gamma$. So we can put $h'(v) := \Gamma$. ■

If Γ, Δ are L-places, $\Gamma \restriction \Delta$ denotes the restriction of Γ to the language of Δ.

Lemma 4.5 *(Skvortsov's extension lemma)*

(1) Let Γ, Δ be L-places, $\Gamma_0 = \Gamma \restriction \Delta$ and suppose that $\Box^- \Gamma_0 \subseteq \Delta$. Then there exists an L-place $\Delta' \supseteq \Delta$ such that $\Gamma R_L \Delta'$.

(2) Let h be a finite L-network over a tree F with root v, and let Γ be an L-place, $\Gamma_0 = \Gamma \restriction h(v)$, and suppose that $\Box^- \Gamma_0 \subseteq h(v)$. Let F' be the tree obtained by adding a root u below F. Then there exists a finite L-network $h' \geq h$ over F' such that $\Gamma = h'(u)$.

Proof This is a reformulation of Lemma 6.4.28 from [3], and the proof follows the same lines.

(1) The assumptions imply that the theory $\Box^{-}\Gamma \cup \Delta$ is consistent (see the details in [3]); so it extends to an L-place Δ'.

(2) We can argue by induction on the cardinality of F. By (1) there exists an L-place $\Delta' \supseteq h(v)$ such that $\Gamma R_L \Delta'$. If v has no successors (i.e., F is a singleton), we are done: take h' defined on the chain $\{u, v\}$ such that $h'(u) = \Gamma,\ h'(v) = \Delta'$.

Suppose v has successors $v_1, \dots v_n$, $F_i = F{\uparrow}v_i$. h_i is the restriction of h to F_i. Since we can rename the constants from $D_{\Delta'} - D_{h(v)}$, we may assume that they do not occur in any $h(v_i)$; thus $\Delta = \Delta' \restriction h(v_i)$, and $\Box^{-}\Delta \subseteq h(v_i)$. Now by IH there exists $h'_i \geq h_i$ defined on the tree F_i with the added bottom element v such that $h'_i(v) = \Delta'$. Then we define the following network h' on F':

$$h'(u) = \Gamma,\ h'(v) = \Delta',\ h'|F_i = h'_i.$$

■

Now we assume that L contains $\mathbf{QK4}Ad$.

Lemma 4.6 *(On inserts) Let h be a finite L-network, and let v be a successor of u in $dom(h)$. Then there exists a finite L-network $h' > h$ such that v is not a successor of u in $dom(h')$.*

Proof Suppose $h(u) = \Gamma,\ h(v) = \Delta$, and let $\Delta_0 = \Delta \restriction \Gamma$. It follows that the set $\Gamma' := \Box^{-}\Gamma \cup \{\Diamond A \mid A \in \Delta_0\}$ is L-consistent. In fact, otherwise there exist $B \in \Box^{-}\Gamma$ and $A \in \Delta_0$ such that $\{B, \Diamond A\}$ is inconsistent (since the sets $\Box^{-}\Gamma$, Δ_0 are closed under conjunction and $\Diamond A_1 \wedge \Diamond A_2$ implies $\Diamond(A_1 \wedge A_2)$). So $L \vdash B \supset \neg\Diamond A$, or equivalenty, $L \vdash B \supset \Box\neg A$. Hence by the monotonicity of $\Box$, $L \vdash \Box B \supset \Box\Box\neg A$; thus $L \vdash \Box B \supset \Box\neg A$ by Ad. Since $\Box B \in \Gamma$ and A is in the language of Γ, this implies $\Box\neg A \in \Gamma$. Since $\Gamma R_L \Delta$, it follows that $\neg A \in \Delta$, which is a contradiction.

Then Γ' can be extended to an L-place Θ (with new unused constants). Let $\Theta_0 = \Theta \restriction \Delta$ ($= \Theta \restriction \Delta_0$, since new constants of Θ do not occur in Δ).

It follows that $\Box^{-}\Theta_0 \subseteq \Delta_0$. In fact, $\neg A \in \Delta_0$ implies $\Diamond\neg A \in \Gamma' \subseteq \Theta$.

