On a meaningful axiomatic derivation of the Doppler effect and other scientific equations

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Abstract

The mathematical expression of a scientific or geometric law typically does not depend on the units of measurement. This makes sense because measurement units have no representation in nature. Any mathematical model or law whose form would be fundamentally altered by a change of units would be a poor representation of the empirical world. This paper formalizes this invariance of the form of the laws as a *meaningfulness* axiom. In the context of this axiom, relatively weak, intuitive constraints may suffice to generate standard scientific or geometric formulas, possibly up to some numerical parameters. We give several example of such constructions, with a focus of the Doppler effect and some other relativistic formulas.

When properly formalized, the invariance of the mathematical form of a scientific or geometric law under changes of units becomes a powerful '*meaningfulness*' axiom. Combining this meaningfulness axiom with abstract, intuitive, 'gedanken' properties such as associativity, permutability, bisymmetry, or other conditions in the same vein, enables the derivation of scientific or geometrical laws (possibly up to some parameter values). In the last section of this paper, I will show how, in the context of meaningfulness, the axiom

$$L(L(\lambda, v), w) = L(\lambda, v \oplus w) \tag{1}$$

yields specific numerical expressions for the function L and the operation $\oplus$.

Equation (1) is an *abstract axiom* representing the mechanisms conceivably involved in the Doppler effect (Feynman, Leighton, and Sands, 1963, Vol. 1). The operation $\oplus$ represents the relativistic addition of velocities. The left hand side of Equation (1) formalizes an iteration of the function L. The equation states that such an iteration amounts to adding a velocity via the relativistic addition of velocities operation.

A. Motivating the meaningfulness condition

The trouble with an equation such as (for example)

$$\text{[LF]} \qquad L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2}, \tag{2}$$

representing the Lorentz-FitzGerald Contraction is its ambiguity: the units of ℓ, which denotes the length of an object, and of v and c, for the speed of the observer and the

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speed of light, are not specified. Writing $L(70, 3)$ has no empirical meaning if one does not specify, for example, that the pair $(70, 3)$ refers to 70 meters and 3 kilometers per second, respectively. While such a parenthetical reference is standard in a scientific context, it is not instrumental for our purpose, which is to express, formally, an invariance with respect to any change in the units. To rectify the ambiguity, I propose to interpret

$$L(\ell, v) \qquad \text{as a shorthand notation for} \qquad L_{1,1}(\ell, v),$$

in which ℓ and L on the one hand, and v on the other hand, are measured in terms of two particular *initial* or *anchor* units fixed arbitrarily. Such units could be m (meter) and km/sec, if one wishes. The '1, 1' index of $L_{1,1}$ signifies these initial units.

Describing the phenomenon in terms of other units means that we multiply ℓ and v in any pair (ℓ, v) by some positive constants α and β, respectively. At the same time, L also gets to be multiplied by α, and the speed of light c by β. Doing so defines a new function $L_{\alpha,\beta}$, which is different from $L = L_{1,1}$ if either $\alpha \neq 1$ or $\beta \neq 1$ (or both).

But, from an empirical standpoint, $L_{\alpha,\beta}$ carries exactly the same information as $L_{1,1}$. For example, if our new units are *km* and *m/sec*, then the two expressions

$$L_{10^{-3},10^3}(.07, 3000) \quad \text{and} \quad L(70, 3) = L_{1,1}(70, 3),$$

while numerically not equal, describe the same empirical situation.

This points to the appropriate definition of $L_{\alpha,\beta}$ in the case of the Lorentz-FitzGerald Contraction. It turns out (see Definition 3) that we should write:

$$L_{\alpha,\beta}(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{\beta c}\right)^2}. \tag{3}$$

The connection between L and $L_{\alpha,\beta}$ is actually:

$$\frac{1}{\alpha}L_{\alpha,\beta}(\alpha\ell, \beta v) = \left(\frac{1}{\alpha}\right)\alpha\ell\sqrt{1 - \left(\frac{\beta v}{\beta c}\right)^2} = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2} = L(\ell, v).$$

Writing $\mathbb{R}_{++}$ for the set of positive real numbers and $\mathbb{R}_+$ for the set of non negative real numbers, this implies, for any α, β, ν and μ in $\mathbb{R}_{++}$,

$$\frac{1}{\alpha}L_{\alpha,\beta}(\alpha\ell, \beta v) = \frac{1}{\nu}L_{\nu,\mu}(\nu\ell, \mu v), \qquad (\alpha\ell, \nu\ell \in \mathbb{R}_+, \beta v \in [0, \beta c[\,, , \mu v \in [0, \mu c[\,). \tag{4}$$

which is a special case of the invariance equation we were looking for, in the particular case of the Lorentz-FitzGerald Contraction Equation (or in the cases of the Doppler Effect or Beer's Law).

1 Remark. Looking at Equation (4), one might object that going in that direction would render the scientific or geometric notation very awkward. But the awkwardness is only temporary. When we have extracted all the useful consequences from the meaningfulness axiom, we can go back to the usual notation. In fact, we already have the equation permitting to retrieve our usual notation. Indeed, Equation (4) implies

$$\tfrac{1}{\alpha}L_{\alpha,\beta}(\alpha\ell, \beta v) = L_{1,1}(\ell, v) = L(\ell, v).$$

Note that the concept of meaningfulness is of course related to standard physical concepts such as dimensional analysis. I will not deal with this issue here, but see Narens (1981, 1988, 2002, 2007).

