AXIOMATIZING DOMAIN ALGEBRAS

SZABOLCS MIKULÁS

ABSTRACT. We look at various versions of domain algebras and provide a survey of axiomatizability results. We also present a finite axiomatization for the variety generated by representable upper semilattice-ordered domain–range semigroups.
Keywords: domain algebras, relation algebras, finite axiomatizability

1. Introduction

The family of domain algebras provides an elegant, one-sorted formalism for automated reasoning about program and system verification, see [DS11, DS08] and [HM11] for details and further motivation. Their primary models are algebras of binary relations, viz. representable domain algebras.

There are several variants of domain algebras depending on the choice of the signature. They agree on having a domain operation D and a composition operation ;. In addition, they may have other operations that can be interpreted on binary relations: range R, antidomain A, converse $^\smile$, join + and meet ·, and constants: zero 0, identity $1'$.

Definition 1.1. Let U be a set. We define operations on elements of $\wp(U \times U)$.

Composition:

$$X \,;\, Y = \{(u,v) \mid (u,w) \in X \text{ and } (w,v) \in Y \text{ for some } w \in U\}$$

Domain:

$$\mathrm{D}(X) = \{(u,u) \mid (u,v) \in X \text{ for some } v \in U\}$$

Range:

$$\mathrm{R}(X) = \{(v,v) \mid (u,v) \in X \text{ for some } u \in U\}$$

Antidomain:

$$\mathrm{A}(X) = \{(u,u) \mid (u,v) \notin X \text{ for any } v \in U\}$$

Converse:

$$X^\smile = \{(u,v) \mid (v,u) \in X\}$$

Identity:

$$1' = \{(u,u) \mid u \in U\}$$

for $X, Y \subseteq U \times U$.

Let Λ be a signature such that $(;, \mathrm{D}) \subseteq \Lambda \subseteq (;, \mathrm{D}, \mathrm{R}, \mathrm{A}, {}^\smile, +, \cdot, 0, 1')$. A *representable domain algebra of signature* Λ is a subalgebra of

$$(\wp(U \times U), \lambda : \lambda \in \Lambda)$$

and a *representable ordered domain algebra* is a representable domain algebra augmented with an ordering $\leq$ interpreted as the subset relation $\subseteq$.

We will denote the class of representable domain algebra of signature Λ as $\mathbb{R}(\Lambda)$ and the variety generated by $\mathbb{R}(\Lambda)$ as $\mathbb{V}(\Lambda)$.

2. Axiomatizing the representation class?

The algebraic behaviour of domain algebras have been investigated, e.g. in [DJS09a, DJS09b]. P. Jipsen and G. Struth raised the question whether the class $\mathbb{R}(;, \mathrm{D})$ of representable domain algebras of the minimal signature $(;, \mathrm{D})$ is finitely axiomatizable.

Theorem 2.1 ([HM11])**.** *Let Λ be a similarity type such that $(\mathrm{D}, ;) \subseteq \Lambda \subseteq (;, \mathrm{D}, \mathrm{R}, \mathrm{A}, 1', 0)$. The class $\mathbb{R}(\Lambda)$ of representable Λ-algebras is not finitely axiomatizable in first-order logic.*

Then we looked at whether the representation class can be finitely axiomatized if we include lattice operations into the signature. We start with adding join.

Theorem 2.2 ([HM11] using [AM11])**.** *Let Λ be a similarity type such that $(;, +) \subseteq \Lambda \subseteq (;, \mathrm{D}, \mathrm{R}, \mathrm{A}, +, {}^\smile, {}^*, 1', 0, 1)$. The class $\mathbb{R}(\Lambda)$ of representable Λ-algebras is not finitely axiomatizable in first-order logic.*

Adding meet does not seem more promising.

Theorem 2.3 ([HM07])**.** *The class $\mathbb{R}(;, \cdot, 1')$ is not finitely axiomatizable in first-order logic.*

The proof is an ultraproduct construction of non-representable algebras, where $1'$ is an atom. Thus we can augment these algebras with D, R.

