What structures can numbers have in relativity theory?

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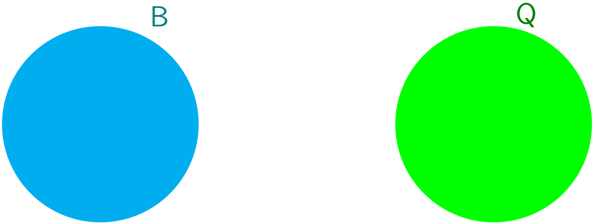
The question at this level is too naive/vague...

What structures can quantities have in certain physical theories?

What structures can quantities have in certain spacetime theories?

What structures can quantities have in special relativity?

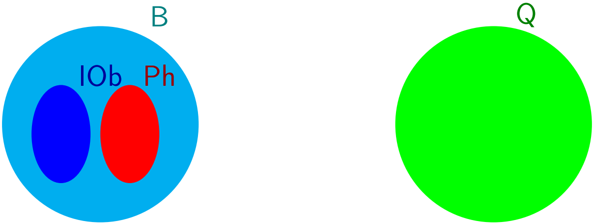
Language: $\{$ B, Q, $\}$



B $\rightsquigarrow$ Bodies (things that move)

Q $\rightsquigarrow$ Quantities

Language: $\{$ B, IOb, Ph, Q, $\}$

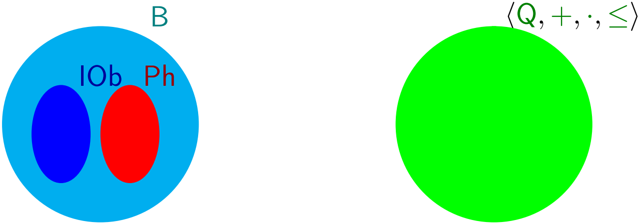


B ⇝ Bodies (things that move)

IOb ⇝ Inertial Observers   Ph ⇝ Photons (light signals)

Q ⇝ Quantities

Language: $\{\, \mathsf{B}, \mathsf{IOb}, \mathsf{Ph}, \mathsf{Q}, +, \cdot, \leq, \quad \}$



B ⇝ Bodies (things that move)
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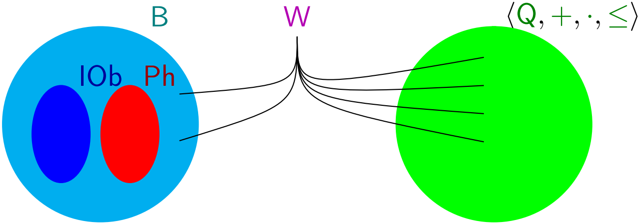
Q ⇝ Quantities $+$, $\cdot$ and $\leq$ ⇝ field operations and ordering

Language: $\{ \mathsf{B}, \mathsf{IOb}, \mathsf{Ph}, \mathsf{Q}, +, \cdot, \leq, \mathsf{W} \}$



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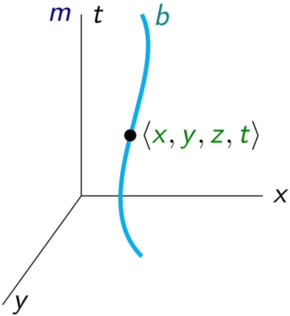
Q $\leftrightsquigarrow$ Quantities $+, \cdot$ and $\leq$ $\leftrightsquigarrow$ field operations and ordering

W $\leftrightsquigarrow$ Worldview (a 6-ary relation of type BBQQQQ)

$\mathsf{W}(m, b, x, y, z, t) \leftrightsquigarrow$ „ observer m coordinatizes body b at spacetime location $\langle x, y, z, t\rangle$."