B. Defining meaningfulness

Our example of the Lorentz-FitzGerald equation made clear that the concept of meaningfulness must apply to a **collection** of scientific or geometric functions (we call them *codes* here), and not to a particular function.

2 Definition. Suppose that J_1, J_2, and J_3 are three non-negative real intervals, and let

$$\mathcal{F} = \{F_{\alpha,\beta} \,|\, \alpha, \beta \in \mathbb{R}_{++}\}$$

be a collection of *codes*, with the *initial code*

$$F = F_{1,1} : J_1 \times J_2 \xrightarrow{\text{onto}} J_3$$

strictly monotonic in both variables.

Each of α and β indexing a code $F_{\alpha,\beta}$ in $\mathcal{F}$ represents a change of the unit of one of the two measurement scales[1].

Let δ_1 and δ_2 be two of rational numbers. The collection of codes $\mathcal{F}$ defined above is (δ_1, δ_2)-*meaningful* if for any $(x_1, x_2) \in J_1 \times J_2$ and $(\alpha, \beta), (\mu, \nu) \in \mathbb{R}^2_{++}$, we have

$$\frac{1}{\alpha^{\delta_1}\beta^{\delta_2}} F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \frac{1}{\mu^{\delta_1}\nu^{\delta_2}} F_{\mu,\nu}(\mu x_1, \nu x_2) = F_{1,1}(x_1, x_2)$$

which yields

$$F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \alpha^{\delta_1}\beta^{\delta_2} F_{1,1}(x_1, x_2) = \alpha^{\delta_1}\beta^{\delta_2} F(x_1, x_2).$$

The role of δ_1 and δ_2 is to specify the measurement scale of the function $F_{\alpha,\beta}$ relative to those of its two variables. In the case of the Lorentz-FitzGerald and similar equations, the measurement scale of the code is the same as that of the first variable. The relevant definition is given below.

3 Definition. A meaningful collection of codes, with $\mathcal{F} = \{F_{\alpha,\beta} \,|\, \alpha, \beta \in \mathbb{R}_{++}\}$ as in the previous definition, is called $(1, 0)$-*meaningful* if it is (δ_1, δ_2)-meaningful with $\delta_1 = 1$ and $\delta_2 = 0$. We have then, for any $(x_1, x_2) \in J_1 \times J_2$ and $(\alpha, \beta), (\mu, \nu) \in \mathbb{R}^2_{++}$,

$$\frac{1}{\alpha^1\beta^0} F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \frac{1}{\mu^1\nu^0} F_{\mu,\nu}(\mu x_1, \nu x_2),$$

$$\Longleftrightarrow$$

$$\frac{1}{\alpha} F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \frac{1}{\mu} F_{\mu,\nu}(\mu x_1, \nu x_2), \qquad (\alpha^1\beta^0 = \alpha,\ \mu^1\nu^0 = \mu)$$

$$= F_{1,1}(x_1, x_2)$$

which yields

$$F_{\alpha,\beta}(x_1, x_2) = \alpha F_{1,1}\left(\frac{x_1}{\alpha}, \frac{x_2}{\beta}\right).$$

Such collections are also called *ST-meaningful*, with *ST* standing for *self transforming*.

Many scientific or geometric laws are self transforming. We give several examples in this paper.

[1] In this paper, we only deal with scientific or geometric functions in two variables, and with ratio measurement scales.

C. As an introduction: the Pythagorean Theorem

One example of an abstract axiom is the *associativity equation:*

$$F(F(x,y),z) = F(x,F(y,z)) \qquad\qquad (x,y,z \in \mathbb{R}_{++})$$

which can be shown to hold for right triangles, with each of

$$F(x,y), \quad F(x,z), \quad F(F(x,y),z) \quad \text{and} \quad F(x,F(y,z))$$

denoting the measures of the hypothenuses of a right triangle as functions of the two sides of the respective right angles. In the figure below, $F(x,y)$ denotes the length of the hypothenuse of the right triangle $\triangle ABC$, with sides lengths x and y, while $F(y,z)$ denotes the length of the hypothenuse of the right triangle $\triangle BCD$.

The two remaining triangles: $\triangle ABD$, with sides lengths x and $F(y,z)$, and $\triangle ACD$, with sides lengths z and $F(x,y)$, have the common hypothenuse AD. Its length is

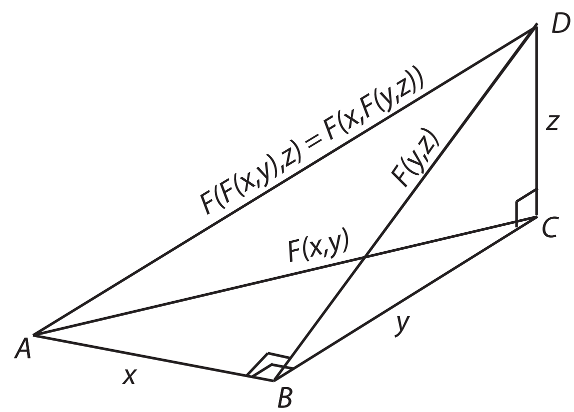
$$F(F(x,y)z) = F(x,F(y,z)). \tag{5}$$

This shows that the hypothenuse of a right triangle is an associative function of (the lengths of) its two sides.

Using functional equations arguments (Aczél, 1966, Section 6.2), we can prove that, for some continuous strictly increasing function f on the set $\mathbb{R}$ of real numbers, the associativity equation (5) has a representation

$$F(x,y) = f^{-1}(f(x) + f(y))$$

an equation generalizing the Pythagorean Theorem.