Corollary 2.4. *Let Λ be a similarity type such that $(;, \mathrm{D}, \cdot) \subseteq \Lambda \subseteq (;, \mathrm{D}, \mathrm{R}, \cdot, 1')$. The class $\mathbb{R}(\Lambda)$ of representable Λ-algebras is not finitely axiomatizable in first-order logic.*

Adding a (distributive) lattice structure does not work either.

Theorem 2.5 ([An91])**.** *Let Λ be a similarity type such that $(;, +, \cdot) \subseteq \Lambda \subseteq (;, +, \cdot, -, {}^\smile, {}^*, 0, 1)$. The class $\mathbb{R}(\Lambda)$ of representable Λ-algebras is not finitely axiomatizable in first-order logic.*

The proof is another ultraproduct construction. Observe that we can define $\mathrm{D}(x) = (x \,;\, x^\smile) \cdot 1'$, $\mathrm{R}(x) = (x^\smile \,;\, x) \cdot 1'$ and $\mathrm{A}(x) = -\,\mathrm{D}(x) \cdot 1'$. Thus we get the following.

Corollary 2.6. *Let Λ be a similarity type such that $(;, \mathrm{D}, +, \cdot) \subseteq \Lambda \subseteq (;, \mathrm{D}, \mathrm{R}, \mathrm{A}, +, \cdot, {}^\smile, 0, 1)$. The representation class $\mathbb{R}(\Lambda)$ is not finitely axiomatizable.*

However, if we restrict ourselves to ordered algebraic structures, then a Cayley-type representation works.

Theorem 2.7 ([Br75] and [HM13])**.** *The representation class $\mathbb{R}(;, \mathrm{D}, \mathrm{R}, {}^\smile, 0, 1', \leq)$ is finitely axiomatizable (by quasiequations) and has the finite representation property.*

3. Axiomatizing the generated variety

Since the representation classes are not finitely axiomatizable in general, it is a natural task to look at the generated varieties and see wehther they are finitely based. M. Hollenberg looked at the case of antidomain.

Theorem 3.1 ([Ho97]). *The variety* $\mathbb{V}(;, \mathrm{A})$ *generated by* $\mathbb{R}(;, \mathrm{A})$ *is finitely axiomatizable.*

Recently, we looked at the case of upper semilattice-ordered domain–range semigroups.

Theorem 3.2 (M. Jackson and Sz. Mikulás). *The variety* $\mathbb{V}(;, \mathrm{D}, \mathrm{R}, +)$ *generated by* $\mathbb{R}(;, \mathrm{D}, \mathrm{R}, +)$ *is finitely axiomatizable.*

The axioms Ax are:

$(D1)$	$\mathrm{D}(x) ; x = x$	$(R1)$	$x ; \mathrm{R}(x) = x$
$(D2)$	$\mathrm{D}(x ; y) = \mathrm{D}(x ; \mathrm{D}(y))$	$(R2)$	$\mathrm{R}(x ; y) = \mathrm{R}(\mathrm{R}(x) ; y)$
$(D3)$	$\mathrm{D}(\mathrm{D}(x) ; y) = \mathrm{D}(x) ; \mathrm{D}(y)$	$(R3)$	$\mathrm{R}(x ; \mathrm{R}(y)) = \mathrm{R}(x) ; \mathrm{R}(y)$
$(D4)$	$\mathrm{D}(x) ; \mathrm{D}(y) = \mathrm{D}(y) ; \mathrm{D}(x)$	$(R4)$	$\mathrm{R}(x) ; \mathrm{R}(y) = \mathrm{R}(y) ; \mathrm{R}(x)$
$(D5)$	$\mathrm{D}(\mathrm{R}(x)) = \mathrm{R}(x)$	$(R5)$	$\mathrm{R}(\mathrm{D}(x)) = \mathrm{D}(x)$
$(D6)$	$\mathrm{D}(x) ; y \leq y$	$(R6)$	$x ; \mathrm{R}(y) \leq x$

together with associativity of ; and +, idempotency of + and additivity of $;, \mathrm{D}, \mathrm{R}$.

