Are non-Hausdorff space-times physically reasonable?

Which of mathematical constructions can be regarded as representing (possible) physical space-times? The answer to this question is provided by physical theories like General Relativity. According to this theory physical space-time is represented by a differential manifold which satisfies Einstein's equations. However, usually not all solutions of these equations are treated as representing physical space-times in the proper sense – additional conditions of 'physical reasonability' are imposed. They can occupy various levels of the theory: they can be accommodated as a part of definition of differential manifold (like the Hausdorff condition, second countability, connectedness, paracompactness), they can take a form of additional constraints on the energy-momentum tensor (conservation law and various energy conditions), on metric (its signature) or on global structure of space-time (causality conditions, lack of some types of singularities, lack of 'holes'). There is no consensus which of these conditions really should be imposed and it seems that there is no simple general argument here. In my talk I would like to concentrate on one of the above conditions, namely the Hausdorff condition. I would describe examples of space-times which do not satisfy this condition, discuss some of their properties and consider arguments for and against taking them as physically reasonable.

One of the simplest examples of non-Hausdorff manifold is the following: take two copies of real numbers (each of them forms a manifold) and identify them up to some point, excluding this point. More advanced examples are described in the physical literature: some extensions of Misner space-time, of Taub-NUT space-time and of Gowdy polarized space-time are non-Hausdorff (see e.g. Hawking and Ellis 1973, Chruściel and Isenberg 1991). All of these non-Hausdorff manifolds are obtained as quotient structures made from other manifolds which satisfy the Hausdorff condition. For the purposes of illustration, I will sketch the construction of non-Hausdorff extensions of Taub-NUT space-time.

The interesting fact is that the first of the examples of non-Hausdorff manifolds admits bifurcate geodesics and all of the mentioned more advanced examples do not admit them. This is because non-Hausdorffness is a necessary but not sufficient condition for presence of such geodesics. The details of connection between non-Hausdorffness and bifurcating curves are analysed in (Hajicek 1971a, Hajicek 1971b, Clarke 1976). Two main results are as follows: the necessary and sufficient condition for a manifold constructed by gluing together Hausdorff manifolds to admit bifurcate curves of the second kind (that is, a pair of curves which agree up to some point, excluding this point) is that the gluing be continuously extendable; a connected 4-dimensional Riemannian manifold which is non-Hausdorff either is not strongly causal or admits bifurcate curves of the second kind. Some extensions of Taub-NUT space-time are non-Hausdorff but the gluing is not continuously extendable, so in this case non-Hausdorffness does not imply existence of bifurcate geodesics.

In the literature the presence of bifurcating geodesics is the main argument invoked against non-Hausdorff space-times. The reason is that in such cases the equation of geodesics does not have a unique global solution (although local uniqueness is still satisfied) and that is the breakdown of determinism because geodesics are assumed to be (potential) worldliness of free test particles. However, as we have seen, in many non-Hausdorff space-times there are no bifurcate geodesics and therefore some physicist consider liberalizations of the Hausdorff condition. For example, (Hawking and Ellis 1973:174) allow for these non-Hausdorff space-times which do not admit bifurcating geodesics; similarly (Geroch 1968: 465) allows for non-Hausdorff space-times in which every geodesic has a unique extension and every curve has no more than one end point.

There are also other arguments against non-Hausdorff space-times, put forward in (Earman 2008) and (Penrose 1979). Earman firstly invokes some mathematical theorems which depend on the Hausdorff condition: every compact set of a topological space is closed and if a sequence of points of a topological space converges, the limit point is unique. His second, more physical worry is about local and global conservation of energy. In order to properly formulate local conservation law, energy-momentum tensor should be continuous and differentiable. However, this entails that when energy ‘travels’ along bifurcate curve, it has to take both branches, because if it went along only one of them, the tensor on another one would be discontinuous. But then global energy conservation would be violated. The third Earman’s argument concerns existence and uniqueness of maximal solutions of Einstein’s equations (given the appropriate initial data) – the theorem which guarantees them relies on the Hausdorff condition. The uniqueness result fails if non-Hausdorff branching is allowed – we may attach non-Haussdorffly additional branches at some given moment of time. The fourth and most philosophical Earman’s argument can be summed as follows: both types of branching (on the level of geodesics and of the whole space-time) include a kind of arbitrariness connected with indeterminism. As concerns geodesics branching, he asks rhetorically: “how would such a particle know which branch of a bifurcating geodesic to follow?”, suggesting that there is no good answer to this question. As concerns space-time branching, he claims that we need some physical theory that prescribes the dynamics of branching – there should be something that determines which of possible branches are realised. Branching cannot, according to Earman, be regarded as explanatory term; quite the opposite – it requires explanation in other terms.

Some of Earman’s objections turn out to be harmless if we carefully interpret branching structures as representing possible evolutions, where at most one of branches can be actualised. For example, there is no problem with discontinuity of energy-momentum tensor: in actual reality it is wholly contained in one branch and the discontinuity concerns only branches which are not realised. The more subtle issue is indeterminism on the level of geodesics (curves followed by free test particles) and space-times which is allowed in some non-Hausdorff cases. There are some principal objections against it: lack of control, lack of factor which determinates the actual evolution (Earman 2008) or breaking “classical causality conception coinciding with determinism” (Hajicek 1971). However, it seems that all of these objections come down to simple rejection of indeterminism, which begs the question.

The only known attempt to use non-Hausdorff space-times to model indeterministic processes can be found in (Penrose 1979). His idea is to model quantum mechanics in Everett’s interpretation by non-Hausdorff branching. Penrose rejects this idea as implausible but for the reasons connected with details of Everett interpretation, not with properties of non-Hausdorff space-times.

Non-Hausdorff space-times are not well examined, so we do not have enough information to settle the issue of their physical reasonability. However, we can conclude that the known partial results cannot be taken as a basis for discrediting these space-times and that the idea to use such non-Hausdorff manifolds to model indeterministic processes within General Relativity is still not explored enough.

Bibliography

Chruściel, P. T. and Isenberg, J. (1991). Nonisometric vacuuum extensions of vacuum maximal globally hyperbolic spacetimes. *Physical Review D*, 48(4):16616–1628.

Clarke, C. J. S. (1976). Space-Time Singularities. *Communications in Mathematical Physics* 49:17–23.

Earman, J. (2008). Pruning some branches from branching spacetimes. In: Dieks, D. (Ed.), *The ontology of spacetime II*. Amsterdam: Elsevier, 187–206.

Geroch, R. (1968). Local Characterization of Singularities in General Relativity. *Journal of Mathematical Physics* 9: 450–465.

Hajicek, P. (1971a). Bifurcate space-time. *Journal of Mathematical Physics* 12:157–160.
Hajicek, P. (1971b). Causality in Non-Hausdorff Space-Times. *Communications in Mathematical Physics* 21:75–84.
Hawking, S. W. and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time*. Cambridge: Cambridge University Press.
Penrose, R. (1979). Singularities and time-asymmetry. In: Hawking, S.W., Israel, W. (Eds.), *General Relativity: An Einstein Centenary Survey*. Cambridge: Cambridge University Press, 581–638.
Ringström, H. (2009). *The Cauchy Problem in General Relativity*. Zürich: European Mathematical Society Publishing House.