Under meaningfulness, and in the context of reasonable background conditions, we can prove that the function F has the form

$$F(x,y) = (x^\eta + y^\eta)^{\frac{1}{\eta}},$$

(see Theorem 5 below). The exact statement requires recalling some conditions.

4 Definition. A code $F : \mathbb{R}_{++} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is

symmetric if $\qquad F(x,y) = F(y,x) \qquad$ for $x, y \in \mathbb{R}_{++}$.

Such a code is

homogeneous if $\qquad F(\theta x, \theta y) = \theta F(y,x) \qquad$ for $x, y, \theta \in \mathbb{R}_{++}$.

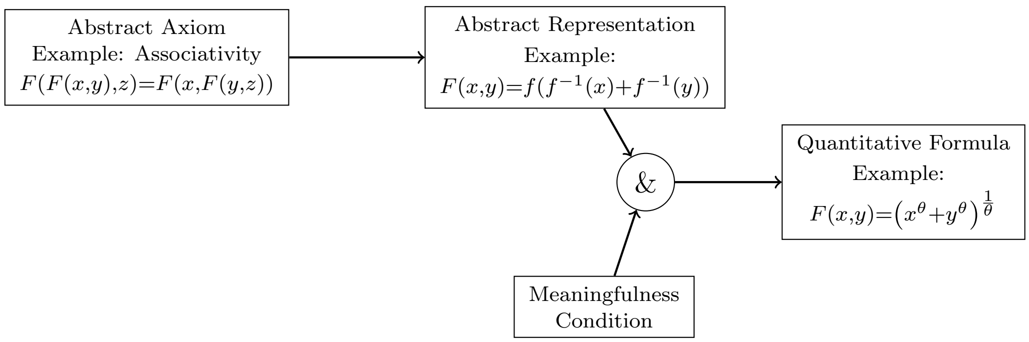
5 Theorem. *Suppose that $\mathcal{F} = \{F_\alpha \,|\, \alpha \in \mathbb{R}_{++}\}$ is a $(\frac{1}{2}, \frac{1}{2})$-ST-meaningful collection of codes, with $F_\alpha : \mathbb{R}_{++} \times \mathbb{R}_{++} \xrightarrow{onto} \mathbb{R}_{++}$ for all α in $\mathbb{R}_{++}$. If one of these codes is strictly increasing in both variables, symmetric, homogeneous and associative, then any code $F_\alpha \in \mathcal{F}$ must have the form*

$$F_\alpha(x,y) = \left(x^\theta + y^\theta\right)^{\frac{1}{\theta}} = F(x,y)\,,$$

for some constant $\theta \in \mathbb{R}_{++}$.

For a proof, see Falmagne and Doble (2015a, Theorem 7.1.1, page 85). The fact that we must have $\theta = 2$ can be derived from the Area of the Square Postulate and a couple of other intuitively obvious postulates of geometry.

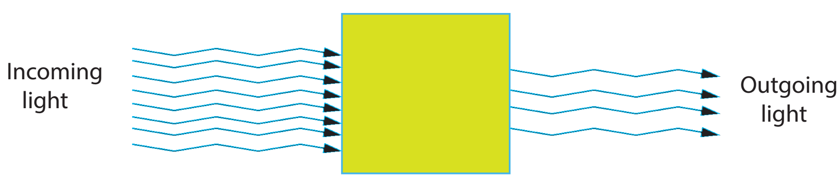
The proofs of Theorem 5 and a couple of other results given in this paper follow the schema illustrated by the next graph.



Proof schema: An abstract axiom yields an abstract representation. The latter, paired with a meaningfulness condition leads, via functional equation arguments, to one or a couple of potential scientific laws specified up to the value(s) of numerical parameter(s).

D. Another example: The Translation Equation for Beer's law

Beer's law, also known as Beer-Lambert law, Lambert-Beer law, or Beer-Lambert-Bouguer law is an equation describing the attenuation of light resulting from the properties of the material through which the light is traveling. (See the figure below.)



Following the guidelines of the Proof Schema, we first formulate the abstract axiom.

6 Definition. Let J and J' be two non-negative real intervals. A code $F : J \times J' \to J$ is *translatable*, or equivalently, satisfies the *translation equation*[2] if

$$F(F(x,y),z) = F(x, y+z) \qquad (x \in J,\ y, z, y+z \in J'). \tag{6}$$

An example of a translatable code is Beer's Law:

$$I(x,y) = x\,\mathrm{e}^{-\frac{y}{c}}. \tag{7}$$

Indeed, we have

$$I(I(x,y),z) = I(x,y)\,\mathrm{e}^{-\frac{z}{c}} = x\,\mathrm{e}^{-\frac{y}{c}}\,\mathrm{e}^{-\frac{z}{c}} = x\,\mathrm{e}^{-\frac{y+z}{c}} = I(x, y+z).$$

Next, we need the abstract representation in this case. It is formulated in the next lemma.

[2] See Aczél (1966, page 245) for this concept and for the proof of Lemma 7.

7 Lemma. *Let $F : J \times J' \to H$ be a code such that $J' =]d, \infty[$ for some $d \in \mathbb{R}_+$, and for some $a \in \mathbb{R}_+$, either $J =]a, b]$ for some $b \in \mathbb{R}_{++}$ or $J =]a, \infty[$, with $F(x, y)$ strictly decreasing in y.*

Then, the code $F : J \times J' \to H$ is translatable if and only if there exists a function f satisfying the equation

$$F(x, y) = f(f^{-1}(x) + y) .$$

Injecting now the meaningfulness condition, we obtain our quantitative formula.

