Atom-canonicity in varieties of relation and cylindric algebras with applications to omitting types in multi-modal logic

Tarek Sayed Ahmed
Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt.

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Abstract . Fix $2 < n < \omega$. L_n denotes first order logic restricted to the first n variables and for any ordinals $\alpha < \beta$, $(\mathsf{R})\mathsf{CA}_\alpha$ denotes the class of (representable) cylindric algebras of dimension α, and $\mathsf{Nr}_\alpha\mathsf{CA}_\beta$ denotes the class of α-neat reducts of CA_β. Certain CA_ns constructed from relation algebras having an n–dimensional cylindric basis are used to show that Vaught's Theorem (VT) looked upon as a special case of the omitting types theorem (OTT) fails in the m-clique guarded fragment (CGF_m) of L_n, when $m \geq n+3$. For infinitely many values of $n \leq l < m \leq \omega$, there is an atomic, countable and complete L_n theory T such that the type of co-atoms (of the formula algebra $\mathfrak{Fm}_T$) is realizable in every m-square model of T but cannot be isolated using l variables. Here 'm-squareness' is the locally well behaved clique-guarded semantics of CGF_m; an m-square model is l-square, but the converse may be false. The limiting case, an ω-square model, is an ordinary model. This is proved algebraically by constructing a countable, atomic and simple algebra $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ whose Dedekind-MacNeille completion ($\mathfrak{Cm}\mathsf{At}\mathfrak{A}$) does not have an m-square representation, a *fortiorti* $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m(\supseteq \mathsf{RCA}_n)$. OTTs are proved with respect to standard semantics for L_n countable theories that have quantifier elimination; it is shown that $< 2^\omega$ many non-principal types can be omitted in case they are maximal. Our purpose throughout the paper is twofold. Apart from presenting novel ideas of applying algebra to logic, we present our new results in both algebraic and modal logic in an integrated format.[1]

Fix $2 < n < \omega$. We use blow up and blur constructions to proving *non*-atom canonicity of several varities of relation and cylindric algebras. We recall that a class K of Boolean algebras with operators (BAOs) is *atom–canonical* if whenever $\mathfrak{A} \in \mathsf{K}$ with atom structure $\mathsf{At}\mathfrak{A}$ is completey additive, then its Dedekind-MacNeille completion, namely, the complex algebra of its atom structure $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is also in K. This subtle construction may be applied to any two classes $\mathbf{L} \subseteq \mathbf{K}$ of completely additive BAOs. One takes an atomic $\mathfrak{A} \notin \mathbf{K}$ (usually but not always finite), blows it up, by splitting one or more of its atoms each to infinitely many subatoms, obtaining an (infinite) *countable* atomic $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}$, such that $\mathfrak{A}$ *is blurred* in $\mathfrak{Bb}(\mathfrak{A})$ meaning that $\mathfrak{A}$ *does not* embed in $\mathfrak{Bb}(\mathfrak{A})$, but $\mathfrak{A}$ embeds in the Dedekind-MacNeille completion of $\mathfrak{Bb}(\mathfrak{A})$, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A})$.

Then any class $\mathbf{M}$ say, between $\mathbf{L}$ and $\mathbf{K}$ that is closed under forming subalgebras will not be atom–canonical, for $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}(\subseteq \mathbf{M})$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A}) \notin \mathbf{K}(\supseteq \mathbf{M})$ because $\mathfrak{A} \notin \mathbf{M}$ and $\mathbf{SM} = \mathbf{M}$. We say, in this case, that $\mathbf{L}$ *is not atom–canonical with respect to*

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$\mathbf{K}$. This method is applied to $\mathbf{K} = \mathbf{S}\mathsf{RaCA}_l$, $l \geq 5$ and $\mathbf{L} = \mathsf{RRA}$ in [7] and to $\mathbf{K} = \mathsf{RRA}$ and $\mathbf{L} = \mathsf{RRA} \cap \mathsf{RaCA}_k$ for all $k \geq 3$ in [3], and will applied below to $\mathbf{K} = \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$, $k \geq 3$ and $\mathbf{L} = \mathsf{RCA}_n$, where Nr_n and Ra denote the operator of forming n–neat reducts and relation algebra reducts, respectively, [4, Definition 2.6.28, Definition 5.2.7].

Using variations on several blow up and blur constructions, we obtain negative results of the form (described in the abstract): *There exists a countable, complete and atomic L_n theory T such that the type Γ consisting of co–atoms is realizable in every m–square model, but Γ cannot be isolated using $\leq l$ variables, where $n \leq l < m \leq \omega$.* Call it $\Psi(l, m)$, short for Vaught's Theorem (VT*) fails at (the parameters) l and m.* Let $\mathsf{VT}(l, m)$ stand for VT *holds at l and m*, so that by definition $\Psi(l, m) \iff \neg\mathsf{VT}(l, m)$. We also include $l = \omega$ in the equation by defining $\mathsf{VT}(\omega, \omega)$ as VT holds for $L_{\omega,\omega}$: Atomic countable first order theories have atomic countable models. In this paper, we provide strong evidence that VT *fails everywhere* in the sense that for the permitted values $n \leq l, m \leq \omega$, namely, for $n \leq l < m \leq \omega$ and $l = m = \omega$, $\mathsf{VT}(l, m) \iff l = m = \omega$. From known algebraic results such as non-atom–canonicity of RCA_n [9] and non-first order definability of the class of completely representable CA_ns [6], it can be easily inferred that $\mathsf{VT}(n, \omega)$ is false, that is to say, VT fails for L_n with respect to (usual) Tarskian semantics [13]. From sharper algebraic results, we prove many other special cases for specific values of l and m, with $l < m$, that support the last equivalence.

For example from the non–atom canonicity of RCA_n with respect to the variety of CA_ns having $n+3$–square representations ($\supseteq \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$), we prove $\Psi(n, n+k)$ for $k \geq 3$ and from the non–atom canonicity of $\mathsf{Nr}_n\mathsf{CA}_{n+k} \cap \mathsf{RCA}_n$ with respect to RCA_n for all $k \in \omega$, we prove $\Psi(l, \omega)$ for all finite $l \geq n$. Both results are obtained by blowing up and blurring finite algebras; a rainbow CA_n in the former case, and a finite RA (whose number of atoms depend on k) in the second case. In this case, we say (and prove) that VT fails *almost* everywhere. The non atom–canonicity of $\mathsf{Nr}_n\mathsf{CA}_{m-1} \cap \mathsf{RCA}_n$ with respect to the variety of CA_ns having m–square representations ($\supseteq \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$) for all $2 < n < m < \omega$, implies that $\Psi(l, m)$ holds for all $2 < n \leq l < m \leq \omega$, in which case VT *fails everywhere.* This is reduced to (finding then) blowing up and blurring a finite relation algebra having a so-called strong $m-1$ blur and no m-dimensional relational basis for each $2 < n < m < \omega$.