The proof uses the following observations.

3.1. **Eliminating join.** Assume

$$\mathbb{V}(;, \mathrm{D}, \mathrm{R}, +) \models s \leq t$$

and we need $Ax \vdash s \leq t$, for all terms s, t.

Using additivity of the operations we have that

$$\mathbb{V}(;, \mathrm{D}, \mathrm{R}, +) \models s_1 + \ldots + s_n = s \leq t = t_1 + \ldots + t_m$$

for some join-free terms $s_1, \ldots, s_n, t_1, \ldots, t_m$.

It is not difficult to show that this happens iff for every i there is j such that

$$\mathbb{V}(;, \mathrm{D}, \mathrm{R}, +) \models s_i \leq t_j$$

Thus it is enough to show $Ax \vdash s_i \leq t_j$ for join-free terms.

3.2. **Domain elements (in the free algebra).** Let $\mathfrak{A}$ be a model of Ax. The set $\mathrm{D}(A)$ of *domain elements* is defined as

$$\mathrm{D}(A) = \{a = \mathrm{D}(a) \mid a \in A\}$$

Lemma 3.3. (1) *The algebra* $(\mathrm{D}(A), ;)$ *of domain elements is a (lower) semilattice and the semilattice ordering coincides with* $\leq$.
(2) *For every* $a \in A$, $\mathrm{D}(a)$ *(resp.* $\mathrm{R}(a)$*) is the minimal element* d *in* $\mathrm{D}(A)$ *such that* $d ; a = a$ *(resp.* $a ; d = a$*).*

Let $\mathcal{F}_{Var} = (F_{Var}, ;, \mathrm{D}, \mathrm{R}, +)$ be the free algebra of the variety defined by Ax that is freely generated by a set Var of variables.

Lemma 3.4. *Let* r, s, t *be join-free terms such that* $\mathcal{F}_{Var} \models \mathrm{D}(r) \leq s ; t$. *Then* $\mathcal{F}_{Var} \models \mathrm{D}(r) \leq s = \mathrm{D}(s)$ *and* $\mathcal{F}_{Var} \models \mathrm{D}(r) \leq t = \mathrm{D}(t)$.

Lemma 3.5. *Let s, t be join-free terms such that $\mathcal{F}_{Var} \models s \leq \mathrm{D}(t)$. Then $\mathcal{F}_{Var} \models s = \mathrm{D}(s)$.*

3.3. **Step-by-step construction.** Using the above observations, we can construct an antisymmetric graph G_ω labelled by join-free terms T^-_{Var} that yields a representation of the free algebra $\mathcal{F}_{Var}$.

We will define a labelled, directed graph G_ω as the union of a chain of labelled, directed graphs $G_n = (U_n, \ell_n, E_n)$ for $n \in \omega$, where

- U_n is the set of nodes,
- $\ell_n \colon U_n \times U_n \to T^-_{Var}$ is a labelling of edges,
- $E_n = \{(u, v) \in U_n \times U_n \mid \ell_n(u, v) \neq \emptyset\}$ is a reflexive, transitive and antisymmetric set of edges.

Initial step. In the 0th step of the step-by-step construction we define $G_0 = (U_0, \ell_0, W_0)$ by creating an edge for every element of T^-_{Var}. We define U_0 by choosing elements $u_a, v_a, \ldots \in \omega$ so that $\{u_a, v_a\} \cap \{u_b, v_b\} = \emptyset$ for distinct a, b, and $u_a = v_a$ iff $\mathrm{D}(a) = a$ (i.e., a is a domain element of $\mathcal{F}_{Var}$). We can assume that $|\omega \setminus U_0| = \omega$. We define

$$\ell_0(u_a, v_a) = a$$
$$\ell_0(u_a, u_a) = \mathrm{D}(a)$$
$$\ell_0(v_a, v_a) = \mathrm{R}(a)$$

and we label all other edges by $\emptyset$.