8 Theorem. *Let $\mathcal{F} = \{F_{\mu,\nu} \mid \mu, \nu \in \mathbb{R}_{++}\}$ be a $(1, 0)$-meaningful ST-collection of codes, with $F_{\mu,\nu} : \mathbb{R}_{++} \times \mathbb{R}_{++} \xrightarrow{onto} \mathbb{R}_{++}$. Suppose that one of these codes, say the code $F_{\mu,\nu}$, is strictly decreasing in the second variable, translatable, and left homogeneous of degree one, that is: for any a in $\mathbb{R}_{++}$, we have $F_{\mu,\nu}(ax, y) = aF_{\mu,\nu}(x, y)$. Then there is a positive constant c such that the initial code F has the form*

$$F(x, y) = x\, \mathrm{e}^{-\frac{y}{c}};$$

so for any code $F_{\alpha,\beta} \in \mathcal{F}$, we have

$$F_{\alpha,\beta}(x, y) = x\, \mathrm{e}^{-\frac{y}{\beta c}}.$$

We summarize below the proof contained in Falmagne and Doble (see the proof of Theorem 7.4.1, page 98, 2015a).

Sketch of proof. We first show that, if one of the codes in the collection $\mathcal{F}$ is translatable, then by the meaningfulness condition, the translatability condition *propagates* to all the codes in the collection. Without loss of generality, we suppose that the initial code $F = F_{1,1}$ is translatable.
Successively, we have for any code $F_{\alpha,\beta}$ in $\mathcal{F}$:

$$\begin{aligned}
F_{\alpha,\beta}(F_{\alpha,\beta}(x, y), z) &= \alpha F\left(\frac{F_{\alpha,\beta}(x, y)}{\alpha}, \frac{z}{\beta}\right) && \text{(by ST-meaningfulness)}\\
&= \alpha F\left(F\left(\frac{x}{\alpha}, \frac{y}{\beta}\right), \frac{z}{\beta}\right) && \text{(by ST-meaningfulness)}\\
&= \alpha F\left(\frac{x}{\alpha}, \frac{y}{\beta} + \frac{z}{\beta}\right) && \text{(by the translatability of } F\text{)}\\
&= F_{\alpha,\beta}(x, y + z) && \text{(by ST-meaningfulness)} .
\end{aligned}$$

So, $F_{\alpha,\beta}$ is translatable. By meaningfulness, we can also show that left homogeneity of degree one propagates to all the codes in the collection $\mathcal{F}$. (We omit this part of the proof.)

Because $F_{\alpha,\beta}$ is translatable, Lemma 7 implies that there exists a strictly decreasing function $f_{\alpha,\beta} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that

$$\begin{aligned}
F_{\alpha,\beta}(ax, y) &= f_{\alpha,\beta}(f^{-1}_{\alpha,\beta}(ax) + y)\\
&= af_{\alpha,\beta}(f^{-1}_{\alpha,\beta}(x) + y) = aF_{\alpha,\beta}(x, y) \qquad \left(\begin{smallmatrix}\text{by left homogeneity}\\ \text{of } F_{\alpha,\beta}\end{smallmatrix}\right).
\end{aligned}$$

Set $f^{-1}_{\alpha,\beta}(x) = w$, and so $f_{\alpha,\beta}(w) = x$. Applying $f^{-1}_{\alpha,\beta}$ on both sides of the second equation above, we get

$$(f_{\alpha,\beta}^{-1} \circ af_{\alpha,\beta})(w) + y = (f_{\alpha,\beta}^{-1} \circ af_{\alpha,\beta})(w + y),$$

or with $g_{a,\alpha,\beta} = (f_{\alpha,\beta}^{-1} \circ af_{\alpha,\beta})$,

$$g_{a,\alpha,\beta}(w) + y = g_{a,\alpha,\beta}(w + y),$$

a Pexider equation in the variables w and y. So,the function $g_{a,\alpha,\beta}$ is of the form

$$g_{a,\alpha,\beta}(w) = w + B(a, \alpha, \beta).$$

for some function $B(a, \alpha, \beta)$ which must be decreasing in a. Rewriting the last equation in terms of the function $f_{\alpha,\beta}$ yields

$$(f_{\alpha,\beta}^{-1} \circ af_{\alpha,\beta})(w) = w + B(a, \alpha, \beta)$$

or equivalently, with $x = f_{\alpha,\beta}(w)$, we get

$$f_{\alpha,\beta}^{-1}(ax) = f_{\alpha,\beta}^{-1}(x) + B(a, \alpha, \beta),$$

another Pexider equation (c.f. Aczél, 1966, page 141) that is, an equation of the form: $h(ax) = h(x) + g(a)$. By functional equations arguments, the equation

$$f_{\alpha,\beta}^{-1}(ax) = f_{\alpha,\beta}^{-1}(x) + B(a, \alpha, \beta),$$

implies for some constants $k(\alpha, \beta) > 0$ and $b(\alpha, \beta)$,

$$f_{\alpha,\beta}^{-1}(x) = -k(\alpha, \beta) \ln x + b(\alpha, \beta)$$

which gives us, with $t = f_{\alpha,\beta}^{-1}(x)$,

$$f_{\alpha,\beta}(t) = \mathrm{e}^{\frac{t-b(\alpha,\beta)}{-k(\alpha,\beta)}}.$$