Figuratively speaking, VT *holds only at the limit when $l \to \infty$ and $m \to \infty$. So we can express the situation (using elementary Calculas terminology) as follows: For $2 < n \leq l < m < \omega$,* $\mathsf{VT}(l, m)$ *is false, but as l and m gets larger,* $\mathsf{VT}(l, m)$ *gets closer to* VT*, in symbols,* $\mathsf{lim}_{l,m\to\infty}\mathsf{VT}(l, m) = \mathsf{VT}(\mathsf{lim}_{l\to\infty}l, \mathsf{lim}_{m\to\infty}m) = \mathsf{VT}(\omega, \omega)$.

Throughout the paper we use the notation of [2].

Layout: In §1 a blow up and blur construction is presented showing that RCA_n is not atom-canonical with respect to $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$, cf. Thm 1.3. For $n \leq l < m \leq \omega$, a chain of implications starting from the existence of finite RAs with strong l-bur and no m-dimensional relational basis leading up to $\Psi(l, m)$ is given, cf. Thm 1.7, ultimately showing that VT fails almost everywhere. VT is shown to fail for any finite first order definable expansion of L_n, cf. Thm 1.12. Classical results of Biro, Maddux and Monk on non-finite axiomatizability of RRA and RCA_n are reproved, cf. Cor. 1.13. In §2 positive results on OTT for L_n are proved, cf. Thm 2.2 and Cor. 2.3. The non-first order definability of the classes of completely representable CA_ns and RAs is reproved differently in Cor. 2.5. In §3 complete representations are studied in connection to neat embeddings cf. Thm 3.1.

1 Non-atom canonicity and applications in the clique–guarded fragments

1.1 Non atom-canonicity of $\mathsf{SNr}_n\mathsf{CA}_{n+3}$

We encounter our first instance of a blow up and blur construction. From now on, unless otherwise indicated, n is fixed to be a finite ordinal > 2.

Definition 1.1. Let $\mathfrak{A} \in \mathsf{CA}_n$ be atomic. Assume that $m, k \leq \omega$. The *atomic game* $G^m_k(\mathsf{At}\mathfrak{A})$*, or simply* G^m_k, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds [8, Definition 3.3.2]. The ω–rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the advantage to reuse the m nodes in play.

In the following lemma and elsewhere throughout the paper $\mathbf{S}_c$ denotes the operation of forming *complete* subalgebras.

Lemma 1.2. *[14] If* $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$ *is atomic, then* $\exists$ *has a winning strategy in* $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$.

For rainbow constructions for CAs, we follow [6, 8]. We use the graph version of the games $G^m_\omega(\beta)$ and $\mathbf{G}^m(\beta)$ where β is a CA_n rainbow atom structure, cf. [6, 4.3.3]; the board of this game consists of coloured graphs. The (complex) rainbow CA_n based on G and R is denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$.

Theorem 1.3. *1. The variety* RRA *is not atom-canonical with respect to* $\mathbf{S}\mathsf{RaCA}_k$*, for any* $k \geq 6$*,*

2. Let $m \geq n+3$*. Then* RCA_n *is not-atom canonical with respect to* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$*.*

Proof. For the first item concerning RAs, cf.[8, Lemmata 17.32, 17.34, 17.35, 17.36].

For item (2): The idea for CAs is like that for RAs by blowing up and blurring the rainbow algebra $\mathfrak{A}_{n+1,n}$ in place of the rainbow relation algebra $\mathbf{R}_{4,3}$ blown up and blurred in the RA case. We work with $m = n+3$. This gives the result for any larger m. We give a fairly complete sketch of the proof detailed in [14, Theorem 5.9].

Blowing up and blurring $\mathfrak{A}_{n+1,n}$ forming a weakly representable atom structure $\mathbf{At}$: Take the finite rainbow CA_n, $\mathfrak{A}_{n+1,n}$ where the reds R is the complete irreflexive graph n, and the greens are $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}^i_0 : 1 \leq i \leq n+1\}$, so that $\mathsf{G} = n+1$. Denote the finite atom structure of $\mathfrak{A}_{n+1,n}$ by $\mathbf{At}_f$. One then replaces the red colours of the finite rainbow algebra of $\mathfrak{A}_{n+1,n}$ each by infinitely many countable reds (getting their superscripts from ω), obtaining this way a weakly representable atom structure $\mathbf{At}$. The atom structure $\mathbf{At}$ is like the weakly (but not strongly) representable atom structure of the atomic and countable and simple $\mathfrak{A} \in \mathsf{Cs}_n$ as defined in [9, Definition 4.1]; the sole difference is that we have $n+1$ greens and not ω–many as is the case in [9]. We denote the resulting term CA_n, $\mathfrak{Tm}\mathbf{At}$ by $\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)$ short hand for blowing up $\mathfrak{A}_{n+1,n}$ by splitting each *red graph (atom)* into ω many. By a red graph is meant (an equivalence class of) a surjection $a : n \to \Delta$, where Δ is a coloured graph in the rainbow signature of $\mathfrak{A}_{n+1,n}$ with at least one edge labelled by a red label (some r_{ij}, $i < j < n$). It can be shown exactly like in [9] that $\exists$ can win the rainbow ω–rounded game and build an n–homogeneous model M by using a shade of red ρ *outside* the rainbow signature, when she is forced a red; [9, Proposition 2.6, Lemma 2.7]. Using this, one proves like in *op.cit* that $\mathfrak{Bb}(\mathfrak{A}_{n+1,n},\mathsf{r},\omega)$ is representable as a set algebra having top element ${}^n\mathsf{M}$.