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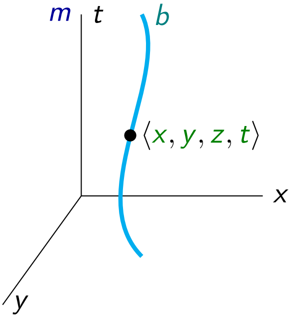

The *worldline* of body b according to observer m

$$wline_m(b) = \{\langle x, y, z, t\rangle \in \mathsf{Q}^4 \ : \ \mathsf{W}(m, b, x, y, z, t)\}$$

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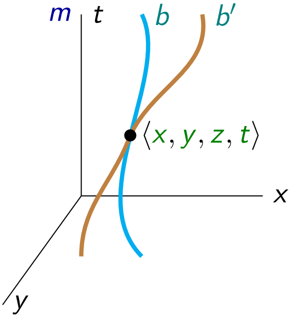
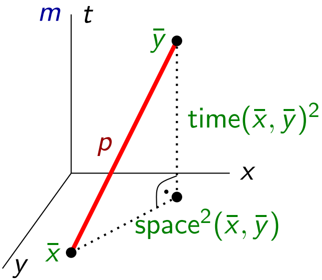


The *event* coordinatized by obserer m at $\langle x, y, z, t\rangle$:

$$ev_m(\langle x, y, z, t\rangle) = \{b \in \mathsf{B} \;:\; \mathsf{W}(m, b, x, y, z, t)\}$$

Axiom [AxPh]:

For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.



$$\mathsf{IOb}(m) \to \exists c\Big[c > 0 \land \forall \bar{x}\bar{y}\Big(\exists p\big[\mathsf{Ph}(p) \land \mathsf{W}(m,p,\bar{x}) \land \mathsf{W}(m,p,\bar{y})\big] \leftrightarrow \mathsf{space}^2(\bar{x},\bar{y}) = c^2 \cdot \mathsf{time}(\bar{x},\bar{y})^2\Big)\Big]$$

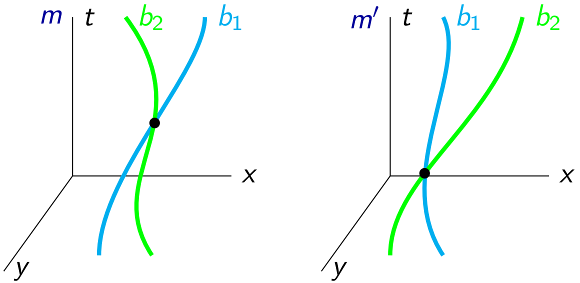
Axiom [AxField]:

The **structure of quantities** $\langle Q, +, \cdot, \leq \rangle$ *is an ordered field.*

- Rational numbers: $\mathbb{Q}$,
- $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\pi)$, ...
- Computable numbers,
- Constructable numbers,
- Real algebraic numbers: $\overline{\mathbb{Q}} \cap \mathbb{R}$,
- **Real numbers**: $\mathbb{R}$,
- Hyperrational numbers: $\mathbb{Q}^*$,
- Hyperreal numbers: $\mathbb{R}^*$,
- Etc.

Axiom [AxEv]:

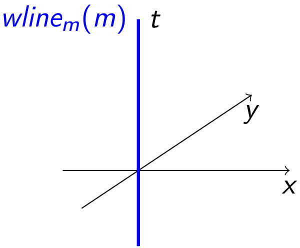
Inertial observers coordinatize the same events (meetings of bodies).



$$\forall m\, m' \bar{x}\;\; \mathsf{IOb}(m) \wedge \mathsf{IOb}(m') \rightarrow \left[\exists \bar{x}'\;\; \forall b\;\; \mathsf{W}(m, b, \bar{x}) \leftrightarrow \mathsf{W}(m', b, \bar{x}')\right].$$

Axiom [AxSelf]:

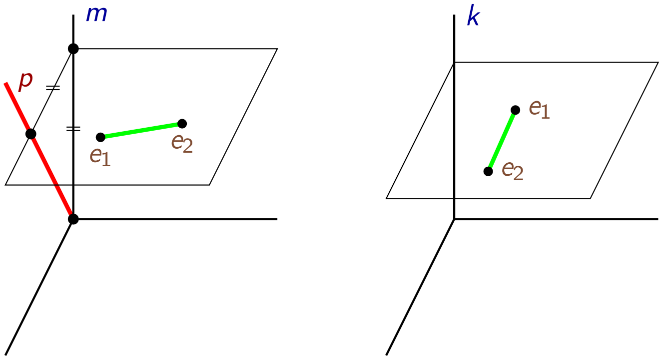
Inertial observers are stationary according to themselves.