So, we get

$$F_{\alpha,\beta}(x, y) = f_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(x) + y) = x\,\mathrm{e}^{-\frac{y}{k(\alpha,\beta)}}$$

after some manipulation. By the left homogeneity of $F_{\alpha,\beta}$ and the ST-meaningfulness of the family $\mathcal{F}$, we must have

$$\tfrac{1}{\alpha}F_{\alpha,\beta}(\alpha x, \beta y) = F_{\alpha,\beta}(x, \beta y) = x\,e^{-\frac{\beta y}{k(\alpha,\beta)}} = F(x, y).$$

The last equation shows that $\frac{\beta}{k(\alpha,\beta)}$ does not depend upon α or β.
Defining $c = \frac{k(\alpha,\beta)}{\beta}$, we finally obtain

$$F(x, y) = x\,\mathrm{e}^{-\frac{y}{c}}.$$

Accordingly, we obtain for any code $F_{\mu,\nu} \in \mathcal{F}$, using left homogeneity of degree 1 in the second equation below

$$F_{\mu,\nu}(x, y) = \mu F\left(\tfrac{x}{\mu}, \tfrac{y}{\nu}\right) = F\left(x, \tfrac{y}{\nu}\right) = x\,\mathrm{e}^{-\frac{y}{\nu c}}. \qquad \square$$

Various other results in the same vein are reported in Falmagne and Doble (2015a) (see also Falmagne, 2015b).

The last two lines of the table below summarizes some of these results. The functional equations results mentioned in the second (abstract representation) column of the table may be found, together with a considerable list of other results and extended references, in Janos Azcél's classic volume (Aczél, 1966).

Name and formula of abstract axiom	Abstract representation: $\exists$ functions f, m, g, etc.	Resulting possible scientific laws[2]
Associativity $F(F(x,y),z)=F(x,F(y,z))$	$F(x,y)=f(f^{-1}(x)+f^{-1}(y))$	$F(x,y)=(y^{\eta}+x^{\eta})^{\frac{1}{\eta}}$
Translatability $F(F(x,y),z)=F(x,y+z)$	$F(x,y)=f(f^{-1}(x)+y)$	$F(x,y)=x\mathrm{e}^{-\frac{y}{c}}$
Quasi-permutability $F(G(x,y),z)=F(G(x,z),y)$	$F(x,y)=m(f(x)+g(y))$	$F(x,y)=(x^{\eta}+\lambda y^{\eta}+\theta)^{\frac{1}{\eta}}$ or $F(x,y)=\phi x y^{\gamma}$ or $(x^{\eta}+y^{\eta})^{\frac{1}{\eta}}$
Bisymmetry $F(F(x,y),F(z,w))=F(F(x,z),F(y,w))$	$F(x,y)=f\big((1-q)f^{-1}(x)+qf^{-1}(y)\big)$	$F(x,y)=(1-q)x^{\eta}+qy^{\eta})^{\frac{1}{\eta}}$ or $F(x,y)=x^{1-q}y^{q}$

E. Meaningful derivation of the relativistic Doppler effect formula

A *relativistic Doppler effect* occurs when an observer of a source of light with wavelength λ is in relative motion with respect to that source. Suppose that the observer and the source are moving toward each other at the speed v. The perceived wavelength $L(\lambda, v)$ increases in λ and decreases in v, according to the special relativity formula

$$\text{[DE]} \qquad L(\lambda, v) = \lambda\sqrt{\frac{c-v}{c+v}} \qquad (\lambda \in \mathbb{R}_{++},\ \ v \in [0, c[\,),$$

in which: c is the speed of light, λ is the wavelength of the light emitted by the source, and $L(\lambda, v)$ is the wavelength of that light measured by the observer (cf. Ellis and Williams, 1966; Feynman, Leighton, and Sands, 1963).

Our goal is this section is to derive Formula [DE] (up to its $\frac{1}{2}$ exponent), from a meaningfulness axiom, some background constrains, and the abstract condition

$$\text{[R]} \qquad L(L(\lambda, v), w) = L(\lambda, v \oplus w).$$

mentioned earlier in this paper.

Some recent papers dealing with the axiomatization of special relativity concepts are Andréka et al. (2006a,b, 2008) and Moriconi (2006). In the first three papers, the axiomatization is based on a logical analysis, while in the last one, it is grounded in physical principles. The motivation of the present paper is different in that meaningfulness plays a key role. As mentioned in our introductory paragraph, our aim was to show how the combination of a meaningfulness axiom with an abstract, possibly intuitive condition, would result—via an abstract representation of the abstract condition—in an explicit physical or geometric law (possibly up to real parameters).

It may not be obvious why Condition [R] is relevant to the situation inducing the Doppler effect in the guise of Formula [DE]. However we will see in Theorem 10 that Condition [R] is equivalent to the formula

$$\text{[M]} \qquad L(\lambda, v) \leq L(\lambda', v') \iff L(\lambda, v \oplus w) \leq L(\lambda', v' \oplus w),$$

which may seem intuitively more consistent with that situation.