Embedding $\mathfrak{A}_{n+1,n}$ into $\mathfrak{Cm}(\mathsf{At}(\mathfrak{Bb}(\mathfrak{A}_{n+1,n},\mathsf{r},\omega)))$: Let CRG_f be the class of coloured graphs on $\mathbf{At}_f$ and CRG be the class of coloured graph on $\mathbf{At}$. Write M_a for the atom that is the (equivalence class of the) surjection $a:n\to M$, $M\in\mathsf{CGR}$. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_a\sim N_b$, $(M,N\in\mathsf{CGR})\iff$ they are identical everywhere except at possibly at red edges: $M_a(a(i),a(j))=\mathsf{r}^l\iff N_b(b(i),b(j))=\mathsf{r}^k$, for some $l,k\in\omega$. We say that M_a is a *copy of* N_b if $M_a\sim N_b$. Define the map Θ from $\mathfrak{A}_{n+1,n}=\mathfrak{Cm}\mathbf{At}_f$ to $\mathfrak{Cm}\mathbf{At}$, by specifing first its values on $\mathbf{At}_f$, via $M_a\mapsto\bigvee_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of M_a. So each atom maps to the suprema of its copies. This map is well-defined because $\mathfrak{Cm}\mathbf{At}$ is complete. Furthermore, it can be checked that Θ is an injective a homomorphism.

$\forall$ has a winning strategy in $\mathbf{G}^{n+3}\mathsf{At}(\mathfrak{A}_{n+1,n})$: For him to win, $\forall$ lifts his winning strategy from the private Ehrenfeucht–Fraïssé forth game $\mathsf{EF}_{n+1}^{n+1}(n+1,n)$ (in $n+1$ rounds), to the graph game on $\mathbf{At}_f=\mathsf{At}(\mathfrak{A}_{n+1,n})$ [6, p. 841] forcing a win using $n+3$ nodes. He bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). By Lemma 1.2, $\mathfrak{A}_{n+1,n}\notin\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Since $\mathfrak{A}_{n+1,n}$ is finite, then $\mathfrak{A}_{n+1,n}\notin\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$, for else $\mathfrak{A}^+_{n+1,n}=\mathfrak{A}_{n+1,n}\in\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$. But $\mathfrak{A}_{n+1,n}$ embeds into $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, hence $\mathfrak{Cm}\mathbf{At}=\mathfrak{Cm}(\mathsf{At}\mathfrak{Bb}(\mathfrak{A}_{n+1,n},\mathsf{r},\omega))$ is outside the variety $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$, as well. □

1.2 Clique-guarded semantics

Fix $2<n<\omega$. The reader is referred to [7, Definitions 13.4, 13.6] for the notions of m–flat and m–square representations for relation algebras ($m>2$) to be generalized next to CA_ns.

Definition 1.4. [14, §5, p.14] Assume that $2<n<m<\omega$. Let M be the base of a relativized representation of $\mathfrak{A}\in\mathsf{CA}_n$ witnessed by an injective homomorphism $f:\mathfrak{A}\to\wp(V)$, where $V\subseteq{}^n\mathsf{M}$ and $\bigcup_{s\in V}\mathsf{rng}(s)=\mathsf{M}$. We write $\mathsf{M}\models a(s)$ for $s\in f(a)$. Let $\mathfrak{L}(\mathfrak{A})^m$ be the first order signature using m variables and one n-ary relation symbol for each element in A. Let $\mathfrak{L}(\mathfrak{A})^m_{\infty,\omega}$ be the infinitary extension of $\mathfrak{L}(\mathfrak{A})^m$ allowing infinite conjunctions. Then *an n-clique* is a set $C\subseteq\mathsf{M}$ such that $(a_1,\ldots,a_{n-1})\in V=1^{\mathsf{M}}$ for distinct $a_1,\ldots,a_n\in C$. Let $\mathsf{C}^m(\mathsf{M})=\{s\in{}^m\mathsf{M}:\mathsf{rng}(s)\text{ is an }n\text{-clique}\}$. $\mathsf{C}^m(\mathsf{M})$ is called the *n-Gaifman hypergraph of* M, with the n–hyperedge relation 1^{M}.

The *clique guarded semantics* $\models_c$ are defined inductively. We give only existential quantifiers (cylindrifiers): for $\bar{s}\in{}^m\mathsf{M}$, $i<m$, $\mathsf{M},\bar{s}\models_c\exists x_i\phi\iff$ there is a $\bar{t}\in\mathsf{C}^m(\mathsf{M})$, $\bar{t}\equiv_i\bar{s}$ such that $\mathsf{M},\bar{t}\models\phi$.

We say that M is *an m–square representation* of $\mathfrak{A}$, if for all $\bar{s}\in\mathsf{C}^m(\mathsf{M}),a\in\mathfrak{A}$, $i<n$, and injective map $l:n\to m$, whenever $\mathsf{M}\models\mathsf{c}_ia(s_{l(0)},\ldots,s_{l(n-1)})$, then there is a $\bar{t}\in\mathsf{C}^m(\mathsf{M})$ with $\bar{t}\equiv_i\bar{s}$, and $\mathsf{M}\models a(t_{l(0)},\ldots,t_{l(n-1)})$; M is an *(infinitary) m–flat representation* if it is m–square and for all $\bar{s}\in\mathsf{C}^m(\mathsf{M})$, for all distinct $i,j<m$, $\mathsf{M}\models_c[\exists x_i\exists x_j\phi\longleftrightarrow\exists x_j\exists x_i\phi](\bar{s})$, where $\phi\in(\mathfrak{L}(\mathfrak{A})^m_{\infty,\omega})\mathfrak{L}(\mathfrak{A})^m$. Complete representability for m-squareness and m-flatness is defined like the classical case.

The main ideas used in the next Theorem can be found in [7, Definitions 12.1, 12.9, 12.10, 12.25, Propositions 12.25, 12.27] adapted to the CA case. In all cases, the m–

dimensional dilation stipulated in the statement of the Theorem, will have top element $\mathsf{C}^m(\mathsf{M})$, where M is the m–relativized representation of the given algebra, and the operations of the dilation are induced by the n-clique–guarded semantics.

Proposition 1.5. *[7, Theorems 13.45, 13.36]. Assume that $2 < n < m < \omega$ and let $\mathfrak{A} \in \mathsf{CA}_n$. Then $\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m \iff \mathfrak{A}$ has an infinitary m–flat representation $\iff \mathfrak{A}$ has an m–flat representation. Furthermore, if $\mathfrak{A}$ is atomic, then $\mathfrak{A}$ has a complete infinitary m–flat representation $\iff \mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n(\mathsf{CA}_m \cap \mathbf{At})$.*

1.3 VT for the clique guarded fragments

Fix $2 < n \leq l < m \leq \omega$. We turn to the statement $\Psi(l, m)$ as defined in the introduction. By an m–square model M of a theory T we understand an m–square representation of the algebra $\mathfrak{Fm}_T$ with base M.

Let $\mathsf{VT}(l, m) = \neg\Psi(l, m)$, short for VT *holds 'at the parameters l and m'* where by definition, we stipulate that $\mathsf{VT}(\omega, \omega)$ is just VT for $L_{\omega,\omega}$. For $2 < n \leq l < m \leq \omega$ and $l = m = \omega$, we investigate the plausability of the following statement which we abbreviate by (**): $\mathsf{VT}(l, m) \iff l = m = \omega$.