$$\forall mxyzt\ \big(\mathsf{IOb}(m) \rightarrow \big[\mathsf{W}(m,m,x,y,z,t) \leftrightarrow x = y = z = 0\big]\big).$$

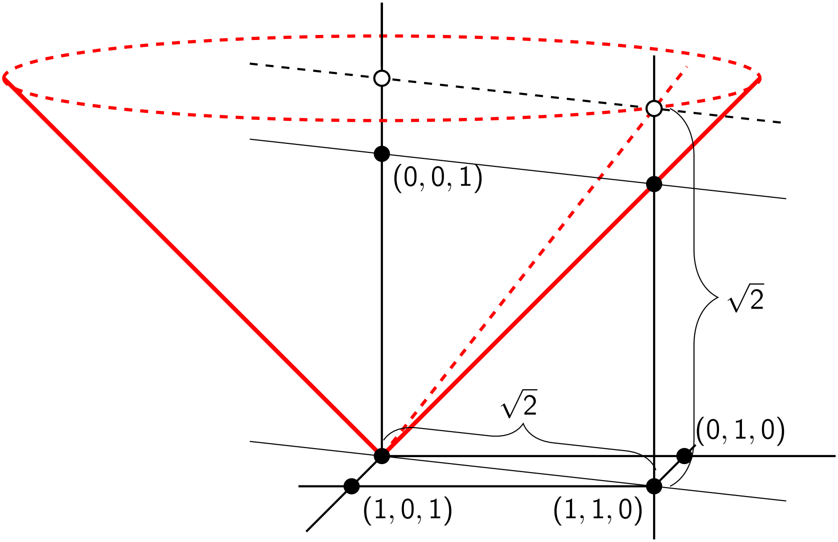
Axiom [AxSym]:

Inertial observers can use the same units of measurements.



SpecRel= **AxPh**+ AxField+ AxEv+ AxSelf+ AxSym

Missing lines over $\mathbb{Q}$



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Theorem: (Andréka–Madarász–Németi,1998)

SpecRel $\models$ *„moving clocks get out of synchronism,"*
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„moving spaceships shrink,"
etc.

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SpecRel $\models$ *„moving clocks get out of synchronism,"*
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Question: (Research direction)

What would remain from the theorems of SpecRel*, if we replaced ordered fields with other algebraic structures, e.g., with ordered rings?*

This framework generalizes naturally over any dimension $d \geq 2$.

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Def.:

$\mathrm{Numbers}_n(\mathrm{Th}) = \{$ *The quantity parts* $\langle \mathrm{Q}, +, \cdot, \leq \rangle$ *of the models of* Th *if* $d = n\}$.

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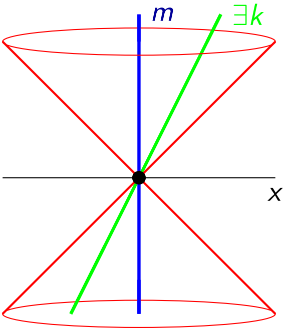
Example:

$$\mathsf{Numbers}_n(\mathsf{SpecRel}) = \textit{ordered fields}$$

Axiom [AxThExp]:

Inertial observers can move in any direction at any speed less than the speed of light.



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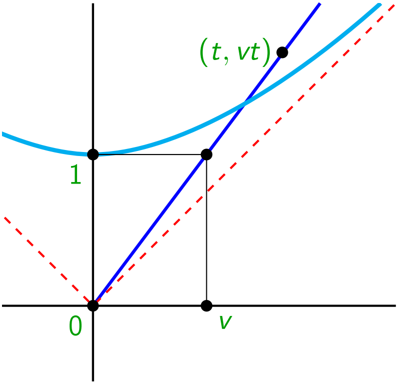

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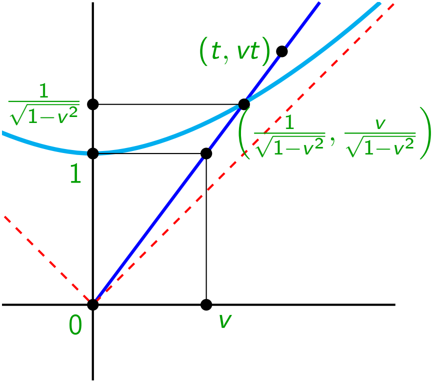