9 Definition. Let $L : \mathbb{R}_{++} \times [0, c[\to \mathbb{R}_{++}$ be a *code*, with $c > 0$ a constant standing for the speed of light. The code L is a *Doppler Function* if there is a binary operator $\oplus : [0, c[\times [0, c[\to [0, c[$ such that the pair $(L, \oplus)$ satisfies the following five conditions:

1. The function L is strictly increasing in the first variable, strictly decreasing in the second variable, continuous in both variables, and for all $\lambda, \lambda' \in \mathbb{R}_+$ and $v, v' \in [0, c]$, and for any $a > 0$, we have
$$L(\lambda, v) \leq L(\lambda', v') \quad \Longleftrightarrow \quad L(a\lambda, v) \leq L(a\lambda', v').$$
2. $L(\lambda, 0) = \lambda$ for all $\lambda \in \mathbb{R}_+$.
3. $\lim_{v \to c} L(\lambda, v) = 0$.
4. The operation $\oplus$ is continuous, commutative, strictly increasing in both variables, and has 0 as an identity element.
5. Either Axiom [R] or Axiom [M] below is satisfied for $\lambda, \lambda' > 0$, and v, v', $w \in [0, c[$:

   [R] $L(L(\lambda, v), w) = L(\lambda, v \oplus w)$;

   [M] $L(\lambda, v) \leq L(\lambda', v') \Longleftrightarrow L(\lambda, v \oplus w) \leq L(\lambda', v' \oplus w)$.

When these five conditions are satisfied, the pair $(L, \oplus)$ is called an *abstract Doppler-pair.*

In words, Axioms [R] and [M] state the following ideas.

> Axiom [R]: *One iteration of the function L involving two velocities v and w has the same effect on the perceived length as adding v and w via the operation $\oplus$.*
>
> Axiom [M]: *Adding a velocity via the operation $\oplus$ preserves the order of the function L.*

10 Theorem. *Suppose that $(L, \oplus)$ is an abstract Doppler-pair. Then the following equivalences hold:*
$$[\text{R}] \quad \Longleftrightarrow \quad \big([\text{DE}^\dagger] \ \& \ [\text{AV}^\dagger]\big) \quad \Longleftrightarrow \quad [\text{M}],$$
with for some strictly increasing and continuous function u and some positive constant ξ:

$[\text{DE}^\dagger]$ $\qquad L(\lambda, v) = \lambda \left(\frac{c - u(v)}{c + u(v)}\right)^{\xi};$

$[\text{AV}^\dagger]$ $\qquad v \oplus w = u^{-1}\left(\frac{u(v) + u(w)}{1 + \frac{u(v)u(w)}{c^2}}\right).$

(For a proof, see Falmagne and Doignon, 2010).

We now have the representation formulas for the abstract axioms [R] and [M]. The next definition introduces the meaningful collection with initial pair $(L, \oplus)$.

11 Definition. Let $\mathcal{L} = \{L_{\mu,\nu} \,|\, \mu, \nu \in \mathbb{R}_{++}\}$ be a ST-meaningful collection of codes, with $L_{\mu,\nu} : \mathbb{R}_{++} \times [0, c[\xrightarrow{\text{onto}} \mathbb{R}_{++}$ and $c \in \mathbb{R}_{++}$. Let $\mathcal{O} = \{\oplus_\nu \,|\, \nu \in \mathbb{R}_{++}\}$ be a $(\frac{1}{2}, \frac{1}{2})$-meaningful collection of operators, with
$$\oplus_\nu : [0, c[\times [0, c[\xrightarrow{\text{onto}} [0, c[\quad \text{and} \quad v \oplus_\nu w = \nu\left(\tfrac{v}{\nu} \oplus \tfrac{w}{\nu}\right) \qquad (\nu \in \mathbb{R}_{++}, \ v, w \in [0, c[\,).$$

Suppose that each code $L_{\mu,\nu} \in \mathcal{L}$ is paired with a binary operation $\oplus_\nu \in \mathcal{O}$, forming an ordered pair $(L_{\mu,\nu}, \oplus_\nu)$, with the initial ordered pair $(L_{1,1}, \oplus_1) = (L, \oplus)$. Then the pair of collections $(\mathcal{L}, \mathcal{O})$ is called a *meaningful Doppler-system.*

Note that the measurement scale of the operation $\oplus_\nu$ is the same as that of the second variable of the function $L_{\mu,\nu}$.

12 Remark. In the proof of the next lemma, we have as the first equation

$$L_{\alpha,\beta}(\lambda, v) = \alpha L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right) \tag{8}$$

which is equivalent to

$$L_{\alpha,\beta}(\alpha\lambda, \beta v) = \alpha L(\lambda, v). \tag{9}$$

By definition, the domain of the function L in Equation (9) is $\mathbb{R}_+ \times [0, c[$ with $v \in [0, c[$. But in the r.h.s. of Equation (8), we cannot have $\frac{v}{\beta} \in [0, c[$ since we have

$$0 \leq v < c \quad \Longleftrightarrow \quad 0 \leq \tfrac{v}{\beta} < \tfrac{c}{\beta}.$$

(Assuming that $\frac{v}{\beta} \in [0, c[$ would lead to a contradiction.) So, the upper bound of the second variable in $L(\frac{\lambda}{\alpha}, \frac{v}{\beta})$ is now $\frac{c}{\beta}$. This point is also relevant to the second equation in Formula (12) in the proof of Theorem 14.

A similar remark applies to the two functions $L_{\alpha,\beta}$ in the l.h.s. of (8) and (9) .

13 Propagation lemma for abstract Doppler-pairs. *Suppose that one ordered pair $(L_{\mu,\nu}, \oplus_\nu)$ from a meaningful Doppler-system $(\mathcal{L}, \mathcal{O})$ is an abstract Doppler-pair, that is, $(L_{\mu,\nu}, \oplus_\nu)$ satisfies Conditions 1-5 of the definition of an abstract Doppler-pair. Then any ordered pair $(L_{\alpha,\beta}, \oplus_\beta)$, with $L_{\alpha,\beta} \in \mathcal{L}$ and $\oplus_\beta \in \mathcal{O}$, is also such an abstract Doppler-pair.*

So, meaningfulness enables the propagation of all five conditions to any ordered pair $(L_{\alpha,\beta}, \oplus_\beta)$ in a meaningful Doppler-system $(\mathcal{L}, \mathcal{O})$.

