Structure Theorem for a Class of Group-like Residuated Chains à la Hahn

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Abstract

Hahn's famous structure theorem states that totally ordered Abelian groups can be embedded in the lexicographic product of real groups. Our main theorem extends this structural description to order-dense, commutative, group-like residuated chains, which has only finitely many idempotents. It is achieved via the so-called partial-lexicographic product construction (to be introduced here) using totally ordered Abelian groups, as building blocks.

Hahn's structure theorem [14] states that totally ordered Abelian groups can be embedded in the *lexicographic product* of *real groups*. Residuated lattices [30] (aka. FL-algebras) are semigroups only, and are algebraic counterparts in the sense of [5] of a wide class of logics, called substructural logics [13]. The focus of our investigation is the class of commutative *group-like* residuated chains, that is, totally ordered, involutive, commutative residuated lattices such that the unit of the monoidal operation coincides with the constant that defines the involution. The latest postulate forces the structure to resemble totally ordered Abelian groups in many ways. Firstly, a cone representation, similar to that of totally ordered groups holds true [20]. Secondly, group-like commutative residuated chains can be characterized as generalizations of totally ordered Abelian groups by weakening the strictly-increasing nature of the partial mappings of the group multiplication to nondecreasing behaviour, see Theorem 1. Thirdly, in quest for establishing a structural description for commutative group-like residuated chains à la Hahn, so-called partial-lexicographic product constructions will be introduced. Roughly, only a cancellative subalgebra of a commutative group-like residuated chain is used as a first component of a lexicographic product, and the rest of the algebra is left unchanged. This results in group-like FL_e-algebras, see Theorem 2. The main theorem is about the structure of order-dense group-like FL_e-chains with a finite number of idempotents: Each such algebra can be constructed by iteratively using the partial-lexicographic product constructions using totally ordered Abelian groups as building blocks, see Theorem 3. This result extends the famous structural description of totally ordered Abelian groups by Hahn [14], to order-dense group-like commutative residuated chains with finitely many idempotents. The result is quite surprising.

Residuated lattices were introduced in the 30s of the last century by Ward and Dilworth [30] to investigate ideal theory of commutative rings with unit. After a few decades of slow development of the field a book dedicated to residuation appeared in 1972 [4]. Examples of residuated lattices include Boolean algebras, Heyting algebras [26], complemented semigroups [8], bricks [6], residuation groupoids [9], semiclans [7], Bezout monoids [3], MV-algebras [10], BL-algebras [15], and lattice-ordered groups; a number of other algebraic structures can be rendered as residuated lattices. Recently the investigation of residuated lattices (roughly, residuated monoids on

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lattices) got a new impetus and has been staying in the focus of strong attention. Beyond the algebraic interest, the reason is that residuated lattices turned out to be algebraic counterparts of substructural logics [29, 28]. Substructural logics encompass among many others, classical logic, intuitionistic logic, relevance logics, many-valued logics, mathematical fuzzy logics, linear logic, and their non-commutative versions. These logics had different motivations and methodology. The theory of substructural logics has put all these logics, along with many others, under the same motivational and methodological umbrella. Residuated lattices themselves have been the key component in this remarkable unification. A monograph devoted to residuated lattices and substructural logics appeared in 2007 [13]. Applications of substructural logics and residuated lattices span across proof theory, algebra, and computer science. FL_e-algebras are commutative residuated lattices with an additional constant [13]. FL_e-algebras with an involutive negation have very interesting symmetry properties [18, 19, 17] and, as a consequence, among involutive FL_e-algebras we have beautiful geometric constructions which are lacking for general FL_e-algebras [18, 24]. Furthermore, not only involutive FL_e-algebras have very interesting symmetry properties, but some of their logical calculi have important symmetry properties, too: both sides of a (Gentzen) sequent may contain more than one formula, while (hyper)sequent calculi for their non-involutive counterparts admit at most one formula on the right. As for the *classification* of residuated lattices, as one naturally expects, this is possible only by imposing additional postulates. Some related results are in [16, 1, 12, 11, 27, 2, 25, 23, 21], for a more detailed discussion, see [21].

A commutative binary operation $\circledast$ on a poset $(X, \leq)$ is called *residuated* if there exists another binary operation $\to_\circledast$ on X such that for $x, y, z \in X$, $x \circledast y \leq z$ iff $y \to_\circledast z \geq x$. $(X, \circledast, \to_\circledast, \wedge, \vee, t, f)$ is an *FL_e-algebra* if $(X, \wedge, \vee)$ is a lattice, $(X, \leq$ [1]$, \circledast, t)$ is a commutative, residuated monoid, and f is an arbitrary constant. An FL_e-algebra is *involutive*, if for $x \in X$, $(x')' = x$ holds, where $'$, the so-called residual complement is defined by $x' = x \to_\circledast f$. An involutive FL_e-algebra is *group-like*, if $t = f$[2]. Denote Γ the lexicographic product.

