How different are classical and relativistic spacetimes?
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This is part of an ongoing joint research with Madarász, J. and Székely, G. This research was inspired by László E. Szabó's paper [S].

We take classical (Newtonian, or pre-relativistic) spacetime to be the geometry determined by the Galilean transformations. In more detail: Let the universe of the structure CST be four-dimensional real space R^4 together with the binary relation of simultaneity, ternary relation of collinearity, and quaternary relation of orthogonality, where four points are said to be orthogonal iff they are distinct and the first two points and the other two points are pairwise simultaneous and they determine orthogonal lines in the Euclidean sense. Let CST represent classical spacetime.

Relativistic spacetime is the geometry determined by the Poincare transformations. In more detail: The universe of the structure RST is four-dimensional real space R^4 and its relations are collinearity and Minkowski-orthogonality (or, equivalently, the only binary relation of light-like separability). Let RST represent special relativistic spacetime.

The question whether two structures are identical except for renaming of basic notions is a central topic in definability theory of mathematical logic. It is formulated as whether the two structures are definitionally equivalent or not (see e.g., [Ho]).

Clearly, CST and RST are not definitionally equivalent in the traditional Tarskian sense, since in CST one can define a nontrivial equivalence relation (the simultaneity), while in RST one cannot define any nontrivial equivalence relation on the universe. However, in "modern" definability theory of mathematical logic one can define new universes of entities, too (cf e.g., [H], [M] or [BH]). In this extended modern sense, in RST one can define a new universe with nontrivial equivalence relations on it (e.g., one can define a field isomorphic to R^4). In fact, both spacetimes can be faithfully interpreted into the other. In the following, by definitional equivalence we always mean definitional equivalence in the modern sense. Definitional equivalence of two theories is a mathematical notion expressing "identiy of" theories. Two theories are definitionally equivalent iff there is a one-to one and onto correspondence between the defined concepts of the two theories such that this correspondence respects the relation of definability. The same notion is applicable to structures.

Theorem 1. CST and RST are not definitionally equivalent.

To prove Theorem 1, it is enough to prove that the automorphism groups (i.e., groups of symmetries) of CST and RST are not isomorphic. The automorphism group of CST is the general inhomogeneous Galilean group, where "inhomogeneous" means that we include translations and "general" means that we include dilations. Analogously, the automorphism group of RST is the general inhomogeneous Lorenz group. The two automorphism groups are not even definitionally equivalent. This follows from the following theorem which seems to be interesting in its own. It sais that the abstract automorphism groups of the two spacetimes contain exactly the same "content" as the geometries themselves, they "do not forget structure".

Theorem 2.
(i) CST is definitionally equivalent to its automorphism group as well as to the inhomogeneous Galilean group.
(ii) RST is definitionally equivalent to its automorphism group as well as to the inhomogeneous Lorenz group.

Similar investigations can be found, e.g., in [E], [EH] and [P].

References

[BH] Barrett, T. W., Halvorson, H., From geometry to conceptual relativity. PhilSci Archive, 2016.
[E] Ellers, E.W., The Minkowski group. Geometriae Dedicata 15 (1984), 363-375.
[EH] Ellers, E.W., Hahl, H., A homogeneous dexctiption of inhomogeneous Minkowski groups. Geometriae Dedicata 17 (1984), 79-85.
[H] Harnik, V., Model theory vs. categorical logic: two approaches to pretopos completion (a.k.a. T^{eq}). In: Models, logics, and higher-dimensional categories: a tribue to the work of Mihály Makkai. CRM Proceedings and Lecture Notes 53, American Mathematical Society, 2011. pp.79-106.
[Ho] Hodges, W., Model theory. Cambridge University Press, 1993.
[M] Madarász, J., Logic and relativity (in the light of definability theory). PhD Dissertation, ELTE Budapest, 2002. xviii+367pp.
[P] Pambuccian, V., Groups and plane geometry. Studia Logica 81 (2005), 387-398.
[S] Szabó, L. E., Does special relativity theory tell us anything new about space and time? Preprint http://philosophy.elte.hu/leszabo/Preprints/lesz_does_d.pdf