Proof. Without loss of generality, we can assume that the ordered pair $(L, \oplus)$ of initial code L is an abstract Doppler-pair, and so satisfies the five conditions of Definition 9. By meaningfulness, we have: $L_{\alpha,\beta}(\lambda, v) = \alpha L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right)$ and $v \oplus_\beta w = \beta\left(\frac{v}{\beta} \oplus \frac{w}{\beta}\right)$. Conditions 1 to 4 readily follow. Condition 1 holds because, successively:

$$\begin{aligned}
L_{\alpha,\beta}(\lambda) \leq L_{\alpha,\beta}(\lambda', v') &\Longleftrightarrow \alpha L\left(a\frac{\lambda}{\alpha}, \frac{v}{\beta}\right) \leq \alpha L\left(a\frac{\lambda'}{\alpha}, \frac{v'}{\beta}\right) && \text{(by ST-meaningfulness)}\\
&\Longleftrightarrow \alpha L\left(a\frac{\lambda}{\alpha}, \frac{v}{\beta}\right) \leq \alpha L\left(a\frac{\lambda'}{\alpha}, \frac{v'}{\beta}\right) && \text{(by Condition 1 for } (L, \oplus))\\
&\Longleftrightarrow L_{\alpha,\beta}(a\lambda, v) \leq L_{\alpha,\beta}(a\lambda', v') && \text{(by ST-meaningfulness).}
\end{aligned}$$

For Condition 3, we have $\lim_{v \to c} L_{\alpha,\beta}(\lambda, v) = \alpha \lim_{\frac{v}{\beta} \to \frac{c}{\beta}} L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right) = 0$ (c.f. Remark 12). We omit the proofs of Conditions 2 and 4 which are straightforward consequences of ST-meaningfulness.

We turn to Condition 5. Since Axioms [R] and [M] are equivalent by Theorem 10, it suffices to prove that the ordered pair $(L_{\alpha,\beta}, \oplus_\beta)$ satisfies Axiom [R].

By the ST-meaningfulness of $\mathcal{L}$,

$$L_{\alpha,\beta}(L_{\alpha,\beta}(\lambda, v), w) = \alpha L\left(\frac{L_{\alpha,\beta}(\lambda, v)}{\alpha}, \frac{w}{\beta}\right) = \alpha L\left(\frac{\alpha L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right)}{\alpha}, \frac{w}{\beta}\right).$$

Canceling the α's in the fraction inside the parentheses in the r.h.s. gives

$$\begin{aligned}
L_{\alpha,\beta}(L_{\alpha,\beta}(\lambda, v), w) &= \alpha L\left(L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right), \frac{w}{\beta}\right) \\
&= \alpha L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta} \oplus \frac{w}{\beta}\right) && \left(\begin{matrix}\text{by Axiom [R]} \\ \text{applied to } L\end{matrix}\right) \\
&= \alpha L\left(\frac{\lambda}{\alpha}, \frac{1}{\beta}(v \oplus_\beta w)\right) && (\text{by the meaningfulness of } \mathcal{O}) \\
&= L_{\alpha,\beta}\left(\lambda, v \oplus_\beta w\right) && (\text{by the ST-meaningfulness of } \mathcal{L}). \quad \square
\end{aligned}$$

14 Representation Theorem. *Suppose that one ordered pair* $(L_{\mu,\nu}, \oplus_\nu)$ *from a meaningful Doppler-system* $(\mathcal{L}, \mathcal{O})$ *is an abstract Doppler-pair, that is,* $(L_{\mu,\nu}, \oplus_\nu)$ *satisfies Conditions* 1-5 *of Definition 9. Suppose also that* $L_{\mu,\nu}$ *does not vary with* ν*. Then, Axioms* [DE†] and [AV†] *of Theorem 10 become for the initial code L:*

$$\text{[DE]} \qquad L(\lambda, v) = \lambda\left(\frac{c-v}{c+v}\right)^{\xi} \qquad (\textit{with } \lambda \in \mathbb{R}_+,\ v \in [0, c[\textit{ and } \xi \in \mathbb{R}_{++}) \tag{10}$$

and for the operation $\oplus$*:*

$$\text{[AV]} \qquad v \oplus w = \frac{v+w}{1+\frac{vw}{c^2}} \qquad (\textit{with } v, w \in [0, c[\,). \tag{11}$$

(See Remark 15 for the condition: "$L_{\mu,\nu}$ *does not vary with* ν.")

Proof. Without loss of generality, we can assume that $(L, \oplus)$ is an abstract Doppler-pair, with L the initial code of the meaningful Doppler-system $(\mathcal{L}, \mathcal{O})$; that is, $(L, \oplus)$ satisfies the five conditions of Definition 9.