In the next Theorem several conditions are given implying $\Psi(l, m)_f$ for various values of the parameters l and m where $\Psi(l, m)_f$ is the formula obtained from $\Psi(l, m)$ replacing square by flat. For an atomic relation algebra $\mathfrak{R}$ and $n > 3$, $\mathsf{Mat}_n(\mathsf{At}\mathfrak{R})$ denotes the set of all n–dimensional basic matrices on $\mathfrak{R}$ [7, Definition 12.35]. The following definition to be used in the sequel is taken from [3]:

Definition 1.6. Let $\mathfrak{R}$ be a relation algebra, with non–identity atoms I and $2 < n < \omega$. Assume that $J \subseteq \wp(I)$ and $E \subseteq {}^3\omega$. We say that (J, E) is a *strong n–blur* for $\mathfrak{R}$, if J is a *complex n–blur* as defined in [3, Definition 3.1] and the tenary relation E is an *index blur* defined as in item (ii) of [3, Definition 3.1], and satisfying condition $(J5)_n$ formulated on [3, p.79], namely, $(\forall V_1, \ldots V_n, W_2, \ldots W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T)$.

Theorem 1.7. *Let $2 < n \leq l < m \leq \omega$. Then every item implies the immediately following one.*

1. *There exists a finite relation algebra $\mathfrak{R}$ with a strong l–blur and no infinite m–dimensional hyperbasis,*

2. *There is a countable atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{CmAt}\mathfrak{A}$ does not have an m–flat representation,*

3. *There is a countable atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{A}$ has no complete infinitary m–flat representation,*

4. *$\Psi(l', m')_f$ is true for any $l' \leq l$ and $m' \geq m$.*

The same implications hold upon replacing infinite m–dimensional hyperbasis by m–dimensional relational basis (not necessarily infinite), m–flat by m–square and $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$ by $\mathbf{S}\mathsf{Nr}_n\mathsf{D}_m$. Furthermore, in the new chain of implications every item implies the corresponding item in Theorem 1.7. In particular, $\Psi(l, m) \implies \Psi(l, m)_f$.

Proof. (1) $\implies$ (2): Let $\mathfrak{R}$ be as in the hypothesis with strong l–blur (J, E). The idea is to 'blow up and blur' $\mathfrak{R}$ in place of the Maddux algebra $\mathfrak{E}_k(2, 3)$ blown up and blurred in [3, Lemma 5.1], where $k < \omega$ is the number of non–identity atoms and k depends recursively on l, giving the desired strong l–blurness, cf. [3, Lemmata 4.2, 4.3]. Let $2 < n \leq l < \omega$. The relation algebra $\mathfrak{R}$ is blown up by splitting all of the atoms each to infinitely many giving a new infinite atom structure $\mathbf{At}$ denoted in [3, p.73] by At. One proves that the blown up and blurred atomic relation algebra $\mathfrak{Bb}(\mathfrak{R}, J, E)$ (as defined in [3]) with atom structure $\mathbf{At}$ is representable; in fact this representation is induced by a complete representation of its canonical extension, cf. [3, Item (1) of Theorem 3.2].

Because (J, E) is a strong l–blur, then, by its definition, it is a strong j–blur for all $n \leq j \leq l$, so the atom structure $\mathbf{At}$ has a j–dimensional cylindric basis for all $n \leq j \leq l$, namely, $\mathsf{Mat}_j(\mathbf{At})$. For all such j, there is an RCA_j denoted on [3, Top of p. 9] by $\mathfrak{Bb}_j(\mathfrak{R}, J, E)$ such that $\mathfrak{TmMat}_j(\mathbf{At}) \subseteq \mathfrak{Bb}_j(\mathfrak{R}, J, E) \subseteq \mathfrak{CmMat}_j(\mathbf{At})$ and $\mathsf{At}\mathfrak{Bb}_j(\mathfrak{R}, J, E)$ is a weakly representable atom structure of dimension j, cf. [3, Lemma 4.3].

Now take $\mathfrak{A} = \mathfrak{Bb}_n(\mathfrak{R}, J, E)$. We claim that $\mathfrak{A}$ is as required. Since $\mathfrak{R}$ has a strong j–blur (J, E) for all $n \leq j \leq l$, then $\mathfrak{A} \cong \mathfrak{Nr}_n\mathfrak{Bb}_j(\mathfrak{R}, J, E)$ for all $n \leq j \leq l$ as proved in [3, item (3) p.80]. In particular, taking $j = l$, $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$. We show that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ does not have an m–flat representation. Assume for contradicton that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ does have an m–flat representation M. Then M is infinite of course. Since $\mathfrak{R}$ embeds into $\mathfrak{Bb}(\mathfrak{R}, J, E)$ which in turn embeds into $\mathfrak{Ra}\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, then $\mathfrak{R}$ has an m–flat representation with base M. But since $\mathfrak{R}$ is finite, $\mathfrak{R} = \mathfrak{R}^+$, so $\mathfrak{R}$ has an infinite m–dimensional hyperbasis, contradiction.

(2) $\implies$ (3): A complete m–flat representation of (any) $\mathfrak{B} \in \mathsf{CA}_n$ induces an m–flat representation of $\mathfrak{Cm}\mathsf{At}\mathfrak{B}$ which implies by Theorem 1.5 that $\mathfrak{Cm}\mathsf{At}\mathfrak{B} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$.

(3) $\implies$ (4): By [4, §4.3], we can (and will) assume that $\mathfrak{A} = \mathfrak{Fm}_T$ for a countable, simple and atomic theory L_n theory T. Let Γ be the n–type consisting of co–atoms of T. Then Γ is realizable in every m–flat model, for if M is an m–flat model omitting Γ, then M would be the base of a complete infinitary m–flat representation of $\mathfrak{A}$, and so $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$ which is impossible. But $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l$, so using exactly the same (terminology and) argument in [3, Theorem 3.1] we get that any witness isolating Γ needs more than l–variables. We have proved $\Psi(l, m)$. The rest follows from the definitions.

For squareness the proofs are essentially the same undergoing the obvious modifications. In the first implication 'infinite' in the hypothesis is not needed because any finite relation algebra having an infinite m–dimensional relational basis has a finite one, cf. [7, Theorem 19.18] which is not the case with hyperbasis, cf. [7, Prop. 19.19]. □

Corollary 1.8. *For $2 < n < \omega$ and $n \leq l < \omega$, $\Psi(n, n+3)$ and $\Psi(l, \omega)$ hold.*

Proof. From Theorem 1.3, 1.7 and [3]. □

It is timely that we tie a few threads together.