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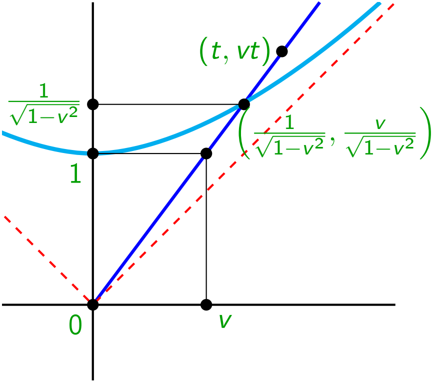

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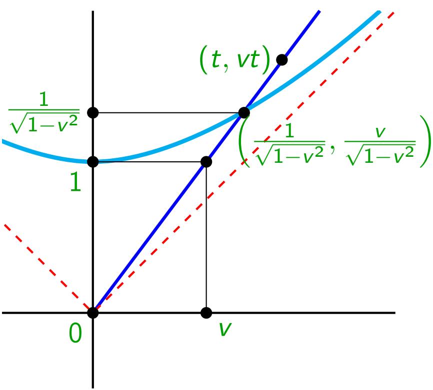

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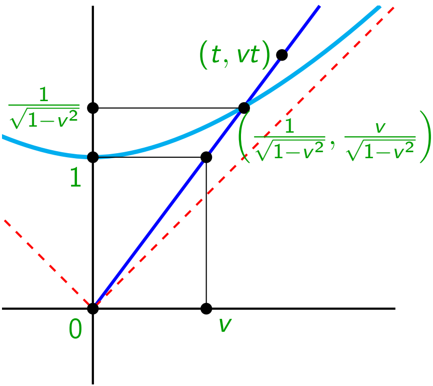

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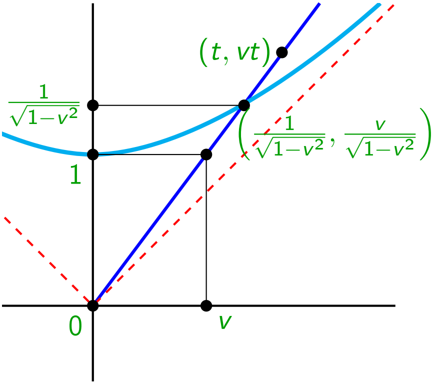

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$\sqrt{x} \in Q$

Theorem: (Andréka–Madarász–Németi, 1998)

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Question: ($n \geq 4$)

$$\text{Numbers}_n(\text{SpecRel}_0 + \text{AxThExp}) = ???$$

Axiom [AxThExp$^-$]:

Inertial observers can move in any direction at a speed which is arbitrarily close to any speed less than the speed of light.

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Conjecture:

$$\mathrm{Numbers}_n(\mathrm{SpecRel} + \mathrm{AxThExp}^-) = \textit{ordered fields}$$

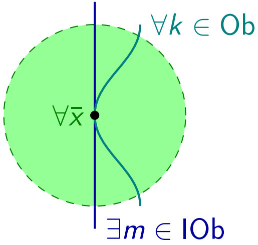
What structures can quantities have in general relativity?

Accelerated observers (AccRel) / general relativity (GenRel)

The language of AccRel/GenRel is the same as that of SpecRel.

Axiom [AxCmv]:

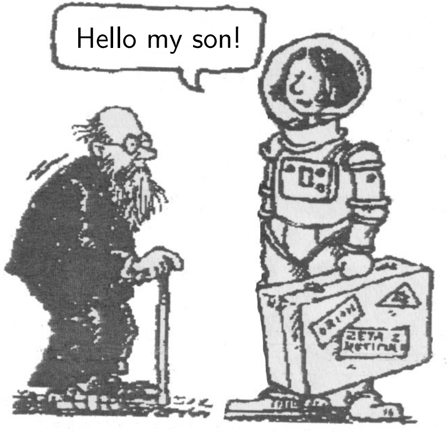
At each moment of its life, every observer coordinatizes the nearby world for a short while in the same way as an inertial observer does.



