Beyond Gödel's incompleteness theorem

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Abstract

In 1931, Kurt Gödel proved his first incompleteness theorem which is considered to be one of the most important results in modern logic. This discovery revolutionized our understanding of mathematics and logic, and had strong impacts in mathematics, physics, psychology, theology and some other fields of philosophy. It also plays a part in modern linguistic theories, which emphasize the power of language to come up with new ways to express ideas. Gödel's incompleteness theorem states that a complete and consistent list of axioms that extends "arithmetic" and is enumerated by an effective method (an algorithm or a computer program) can never be created. Gödel's work depends on arithmetic inside the theories at issue, and it was a task to loosen this marriage to study the same phenomena for arbitrary logics. Although the natural numbers play an essential role in its proof, the statement of the incompleteness theorem is in fact talking only about complete and consistent theories. Thus, replacing arithmetic with a suitable formula, in the logic in the question, yields a meaningful property for arbitrary logics, it is called *Gödel's incompleteness property (GIP)*.

Gödel's incompleteness property is closely connected to undecidability for arbitrary logics. Indeed, Gödel's incompleteness property for a logic that has a "recursively enumerable" set of formulas implies that this logic is in fact undecidable. Otherwise, one can use the decidability algorithm together with the enumeration algorithm to find complete and consistent theory extending any consistent formula. But there are several interesting decidable logics, so GIP fails automatically for these logics. However, if we replace "enumerated by an effective method" with "finitely axiomatizable" in GIP, then the so obtained *weak Gödel's incompleteness property (wGIP)* is still a property worth investigating for these decidable logics. Both wGIP and GIP are about the quality of expressive power, not about the strength of expressive power, just as decidability is not about smallness but about "how easy to define". The credit for defining GIP and wGIP goes back to István Németi in 1985.

The problem of investigating GIP and wGIP is not trivial, it is very involved and usually requires new techniques as well as tricky use of the known techniques. In this talk, we aim to present our latest results in this direction. We will also show what kind of logic-properties we used to achieve these results. Then, we will raise some conjectures that give a complete characterizations of these incompleteness properties. For instance, we claim that any arbitrary logic (that has a propositional part) lacks wGIP on finite languages if it has the finite model property and there is a 'derived' unary connective δ such that, for any formula φ, either $\models \delta(\varphi)$ or $\models \neg\delta(\varphi)$. An interesting conjecture, due to Zalán Gyenis, states that GIP and wGIP are equivalent for undecidable logics. Another surprising claim, at least for those who are familiar with these properties, is that wGIP may fail for some version of FOL on an infinite language. We will support these conjectures by comparing them, not only to the known results concerning these properties, but rather to the techniques used to prove these results.

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