Definition 1.9. Let $2 < n < \omega$. We say that VT *fails for L_n almost everywhere* if there exist positive $l, m \geq n$ such that $\mathsf{V}(k, \omega)$ and $\mathsf{V}(n, t)$ are false for all finite $k \geq l$ and all $t \geq m$. We say that VT *fails for L_n everywhere* if for $3 \leq l < m \leq \omega$ and $l = m = \omega$, $\mathsf{V}(l, m)$ holds $\iff$ $l = m = \omega$, that is to say $(**)$ above holds.

From Corollary 1.8 and the implication (1) $\Longrightarrow$ (6) in Theorem 1.7 (by taking $l = m-1$), we get:

Theorem 1.10. *Let $2 < n < \omega$. Then* VT *fails for L_n almost everywhere. Furthermore, if for each $n < m < \omega$, there exists a finite relation algebra $\mathfrak{R}_m$ having $m-1$ strong blur and no m-dimensional relational basis, then* VT *fails for L_n everywhere.*

Theorem 1.3 says that VT fails for the *packed fragment* of L_n [7, §19.2.3]. For a class $\mathbf{K}$ of BAOs, let $\mathbf{K}\cap\mathsf{Count}$ denote the class of atomic algebras in $\mathbf{K}$ having countably many atoms.

Proposition 1.11. *Let $2 < n < \omega$.*

1. *The variety $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+1}$ is atom-canonical. For $n < m < \omega$ if there exists a finite* RA *with an n-blur (not necessarily strong) and no infinite m-dimensional hyperbasis, then RCA_n is not atom-canonical with respect to $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$,*

2. *For any ordinal $0 \leq j$, $\mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_{n+j} \cap \mathsf{Count}$ is not atom–canonical with respect to $\mathsf{RCA}_n \iff j < \omega$,*

3. *There exists an atomic RCA_n such that its Dedekind-MacNeille (minimal) completion does not embed into its canonical extension.*

Proof. (1): Let $\mathfrak{R}$ be as described; denote its an n-blur by (J, E). Let $\mathfrak{B} = \mathfrak{Bb}_n(\mathfrak{R}, J, E)$. Then, since (J, E) is an n blur, $\mathfrak{B} \in \mathsf{RCA}_n$. But $\mathfrak{Cm}\mathsf{At}\mathfrak{B} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$, for otherwise, $\mathfrak{R}$ will have an infinite m–dimensional basis.

(2): Follows from the first item of Theorem 1.7 by taking $\mathfrak{R} = \mathfrak{E}_k(2, 3)$; the finite Maddux algebra with k generators used in [3, Lemma 5.1], where k is finite tuned to give that $\mathfrak{R}$ has an $n+j$ strong blur (J, E). In this case $\mathfrak{A} = \mathfrak{Bb}_n(\mathfrak{R}, J, E) \in \mathsf{Nr}_n\mathsf{CA}_{n+j} \cap \mathsf{RCA}_n$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$. Conversely, for any infinite ordinal j, $\mathsf{Nr}_n\mathsf{CA}_{n+j} = \mathsf{Nr}_n\mathsf{CA}_\omega$ and if $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{Count}$, then $\mathfrak{Tm}\mathsf{At}\mathfrak{A}$ is countable, atomic, and $\mathfrak{Tm}\mathsf{At}\mathfrak{A} \subseteq_c \mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$, so by [13, Theorem 5.3.6], $\mathfrak{Tm}\mathsf{At}\mathfrak{A} \in \mathsf{CRCA}_n$, so $\mathfrak{A} \in \mathsf{CRCA}_n$, thus $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathsf{RCA}_n$.

(4): Let $\mathfrak{A} = \mathfrak{Tm}\mathbf{At}$ be as defined in the proof of Theorem 1.3. Since $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$, it does not embed into $\mathfrak{A}^+$. $\square$

Summary of results on VT: In the coming table $2 < n < \omega$. For any finite j, 'j-hyp' is short hand for infinite j–dimensional hyperbasis, and j–basis is short hand for j–dimensional relational basis. $\mathsf{VT}(l, m)$ for $n \leq l < m \leq \omega$ and $\mathsf{VT}(\omega, \omega)$ are defined as before. All conditional statements can be recovered from the proof of Theorem 1.7 and item (1) of Proposition 1.11.

$\mathsf{VT}(n, \omega)$	no, [3] and Theorem 1.3
$\mathsf{VT}(n, n+3)$	no, Theorem 1.3
$\mathsf{VT}(n, n+2)_f$	no, if $\exists\ \mathfrak{R}$ with n–blur and no $n+2$-hyp
$\mathsf{VT}(l, \omega)$	no, $\mathfrak{E}_k(2, 3)$ has strong l-blur, and no ω-hyp
$\mathsf{VT}(l, m)_f, l \leq m-1$	no, if $\exists\ \mathfrak{R}$ with strong l-blur, and no m-hyp
$\mathsf{VT}(l, m), l \leq m-1$	no, if $\exists\ \mathfrak{R}$ with strong l-blur, and no m-bases
$\mathsf{VT}(\omega, \omega)$	yes, VT for $L_{\omega,\omega}$.

Now we formulate an algebraic result implying that VT fails for any finite first order definable expansion of L_n as defined in [10]. We deviate from the notation in [10] by writing RCA_n^+ for a first order definable expansion of RCA_n.

Proposition 1.12. *Let $2 < n < \omega$. Let RCA_n^+ be a first order definable expansion of RCA_n such that the non–cylindric operations are first order definable by formulas using only finitely many variables $l > n$. If RCA_n^+ is completely additive, then it is not atom–canonical.*

Proof. Let $\mathfrak{n}$ be the finite number of variables occuring in the first order formulas defining the new connectives and let $l = \mathfrak{n} + 1$. Let $\mathfrak{A}$ be countable and atomic such that $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ and $\mathfrak{A}$ has no complete representation; such an $\mathfrak{A}$ exists, cf. Corollary 1.11. Without loss, we can assume that we have only one extra operation f definable by a first order formula ϕ, say, using $n < k < \omega$ variables with at most n free variables. Now ϕ defines a CA_k term $\tau(\phi)$ which, in turn, defines the unary operation f on $\mathfrak{A}$, via $f(a) = \tau(\phi)^{\mathfrak{B}}(a)$. Call the expanded structure $\mathfrak{A}^*(\in \mathsf{RCA}_n^+)$. By complete additivity, $\mathfrak{Cm}\mathsf{At}\mathfrak{A}^*$ is the Dedekind-MacNeille completion of $\mathfrak{A}^*$. But $\mathfrak{Rd}_{ca}\mathfrak{Cm}\mathsf{At}\mathfrak{A}^* = \mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$, *a fortiori*, $\mathfrak{Cm}(\mathsf{At}\mathfrak{A}^*) \notin \mathsf{RCA}_n^+$, and we are done. □