Consider the tree F' obtained from $F = dom(h)$ by adding a new point z between u and v. By Lemma 4.5 there exists a finite network h^1 over $F'{\uparrow}z$ such that $h^1(z) = \Theta$ and $h^1 \geq h$ on $F{\uparrow}v$. Now we can define h' on F', which coincides with h^1 on $F'{\uparrow}z$ and coincides with h at all other points. This is a network, since $\Box^{-}\Gamma \subseteq \Theta$, i.e., $h'(u) R_L h'(z)$. ■

Definition 4.7 *Let Γ_0 be a small L-place. The* selective game *$SG_L(\Gamma_0)$ is played by two players, $\forall$ (the first) and $\exists$ (the second). A position after the n-th turn is a finite network h_n over a tree $F_n = (W_n, R_n)$.*

At the initial position F_0 is an irreflexive singleton u_0 and $h_0(u_0) = \Gamma_0$.

For the $(n+1)$-th move the player $\forall$ has two options.

1. Selecting a defect*, i.e., a pair (u, A) such that $u \in W_n$ and $\Diamond A \in h_n(u)$.*

2. A query for an insert*, i.e., a pair (u, v) such that uR_nv and there are no points between u and v.*

The player $\exists$ should respond with a network $h_{n+1} \geq h_n$ such that

1. If the move of $\forall$ was a defect (u, A), then there exists v such that $uR_{n+1}v$ and $A \in h_{n+1}(v)$.

2. If the move of $\forall$ was a query for an insert (u, v), then then there exists w such that $uR_{n+1}wR_{n+1}v$.

The player $\exists$ wins if the play continues infinitely or $\forall$ cannot make his move.

Note that $\forall$ cannot make the $(n+1)$th move in the only case when $n = 0$ and h_0 has no defects. This happens if Γ_0 is an endpoint in VM_L, i.e., $R_L(\Gamma_0) = \varnothing$.

Every infinite play of the game generates a sequence of networks $h_0 \leq h_1 \leq \ldots$ Then we define the resulting network h_ω, with $dom(h_\omega) = F_\omega := (W_\omega, R_\omega)$, $W_\omega := \bigcup_n W_n$, $R_\omega := \bigcup_n R_n$, $h_\omega(u) := \bigcup\{h_n(u) \mid u \in W_n\}$. One can easily check that this is really a network (not necessarily finite or small).

Lemma 4.8 *$\exists$ has a winning strategy in $SG_L(\Gamma_0)$.*

Proof If $\forall$ cannot make the first move, there is nothing to prove. If the $(n+1)$-th move of $\forall$ is a defect, $\exists$ can eliminate it by her next move according to Lemma 4.4. If the move of $\forall$ is a query for an insert, $\exists$ can respond according to Lemma 4.6. ■

Lemma 4.9 *If Γ_0 is not an endpoint in VM_L, then there exists a play generating a sequence of networks such that $F_\omega \vDash$ **K4**Ad and for any u, for any A in the language of $h_\omega(u)$*

$$M(h_\omega), u \vDash A \text{ iff } A \in h_\omega(u).$$

Proof A *dense tree* is a rooted strictly ordered set $(W, \prec)$, in which every subset $\{u \mid u \prec w\}$ is a dense chain. Let us construct an infinite play such that F_ω is a dense tree.

The worlds will be just natural numbers. At the initial position $F_0 = (0, \varnothing)$ and $h_0(0) = \Gamma_0$.

Let us choose the further strategy for $\forall$ as follows. Fix an enumeration of the countable set $\omega \times \omega$, and an enumeration of $\omega \times \Phi$, where Φ is the set of all modal sentences with constants from S^*. An odd move $(n+1)$ of $\forall$ chooses the first new pair (u, A), which is a defect in h_n. An even move $(n+1)$ of $\forall$ chooses the first new pair $(u, v) \in \omega \times \omega$, which is a query for an insert in h_n.

By lemma 4.8 there is a winning strategy for $\exists$. For the resulting network we have

$$M(h_\omega), u \models A \text{ iff } A \in h_\omega(u).$$

This is checked by induction. The atomic case holds by the definition of $\xi(h)$; the cases of propositional connectives and quantifiers hold by the properties of L-places.