$$\forall k \in \mathrm{Ob}\ \forall \bar{x} \in wline_k(k)\ \exists m \in \mathrm{IOb} \quad d_{\bar{x}}\mathrm{w}_{mk} = Id, \text{ where}$$

$$d_{\bar{x}}\mathrm{w}_{mk} = L \overset{def}{\iff} \forall \varepsilon > 0\ \exists \delta > 0\ \forall \bar{y}\ \ |\bar{y} - \bar{x}| \le \delta$$
$$\to |\mathrm{w}_{mk}(\bar{y}) - L(\bar{y})| \le \varepsilon |\bar{y} - \bar{x}|.$$

Twin paradox $\rightsquigarrow$ TwP



Theorem: (Sz, 2004)

$$\mathsf{SpecRel} + \mathsf{AxCmv} + \langle \mathsf{Q}, +, \cdot, \leq \rangle \cong \mathbb{R} \quad \models \quad \mathsf{TwP}$$

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$$\mathsf{SpecRel} + \mathsf{AxCmv} + \langle \mathsf{Q}, +, \cdot, \leq \rangle \cong \mathbb{R} \quad \models \quad \mathsf{TwP}$$
$$\mathsf{SpecRel} + \mathsf{AxCmv} + \mathsf{Th}(\mathbb{R}) \quad \not\models \quad \mathsf{TwP}$$

Axiom schema [CONT]:

Every definable, bounded and nonempty subset of Q *has a supremum.*

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- ~~Constructible numbers~~,
- Real algebraic numbers: $\mathbb{A}$,
- **Real numbers**: $\mathbb{R}$,
- ~~Hyperrational numbers: $\mathbb{Q}^*$~~,
- Hyperreal numbers: $\mathbb{R}^*$,
- Etc.

Theorem: (Madarász–Németi–Sz, 2006)

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Axiom [AxEv$^-$]:

Any observer encounters the events in which he was observed.

Axiom [AxSelf$^-$]:

The worldline of an observer is an open interval of the time-axis, in his own worldview.

$$\mathsf{AccRel} = \mathsf{SpecRel} + \mathbf{AxCmv} + \mathsf{CONT} + \mathsf{AxEv}^- + \mathsf{AxSelf}^-$$

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Proposition:

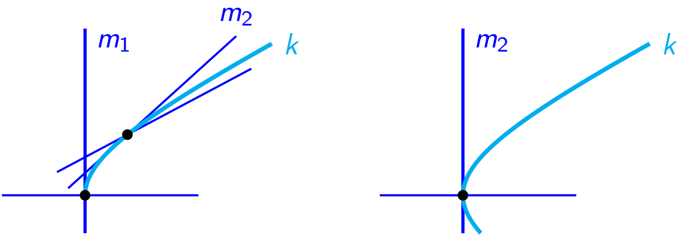
$\mathsf{Numbers}_n(\mathsf{AccRel}) =$ *real closed fields.*

Are real closed fields enough to model every accelerated observer?

Axiom [Ax∃UnifOb]:

Observers can accelerate uniformly.

Uniform acceleration: the acceleration is the same according to every inertial comoving observer.



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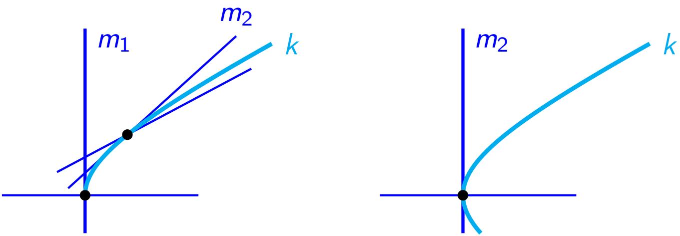

Theorem: (Sz, 2015)

$$\mathsf{AccRel} + \mathsf{Ax\exists UnifOb} \models e \in Q$$

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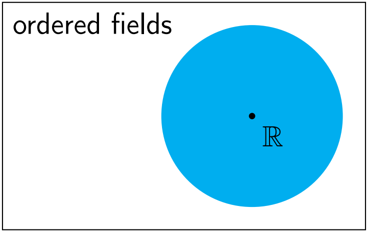
The class of coordinate time structures of theory AccRel + Ax∃UnifOb *is not axiomatizable in the language* $\langle +, \cdot, \leq \rangle$.