Let $2 < n \leq l < m \leq \omega$. In $\mathsf{VT}(l, m)$, while the parameter l measures how close we are to $L_{\omega,\omega}$, m measures the *'degree' of squareness* of permitted models. One can view $\mathsf{lim}_{l\to\infty}\mathsf{VT}(l, \omega) = \mathsf{VT}(\omega, \omega)$ algebraically using ultraproducts as follows. Fix $2 < n < \omega$. For each $2 < n \leq l < \omega$, let $\mathfrak{R}_l$ be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2, 3)$ with strong l–blur (J_l, E_l) and $f(l) \geq l$ as specified in [3, Lemma 5.1] (denoted by k therein). Let $\mathcal{R}_l = \mathfrak{Bb}(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RRA}$ and let $\mathfrak{A}_l = \mathfrak{Nr}_n\mathfrak{Bb}_l(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RCA}_n$. Then $(\mathsf{At}\mathcal{R}_l : l \in \omega \sim n)$, and $(\mathsf{At}\mathfrak{A}_l : l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct. We immediately get:

Corollary 1.13. *Assume that $2 < n < \omega$. Then the following hold:*

1. *The (elementary) class LCA_n of algebras satisfying the Lyndon conditions (which is $\mathbf{El}\mathsf{CRCA}_n$) is not finitely axiomatizable,*

2. *(Biro, Maddux)The set of equations using only one variable that holds in each of the varieties RCA_n and RRA, together with any finite first order definable expansion of each, cannot be derived from any finite set of equations valid in the variety.*

2 Positive OTTs for L_n with standard 'unguarded' semantics

Unless otherwise explicitly indicated, n is finite and > 2.

Definition 2.1. Let λ be a cardinal. If $\mathfrak{A} \in \mathsf{RCA}_n$ and $\mathbf{X} = (X_i : i < \lambda)$ is a family of subsets of $\mathfrak{A}$, we say that $\mathbf{X}$ *is omitted in* $\mathfrak{C} \in \mathsf{Gs}_n$, if there exists an isomorphism $f : \mathfrak{A} \to \mathfrak{C}$ such that $\bigcap f(X_i) = \emptyset$ for all $i < \lambda$. If $X \subseteq \mathfrak{A}$ and $\prod X = 0$, then we refer to X as a *non-principal type* of $\mathfrak{A}$.

Observe that $\mathfrak{A} \in \mathsf{RCA}_n$ is completely representable $\iff$ $\mathfrak{A}$ is atomic, and the single non-principal type of co-atoms can be omitted in a Gs_n. Let covK be the cardinal used in [13, Theorem 3.3.4]. We deal also with the cardinal $\mathfrak{p}$ satisfying $\omega < \mathfrak{p} \leq 2^\omega$ and has the following property: If $\lambda < \mathfrak{p}$, and $(A_i : i < \lambda)$ is a family of meager subsets of a Polish space X (of which Stone spaces of countable Boolean algebras are examples) then $\bigcup_{i\in\lambda} A_i$ is meager. It is consistent that $\omega < \mathfrak{p} < \mathsf{covK} \leq 2^\omega$.

Theorem 2.2. *Let $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$ be countable. Let $\lambda < 2^\omega$ and let $\mathbf{X} = (X_i : i < \lambda)$ be a family of non-principal types of $\mathfrak{A}$. Then the following hold:*

1. *If $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and the X_is are non–principal ultrafilters, then $\mathbf{X}$ can be omitted in a Gs_n,*

2. *Every subfamily of $\mathbf{X}$ of cardinality $< \mathfrak{p}$ can be omitted in a Gs_n. Furthermore, if $\mathfrak{A}$ is simple, then every subfamily of $\mathbf{X}$ of cardinality $< \mathsf{covK}$ can be omitted in a Cs_n.*

Proof. For the first item we assume that $\mathfrak{A}$ is simple (a condition that can be easily removed). We have $\bigwedge^{\mathfrak{B}} X_i = 0$ for all $i < \kappa$ because, $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$, cf. [14, First part of Theorem 2.2]. Since $\mathfrak{B}$ is a locally finite, we can assume that $\mathfrak{B} = \mathfrak{Fm}_T$ for some countable consistent theory T. For each $i < \kappa$, let $\Gamma_i = \{\phi : \phi/T \in X_i\}$. Let $\mathbf{F} = (\Gamma_j : j < \kappa)$ be the corresponding set of types in T. Then each Γ_j $(j < \kappa)$ is a non-principal and *complete n-type* in T, because each X_j is a maximal filter in $\mathfrak{A} = \mathfrak{Nr}_n\mathfrak{B}$.

Let $(\mathsf{M}_i : i < 2^\omega)$ be a set of countable models for T that overlap only on principal maximal types which exist by [16, Theorem 5.16, Chapter IV]: Assume for contradiction that for all $i < 2^\omega$, there exists $\Gamma \in \mathbf{F}$, such that Γ is realized in M_i. Let $\psi : 2^\omega \to \wp(\mathbf{F})$, be defined by $\psi(i) = \{F \in \mathbf{F} : F \text{ is realized in } \mathsf{M}_i\}$. Then for all $i < 2^\omega$, $\psi(i) \neq \emptyset$. Furthermore, for $i \neq j$, $\psi(i) \cap \psi(j) = \emptyset$, for if $F \in \psi(i) \cap \psi(j)$, then it will be realized in M_i and M_j, and so it will be principal. This implies that $|\mathbf{F}| = 2^\omega$ which is impossible. Hence we obtain a model $\mathsf{M} \models T$ omitting $\mathbf{X}$ in which ϕ is satisfiable. The map f defined from $\mathfrak{A} = \mathfrak{Fm}_T$ to $\mathsf{Cs}_n^{\mathsf{M}}$ (the set algebra based on M [4, 4.3.4]) via $\phi_T \mapsto \phi^{\mathsf{M}}$, where the latter is the set of n–ary assignments in M satisfying ϕ, omits $\mathbf{X}$. Injectivity follows from the facts that f is non–zero and $\mathfrak{A}$ is simple.