By ST-meaningulness, we have for any code $L_{\alpha,\beta}$:

$$L_{\alpha,\beta}(\lambda, v) = \alpha L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right) = \alpha\left(\frac{\lambda}{\alpha}\right)\left(\frac{\frac{c}{\beta} - u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta} + u\left(\frac{v}{\beta}\right)}\right)^{\xi} \qquad \left(\text{with } \frac{v}{\beta} \in \left[0, \frac{c}{\beta}\right[\right) \tag{12}$$

by Theorem 10 (c.f. Remark 12 concerning $0 \leq \frac{v}{\beta} < \frac{c}{\beta}$). So, we have

$$L_{\alpha,\beta}(\lambda, v) = \lambda\left(\frac{\frac{c}{\beta} - u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta} + u\left(\frac{v}{\beta}\right)}\right)^{\xi} \qquad \left(\text{with } \frac{v}{\beta} \in \left[0, \frac{c}{\beta}\right[\right). \tag{13}$$

Since $L_{\mu,\nu}$ does not vary with ν, $L_{\alpha,\beta}(\lambda, v)$ in the l.h.s. of (13) cannot depend upon β either. As the ratio

$$\frac{\frac{c}{\beta} - u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta} + u\left(\frac{v}{\beta}\right)} \qquad \left\{\begin{matrix}\text{is a function of } v \text{ only,} \\ \text{independent of } \beta, \\ \text{we must have}\end{matrix}\right\} \qquad g(v) = \frac{\frac{c}{\beta} - u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta} + u\left(\frac{v}{\beta}\right)}$$

for some function $g : [0, c[\to\ [0, c[$. Setting $\frac{1}{\beta} = z$ and rearranging, we get

$$u(vz) = zc\left(\tfrac{1-g(v)}{1+g(v)}\right)$$

and with $h(v) = c\,\frac{1-g(v)}{1+g(v)}$, we get the Pexider equation $u(vz) = zh(v)$, whose solution for the strictly increasing continuous function u is : $u(v) = \theta v$ for all $v \in]0, c[$ with $\theta > 0$.

Using the representation [DE†] from Theorem 10, we get

$$L(\lambda, v) = \lambda \left(\tfrac{c-u(v)}{c+u(v)}\right)^{\xi} = \lambda \left(\tfrac{c-\theta v}{c+\theta v}\right)^{\xi}.$$

But the code L must satisfy Condition 3 of an abstract Doppler-pair (Definition 9), which requires that $\lim_{v\to c} L(\lambda, v) = 0$. This implies

$$\lim_{v\to c} \lambda \left(\frac{c-\theta v}{c+\theta v}\right)^{\xi} = \lambda \left(\frac{c-\theta c}{c+\theta c}\right)^{\xi} = \lambda \left(\frac{1-\theta}{1+\theta}\right)^{\xi} = 0 \qquad \text{which holds only if } \theta = 1.$$

We conclude that the function u of Theorem 10 must be the identity function: $u(v) = v$.

Accordingly, the two equations [DE†] and [AV†] obtained in Theorem 10 from the representation of abstract Doppler-pairs become

$$\text{[DE]} \qquad L(\lambda, v) = \lambda \left(\frac{c-v}{c+v}\right)^{\xi}$$

$$\text{[AV]} \qquad v \oplus w = \frac{v+w}{1+\frac{vw}{c^2}}.$$

□

15 Remark. Recall that in Equation (3)

$$L_{\alpha,\beta}(\ell, v) = \ell\sqrt{1-\left(\tfrac{v}{\beta c}\right)^2}$$

the function $L_{\alpha,\beta}$ was strictly increasing in β. This suggests that if the condition "*$L_{\mu,\nu}$ does not vary with ν*" of Theorem 14 is replaced by : "*$L_{\mu,\nu}$ is strictly increasing with ν*", we might be able to derive the Lorentz-FitzGerald Contraction (up to its exponent) along lines similar to those of the above proof. In fact, if we define the function $g(v) = \frac{2u(v)}{c-u(v)}$, we can derive

$$L(\lambda, v) = \lambda \left(\tfrac{c-u(v)}{c+u(v)}\right)^{\xi} = \lambda \left(1 - g\left(\tfrac{v}{c}\right)\right)^{\xi},$$

whose r.h.s. generalizes the Lorentz-FitzGerald equation. However, this result would be combined with a different formula for the relativistic addition of velocities. Indeed, one of the consequences of Theorem 14 is that the Lorentz-FitzGerald Contraction Equation [LF] is inconsistent with Formula [AV]. One of the results in Falmagne and Doignon (2010, Corollary 7) is the implication

$$\text{[AV]} \implies (\text{[R]} \iff \text{[DE]} \iff \text{[M]}).$$

Accordingly, if the standard formula [AV] for the relativistic addition of velocities is assumed, then [LF] is also inconsistent with either of [R] or [M]. However, the Lorentz-FitzGerald Contraction is consistent with another candidate equation for the representation of the relativistic addition of velocities, namely

$$\text{[AV}^\star\text{]} \qquad v \oplus w = c\sqrt{\left(\tfrac{v}{c}\right)^2 + \left(\tfrac{w}{c}\right)^2 - \left(\tfrac{v}{c}\right)^2\left(\tfrac{w}{c}\right)^2}.$$

which arises in the case of perpendicular motions (see e.g. Ungar, 1991, Eq. (8)). In fact, Falmagne and Doignon (2010, Corollary 9) proved the implication

$$\text{[LF]} \implies (\text{[R]} \iff \text{[AV}^\star\text{]} \iff \text{[M]}).$$

So, [LF] is consistent with both [R] and [M] in that case. In can be shown that [AV*] is a meaningful representation.

Note that I did obtain a representation theorem for the Lorentz-FitzGerald Equation, which was using a different kind of meaningfulness constraints based on the concept of *meaningful transformations* (see Falmagne, 2004).

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