Let us consider the case $A = \Diamond B$.[3] Suppose $M(h_\omega), u \models A$; then $M(h_\omega), v \models B$ for some $v \in R_\omega(u)$. Since A is in the language of $h_\omega(u)$ and h_ω is a network, we have $h_\omega(u) R_L h_\omega(v)$, so A (and B) is also in the language of $h_\omega(v)$. By IH it follows that $B \in h_\omega(v)$; hence $A = \Diamond B \in h_\omega(u)$ by the definition of R_L.

The other way round, suppose $A \in h_\omega(u)$; then $A \in h_n(u)$ (i.e., (u, A) is a defect in h_n) for some finite n. Choose the minimal such n; so (u, A) is a defect in h_m for all $m > n$. Since the defects subsequently appear as odd moves of $\forall$, there exists m such that (u, A) is his $(m+1)$-th move. By the response of $\exists$, we have $B \in h_{m+1}(v)$ for some $v \in R_{m+1}(u)$. Hence $B \in h_\omega(v)$, $v \in R_\omega(u)$. By IH, we have $M(h_\omega), v \models B$. Thus $M(h_\omega), u \models A$.

To check the density for F_ω, we can use even moves. In fact, if $u R_\omega v$, there exists n such that $u R_n v$. If v is a successor of u in R_n, the pair (u, v) must show up as a later even move of $\forall$. By the response of $\exists$ we have w such that $u R_\omega w R_\omega v$. ■

Definition 4.10 *A modal predicate logic L is strongly Kripke complete if every L-consistent set of sentences is satisfiable at some point of a Kripke model over a frame validating L.*

Theorem 4.11 **QK4**Ad *is strongly Kripke complete.*

Proof Every L-consistent theory without constants can be extended to an L-place Γ_0. If Γ_0 is an endpoint in VM_L, then for any A in its language

$$VM_L, \Gamma_0 \models A \text{ iff } A \in \Gamma_0$$

by the canonical model theorem. Since Γ_0 is an endpoint, the truth at this point reduces to the truth in a model over an irreflexive singleton.

In all other cases we can apply lemma 4.9. So there exists a model $M(h_\omega)$ such that $M(h_\omega), u_0 \models \Gamma_0$ and $F_\omega \models \mathbf{K4}Ad$. Hence $F(h_\omega) \models L$. ■

[3]Since $\Box$ is our primitive, we should deal with $A = \neg\Box B$, which is equivalent to $\Diamond\neg B$; the argument is almost the same.

Theorem 4.12 *If S is a set of closed (i.e., constructed only from $\bot$, $\Box$ and $\supset$) propositional formulas, then* $\mathbf{QK4}Ad + S$ *is strongly Kripke complete.*

Proof By the same argument as in the previous theorem. In this case $S \subset \Gamma$ for all L-places Γ (where $L := \mathbf{QK4}Ad + S$), so $M(h_\omega) \vDash S$. Hence $F_\omega \vDash S$, and thus $F(h_\omega) \vDash L$. ■

5 Final remarks

Axiomatizing modal predicate logics of specific frames is usually a nontrivial problem. In particular, we can be interested in predicate logics of relativistic time. The only clear case is the following.

Theorem 5.1 *Let F be the Minkowski lower halfspace with the causal future relation: aRb iff a signal can be sent from a to b. Them* $\mathbf{ML}(\mathcal{K}F) = \mathbf{QS4}$.

Proof Every cone in F can be mapped p-morphically onto the infinite reflexive binary tree IT_2 [6]. It is also well-known that $\mathbf{ML}(\mathcal{K}IT_2) = \mathbf{QS4}$ (cf. [3], section 6.4). Hence the claim follows. ■

The method proposed in the previous section can also be applied to the logic $\mathbf{K4}Ad_2$, where $Ad_2 := \Diamond p \wedge \Diamond q \supset \Diamond(\Diamond p \wedge \Diamond q)$ is the axiom of 2-density appearing in the logic of chronological future, cf. [7]. However, axiomatizing the predicate version of the latter logic seems a difficult problem.

Also note that our method is inapplicable to the case of constant domains. Moreover, the corresponding logic $L' := \mathbf{QK4}Ad + Ba$, where

$$Ba := \forall x \Box P(x) \supset \Box \forall x P(x)$$

is the Barcan axiom, may be Kripke incomplete. In fact, incompleteness is known for the logic $\mathbf{QK}Ad + \Diamond\top$ (cf. [4]), and it probably extends to L' (although the proof from [4] does not fit for L', because of transitivity).

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