For (2), we can assume that $\mathfrak{A} \subseteq_c \mathfrak{Nr}_n\mathfrak{B}$, $\mathfrak{B} \in \mathsf{Lf}_\omega$ and proceed like [13, p. 216 of proof of Theorem 3.2.4] replacing the formula algebra $\mathfrak{Fm}_T$ by $\mathfrak{B}$. $\square$

By observing that if T is an L_n theory that admits elimination of quantifiers $(n < \omega)$, then $\mathfrak{Fm}_T \in \mathsf{Nr}_n\mathsf{CA}_\omega$, we get using Theorem 2.2 the following corollary:

Corollary 2.3. *Let n be any finite ordinal. Let T be a countable and consistent L_n theory and λ be a cardinal $< \mathfrak{p}$. Let $\mathbf{F} = (\Gamma_i : i < \lambda)$ be a family of non-principal types of T. Suppose that T admits elimination of quantifiers. Then the following hold:*

1. *If ϕ is a formula consistent with T, then there is a model M of T that omits $\mathbf{F}$, and ϕ is satisfiable in M. If T is complete, then we can replace $\mathfrak{p}$ by covK,*

2. *If the non-principal types constituting $\mathbf{F}$ are maximal, then we can replace $\mathfrak{p}$ by 2^ω.*

Using the full power of (+) together with the argument in item (1) of Theorem 2.2, one can replace in the last item of the last corollary ω by any regular uncountable cardinal μ. We show (algebraically) that the maximality condition cannot be removed when we consider uncountable theories.

Proposition 2.4. *Let κ be an infinite cardinal. Then there exists a $\mathfrak{C} \in \mathsf{CA}_\omega$ such that for all $2 < n < \omega$, $|\mathfrak{Nr}_n\mathfrak{C}| = 2^\kappa$, $\mathfrak{Nr}_n\mathfrak{C} \in \mathsf{LCA}_n$, but $\mathfrak{Nr}_n\mathfrak{C}$ is not completely representable. Thus the non–principal type of co–atoms of $\mathfrak{Nr}_n\mathfrak{C}$ cannot be omitted.*

Proof. One uses the ideas in [1] replacing ω and ω_1 by κ and 2^κ, respectively, constructing $\mathfrak{C}$ from a relation algebra. The resulting (new) relation algebra $\mathfrak{R}$ has an ω dimensional amalgamation class S, cf. [1, Lemma 3]. Using the notation in [1, Lemma 6], let $\mathfrak{C}$ be the subalgebra of $\mathfrak{Ca}(S)$ generated by X'; the latter is defined just before the lemma. Then $\mathfrak{R} = \mathsf{Ra}(\mathfrak{C})$, cf. [1, Lemmata 6, 7], but $\mathfrak{R}$ has no complete representation [1, Lemma 2]. Then $\mathfrak{Nr}_n\mathfrak{C}$ $(2 < n < \omega)$ is atomic, but has no complete representation and $\mathfrak{Nr}_n\mathfrak{C} \in \mathsf{LCA}_n$. □

Corollary 2.5. *[6] Let $2 < n < \omega$. Then the classes CRRA and CRCA_n are not elementary.*

3 Complete representations, neat embedding properties and the Lyndon conditions

In the following $\mathbf{Up}$, $\mathbf{Ur}$, $\mathbf{P}$ and $\mathbf{H}$ denote the operations of forming ultraproducts, ultraroots, products and homomorphic images, respectively. $\mathbf{S}_d$ denotes the operation of forming dense subalgebrs and for an ordinal α, CRCA_α denotes the class of completely representable CA_αs, and $\mathbf{El}$ denotes 'elementary closure.'

Theorem 3.1. *For $2 < n < \omega$ the following hold:*

1. $\mathsf{CRCA}_n \subseteq \mathbf{S}_c\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. *At least two of these three classes are distinct,*

2. *All reverse inclusions and implications in the previous item hold, if algebras considered have countably many atoms,*

3. *All classes in the first item are closed under $\mathbf{S}_c$ (a fortiori under $\mathbf{S}_d$), $\mathbf{P}$, but are not closed under $\mathbf{S}$, nor $\mathbf{H}$ nor $\mathbf{Ur}$. Their elementary closure coincides with LCA_n,*

4. $\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathbf{El}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathsf{RCA}_n$. *Furthermore, the strictness of inclusions are witnessed by atomic algebras,*

5. *Any class $\mathbf{K}$ such that $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathbf{K} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, $\mathbf{K}$ is not elementary.*

Proof. Throughout the proof, fix $2 < n < \omega$.

(1): The proof of the first inclusion is similar to the proof for (the analogous result on) relation algebras proved in [5, Theorem 29]. The construction in [1] shows that the first and last classes are distinct.

(2): From [12, Theorem 5.3.6] and [8, Theorem 3.3.3].

(3): Closure under $\mathbf{P}$ and $\mathbf{S}_c$ is straightforward. Non-closure under $\mathbf{S}$ is trivial. We prove non–closure under $\mathbf{H}$ for all three classes in one blow. Take a family $(U_i : i \in \mathbb{N})$ of pairwise disjoint non–empty sets. Let $V_i = {}^nU_i (i \in \mathbb{N})$. Take the full Gs_n, $\mathfrak{A}$ with universe $\wp(V)$, where $V = \bigcup_{i\in\mathbb{N}} V_i$. Then $\mathfrak{A} \in \mathsf{CRCA}_n \subseteq \mathsf{C}$. Let I be the ideal consisting of elements of $\mathfrak{A}$ that intersect only finitely many of the V_i's. Then $\mathfrak{A}/I$ is not atomic, so $\mathfrak{A}/I$ is outside all three classes.

Now we approach closure under $\mathbf{Ur}$. Let $\mathfrak{C} \in \mathsf{CA}_n \sim \mathsf{CRCA}_n$ be atomic having countably many atoms and elementary equivalent to a $\mathfrak{B} \in \mathsf{CRCA}_n$. Such algebras exist, cf. [6], [14, Theorem 5.12]. Then $\mathfrak{C} \equiv \mathfrak{B}$, $\mathfrak{C}$ will be outside all three classes (since they coincide on atomic algebras having countably many atoms), while $\mathfrak{B}$ will be inside them all proving that non of the three is elementary, so being closed under $\mathbf{Up}$, since they are psuedo-elementary classes (cf. [5, Theorem 21] and [7, §9.3] for similar cases), by the Keisler-Shelah ultrapower Theorem they are not closed under $\mathbf{Ur}$.