Domain step. We assume that we have a loop (u, u) labelled by a domain element $c = \mathrm{D}(c) \leq a$, such that $\mathrm{D}(c) \,;\, a$ is not a domain element, but we may miss an edge (u, w) witnessing a.

We choose $w \in \omega \setminus U_m$, extend ℓ_m by

$$\ell_{m+1}(u, w) = \mathrm{D}(c) \,;\, a$$
$$\ell_{m+1}(w, w) = \mathrm{R}(\mathrm{D}(c) \,;\, a)$$

and for every $(p, u) \in E_m$ with $\ell_m(p, u) = d$ (some $d \in T^-_{Var}$)

$$\ell_{m+1}(p, w) = d \,;\, a$$

All other edges involving the point w have empty labels. See Figure 1.

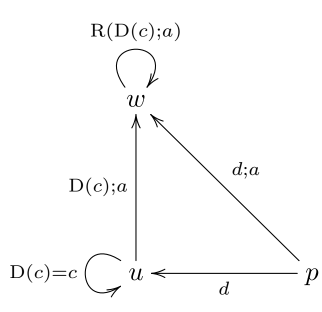

FIGURE 1. Step for domain

Range step. This is completely analogous to the domain step.

Composition step. Our aim is to extend G_m to create edges (u,w) and (w,v) witnessing a and b, provided $a\,;b \geq c = \ell_m(u,v)$.

We assume that

(CC1) $u \neq v$,
(CC2) $\mathrm{D}(c)\,;a\,;\mathrm{D}(b\,;\mathrm{R}(c)) \neq \mathrm{D}(\mathrm{D}(c)\,;a\,;\mathrm{D}(b\,;\mathrm{R}(c)))$,
(CC3) $\mathrm{R}(\mathrm{D}(c)\,;a)\,;b\,;\mathrm{R}(c) \neq \mathrm{R}(\mathrm{R}(\mathrm{D}(c)\,;a)\,;b\,;\mathrm{R}(c))$,

otherwise we define $G_{m+1} = G_m$. If (CC1)–(CC3) hold, then we choose $w \in \omega \backslash U_m$, extend ℓ_m by

$$\begin{aligned}\ell_{m+1}(u,w) &= \mathrm{D}(c)\,;a\,;\mathrm{D}(b\,;\mathrm{R}(c))\\ \ell_{m+1}(w,v) &= \mathrm{R}(\mathrm{D}(c)\,;a)\,;b\,;\mathrm{R}(c)\\ \ell_{m+1}(w,w) &= \mathrm{R}(\mathrm{D}(c)\,;a)\,;\mathrm{D}(b\,;\mathrm{R}(c))\end{aligned}$$

and for $(p,u),(v,q) \in E_m$ with $\ell_m(p,u) = d$ and $\ell_m(v,q) = e$ (some $d,e \in T^-_{Var}$)

$$\begin{aligned}\ell_{m+1}(p,w) &= d\,;a\,;\mathrm{D}(b\,;\mathrm{R}(c))\\ \ell_{m+1}(w,q) &= \mathrm{R}(\mathrm{D}(c)\,;a)\,;b\,;e\end{aligned}$$

All other edges involving w will have empty labels. See Figure 2.

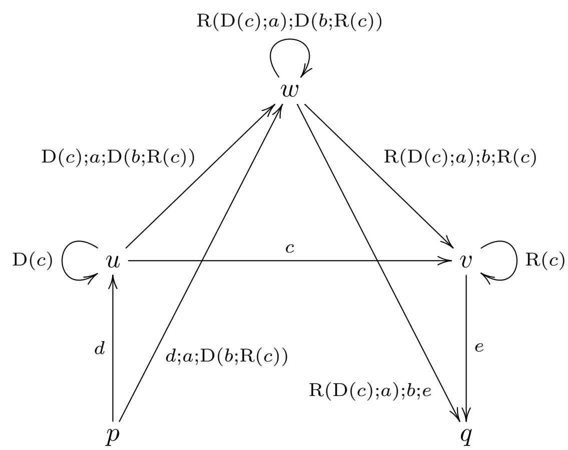

FIGURE 2. Step for composition

Limit step. We define $G_\omega = \bigcup_{n\in\omega} G_n$.