Theorem 1. *For a group-like FL_e-algebra $(X, \wedge, \vee, \circledast, \to_\circledast, t, f)$ the following statements are equivalent: $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered Abelian group if and only if $\circledast$ is cancellative if and only if $x \to_\circledast x = t$ for all $x \in X$ if and only if the only idempotent element in the positive cone of X is t.*

Definition 1. *(Partial-lexicographic products)* Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^{*}$ and $'^{\star}$, respectively. Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$, then add a bottom element $\bot$ to $Y \cup \{\top\}$, and extend $\star$ by $\bot \star y = y \star \bot = \bot$ for $y \in Y \cup \{\bot, \top\}$. Let $\mathbf{X}_1 = (X_1, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be any cancellative subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X}_1$ is a lattice ordered group). We define $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\bot\top})} = \left(X_{\Gamma(X_1, Y^{\bot\top})}, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y)\right)$, where $X_{\Gamma(X_1, Y^{\bot\top})} = (X_1 \times (Y \cup \{\bot, \top\})) \cup ((X \setminus X_1) \times \{\bot\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\bot, \top\}}$ to $X_{\Gamma(X_1, Y^{\bot\top})}$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1, y_1) \to_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} (x'^{*}, y'^{\star}) & \text{if } x \in X_1 \\ (x'^{*}, \bot) & \text{if } x \notin X_1 \end{cases} .$$

Call $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\bot\top})}$ the *(type-I) partial-lexicographic product* of X, X_1, and Y, respectively.

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[1] $\leq$ is derived from the lattice operators.

[2] For instance, lattice ordered groups equipped with $x \to_\circledast y := y \circledast x^{-1}$ are group-like.

Let $\mathbf{X} = (X, \leq_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-chain, $\mathbf{Y} = (Y, \leq_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement ${}'^{*}$ and ${}'^{\star}$, respectively. Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$. Further, let $\mathbf{X_1} = (X_1, \wedge, \vee, *, \to_*, t_X, f_X)$ be a cancellative, discrete, prime[3] subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X_1}$ is a discrete lattice ordered group). We define $\mathbf{X_{\Gamma(X_1,Y^\top)}} = \left(X_{\Gamma(X_1,Y^\top)}, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y)\right)$, where $X_{\Gamma(X_1,Y^\top)} = (X_1 \times (Y \cup \{\top\})) \cup ((X \setminus X_1) \times \{\top\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\top\}}$ to $X_{\Gamma(X_1,Y)}$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1, y_1) \to_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x,y)' = \begin{cases} ((x'^{*}), \top) & \text{if } x \notin X_1 \text{ and } y = \top \\ (x'^{*}, y'^{\star}) & \text{if } x \in X_1 \text{ and } y \in Y \\ ((x'^{*})_\downarrow, \top) & \text{if } x \in X_1 \text{ and } y = \top \end{cases}$$

and

$$x_\downarrow = \begin{cases} u & \text{if there exists } u < x \text{ such that there is no element in } X \text{ between } u \text{ and } x, \\ x & \text{if for any } u < x \text{ there exists } v \in X \text{ such that } u < v < x \text{ holds.} \end{cases} .$$

Call $\mathbf{X_{\Gamma(X_1,Y^\top)}}$ the *(type-II) partial-lexicographic product* of X, X_1, and Y, respectively.

Theorem 2. *$\mathbf{X_{\Gamma(X_1,Y^{\perp\top})}}$ and $\mathbf{X_{\Gamma(X_1,Y^\top)}}$ are involutive FL_e-algebras. If $\mathbf{Y}$ is group-like then also $\mathbf{X_{\Gamma(X_1,Y^{\perp\top})}}$ and $\mathbf{X_{\Gamma(X_1,Y^\top)}}$ are group-like.*

Theorem 3. *Any order-dense group-like FL_e-chain which has only a finite number of idempotents can be built by iterating finitely many times the partial-lexicographic product constructions using only totally ordered groups, as building blocks. More formally, let $\mathbf{X}$ be an order-dense group-like FL_e-chain which has $n \in \mathbf{N}$ ($n \geq 1$) idempotents in its positive cone. Denote $I = \{\perp\top, \top\}$. For $i \in \{1, 2, \ldots, n\}$ there exist totally ordered Abelian groups $\mathbf{G}_i$, $\mathbf{H}_1 \leq \mathbf{G}_1$, $\mathbf{H}_i \leq \mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i)$ ($i \in \{2, \ldots, n-1\}$), and a binary sequence $\iota \in I^{\{2,\ldots,n\}}$ such that $\mathbf{X} \simeq \mathbf{X}_n$, where $\mathbf{X}_1 := \mathbf{G}_1$ and $\mathbf{X}_i := \mathbf{X}_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i{}^{\iota_i})}$ ($i \in \{2, \ldots, n\}$).*

The proof is constructive.

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