For the last required, we show that $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n = \mathbf{El}(\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At})$. Assume that $\mathfrak{A} \in \mathsf{LCA}_n$. Then, by definition, for all $k < \omega$, $\exists$ has a winning strategy in $G_k(\mathsf{At}\mathfrak{A})$. Using ultrapowers followed by an elementary chain argument like in [8, Theorem 3.3.5], $\exists$ has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{B})$ for some countable $\mathfrak{B} \equiv \mathfrak{A}$, and so by [8, Theorem 3.3.3] $\mathfrak{B}$ is completely representable. Thus $\mathfrak{A} \in \mathbf{El}\mathsf{CRCA}_n$. One shows that $\mathbf{El}(\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \subseteq \mathsf{LCA}_n$ using Lemma 1.2. So $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n \subseteq \mathbf{El}(\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \subseteq \mathsf{LCA}_n$, and we are done.

(4): That $\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$ follows from the construction in [15]. The atomic countable $\mathfrak{C} \in \mathsf{RCA}_n$ used in the previous item is in $\mathbf{El}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega \sim \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$. Let $\mathfrak{A} = \mathfrak{Tm}\mathbf{At}$ be the algebra constructed in Theorem 1.3. We know that $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathbf{At}$, but $\mathfrak{A} \notin \mathsf{LCA}_n$, because $\mathsf{At}\mathfrak{A}$ does not satisfy the Lyndon conditions, lest $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathsf{LCA}_n (\subseteq \mathsf{RCA}_n)$. We conclude that $\mathfrak{A} \notin \mathbf{El}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$ proving the strictness of the last inclusion. Since $\mathfrak{E}, \mathfrak{C}$ and $\mathfrak{A}$ are all atomic, we are done.

(5): We use the construction in [14, Theorem 5.12]. The algebra $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} (\in \mathsf{RCA}_n)$ based on $\mathbb{Z}$ (greens) and $\mathbb{N}$ (reds) denotes the rainbow-like algebra used in *op.cit.* One can devise a k rounded game $\mathbf{H}_k$, where k is $\leq \omega$, such that a winning strategy in $\mathbf{H}_\omega(\alpha)$, α a countable atom structure, implies that $\mathfrak{Cm}(\alpha) \in \mathsf{Nr}_n\mathsf{CA}_\omega$, cf. [5] for the RA analogue. Next, it can be shown that $\exists$ has a winning strategy in $\mathbf{H}_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$ for all $k \in \omega$.

Using ultrapowers and an elementary chain argument as in [8, Theorem 3.3.5], one gets a countable and atomic $\mathfrak{B} \in \mathsf{CA}_n$ such $\exists$ has a winning strategy in $\mathbf{H}_\omega(\mathsf{At}(\mathfrak{B}))$, $\mathfrak{B} \equiv \mathfrak{C}_{\mathbb{Z},\mathbb{N}}$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. Since $\mathfrak{B} \subseteq_d \mathfrak{Cm}\mathsf{At}\mathfrak{B}$, $\mathfrak{B} \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$, so $\mathfrak{B} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$. Being countable, it follows by [12, Theorem 5.3.6] that $\mathfrak{B} \in \mathsf{CRCA}_n$. But it is proved in [14, Theorem 5.12], that $\forall$ has a winning strategy in $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$ (denoted in *op.cit* by $F^{n+3}(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$), hence by Lemma 1.2, $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Let $\mathbf{K}$ be a class between $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Then $\mathbf{K}$ is not elementary, because $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3} (\supseteq \mathbf{K})$, $\mathfrak{B} \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n (\subseteq \mathbf{K})$, and $\mathfrak{C}_{\mathbb{Z},\mathbb{N}} \equiv \mathfrak{B}$. We now use the construction in [11], where two atomic algebras $\mathfrak{A}, \mathfrak{B} \in \mathsf{CA}_n$ are constructed such that, $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{B} \notin \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. Thus $\mathfrak{B} \in \mathbf{El}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n) \sim \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$. Since $\mathbf{El}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n) \nsubseteq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$, there can be no elementary class between $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$. Having already eliminated elementary classes between $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, we are done. $\square$

References

[1] T. Sayed Ahmed, *Neat embedding is not sufficient for complete representations* Bulletin Section of Logic **36**(1) (2007) pp. 29–36.

[2] H. Andréka, M. Ferenczi and I. Németi (Editors), **Cylindric-like Algebras and Algebraic Logic.** Bolyai Society Mathematical Studies **22** (2013).

[3] H. Andréka, I. Németi and T. Sayed Ahmed, *Omitting types for finite variable fragments and complete representations.* Journal of Symbolic Logic. **73** (2008) pp. 65–89.

[4] L. Henkin, J.D. Monk and A. Tarski *Cylindric Algebras Part I,II.* North Holland, 1971, 1985.

[5] R. Hirsch, *Relation algebra reducts of cylindric algebras and complete representations*, Journal of Symbolic Logic, **72**(2) (2007), pp. 673–703.

[6] R. Hirsch and I. Hodkinson *Complete representations in algebraic logic*, Journal of Symbolic Logic, **62**(3)(1997) pp. 816–847.

[7] R. Hirsch and I. Hodkinson, *Relation Algebras by Games.* Studies In Logic. North Holland **147** (2002).

[8] R. Hirsch and I. Hodkinson *Completions and complete representations*, in [2] pp. 61–90.

[9] I. Hodkinson, *Atom structures of relation and cylindric algebras.* Annals of pure and applied logic, **89**(1997), p.117–148.

[10] B. Samir and T. Sayed Ahmed *A Neat Embedding Theorem for expansions of cylindric algebras.* Logic Journal of $IGPL$ **15**(2007) pp.41–51.

[11] T. Sayed Ahmed *The class of neat reducts is not elementary.* Logic Journal of $IGPL$, **9**(2001), pp. 593–628.

[12] T. Sayed Ahmed, *Neat reducts and neat embeddings in cylindric algebras.* In [2], pp.90–105.

[13] T. Sayed Ahmed *Completions, Complete representations and Omitting types*, in [2], pp. 186–205.

[14] T. Sayed Ahmed *On notions of representabililty for cylindric polyadic algebras and a solution to the finitizability problem for first order logic with equality.* Mathematical Logic quarterly (2015) **61**(6) pp. 418–447.

[15] T. Sayed Ahmed and I. Németi, *On neat reducts of algebras of logic*, Studia Logica. **68(2)** (2001), pp. 229–262.

[16] S. Shelah, *Classification theory: and the number of non-isomorphic models.* Studies in Logic and the Foundations of Mathematics (1990).