3.4. **Representing the free algebra.** We let

$$x^\flat = \{(u,v) \in U_\omega \times U_\omega : x = \ell_\omega(u,v)\}$$

for every variable $x \in \mathit{Var}$.

Let $\mathfrak{A} = (A, ;, \mathrm{D}, \mathrm{R}, +)$ be the subalgebra of the full algebra $(\wp(U_\omega \times U_\omega), ;, \mathrm{D}, \mathrm{R}, +)$ generated by $\{x^\flat : x \in \mathit{Var}\}$. Clearly $\mathfrak{A}$ is representable. It is not difficult to show that it provides a representation to the free algebra.

Lemma 3.6. *$\mathfrak{A}$ is isomorphic to $\mathcal{F}_{Var}$.*

4. Conclusion

We conclude with some open problems. Are the varieties generated by

- $\mathbb{R}(;, \mathrm{D}, \mathrm{R}, \mathrm{A}, +)$
- $\mathbb{R}(;, \mathrm{D}, \mathrm{R}, +, \cdot)$
- $\mathbb{R}(;, \mathrm{D}, \mathrm{R}, \mathrm{A}, +, \cdot)$

finitely axiomatizable?

References

[An91] H. Andréka, "Representation of distributive lattice-ordered semigroups with binary relations", *Algebra Universalis*, 28:12–25, 1991.

[AM11] H. Andréka and Sz. Mikulás, "Axiomatizability of Positive Algebras of Binary Relations", *Algebra Universalis*, 66:7–34, 2011.

[Br75] D.A. Bredikhin, "A representation of ordered semigroups with involution", *Izv. Vyssh. Uchebn. Zaved. Mat.*, No. 7, pp. 19–29, 1975. [In Russian, English translation: *Soviet Mathematics (Izvestiya VUZ. Matematika)*, 19:7, 15–22, 1975.]

[DJS09a] J. Desharnais, P. Jipsen and G. Struth, "Domain and antidomain semigroups", in R. Berghammer et al. (eds.), *Relations and Kleene Algebra in Computer Science*, pages 73–87, Springer-Verlag, 2009.

[DJS09b] J. Desharnais, P. Jipsen and G. Struth, "Internal Axioms for Domain Semirings", presentation at *TACL'2009*, extended abstract available at `http://staff.science.uva.nl/~gfontain/tacl09-abstracts/tacl2009_submission_73.pdf`.

[DS08] J. Desharnais and G. Struth, "Domain axioms for a family of near-semirings", in J. Meseguer and G. Roşu (eds.), *AMAST 2008*, pages 330–345, Springer-Verlag, 2008.

[DS11] J. Desharnais and G. Struth, "Internal axioms for domain semirings", *Science of Computer Programming*, Vol. 76 (3) (Special issue on the Mathematics of Program Construction MPC-2008), pp. 181–203, Elsevier, 2011.

[HM07] R. Hirsch and Sz. Mikulás, "Representable semilattice-ordered monoids", *Algebra Universalis*, 57:333–370, 2007.

[HM11] R. Hirsch and Sz. Mikulás, "Axiomatizability of representable domain algebras", *Journal of Logic and Algebraic Programming*, 80:75–91, 2011.

[HM13] R. Hirsch and Sz. Mikulás, "Ordered domain algebras", *Journal of Applied Logic*, 11:266–271, 2013.

[Ho97] M. Hollenberg, "An equational axiomatization of dynamic negation and relational composition", *Journal of Logic, Language and Information*, 6:381–401, 1997.