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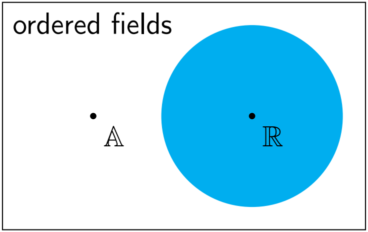

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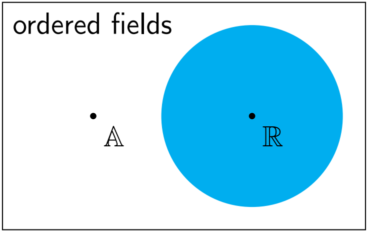

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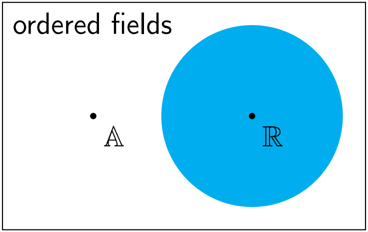

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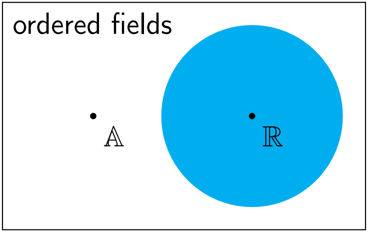
Question:

$$\text{Numbers}_n(\text{AccRel} + \exists\text{UnifOb}) = ???$$

Theorem: (Sz, 2015)

The class of coordinate time structures of theory AccRel + Ax∃UnifOb is not axiomatizable in the language $\langle +, \cdot, \leq \rangle$.



$$\mathbb{A} \sim \mathbb{R}$$

Question:

$$\mathrm{Numbers}_n(\mathrm{AccRel} + \exists\mathrm{UnifOb}) = ???$$

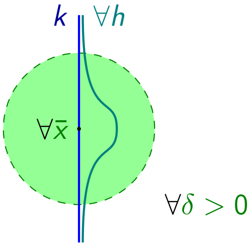
?Exponential ordered fields? (Salma Kuhlmann)

What about even more fancy observers?

Def. (Geodesic):

The worldline of an observer is called timelike geodesic if it „locally maximizes measured time.”

Twin paradox $\rightsquigarrow$



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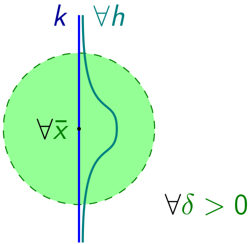

Axiom schema [COMPR]:

For any parametrically definable timelike curve in any observers worldview, there is another observer whose worldline is the range of this curve.

Proposition:

$$\mathsf{Numbers}_n(\mathsf{AccRel} + \mathsf{COMPR}) \subsetneq \mathsf{Numbers}_n(\mathsf{AccRel} + \exists\mathsf{UnifOb})$$

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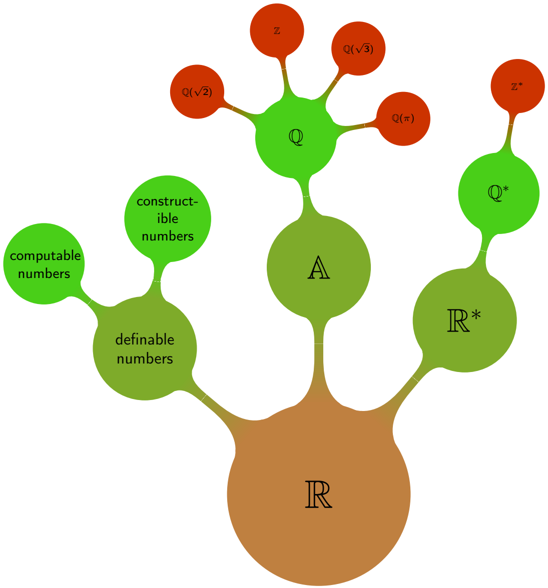
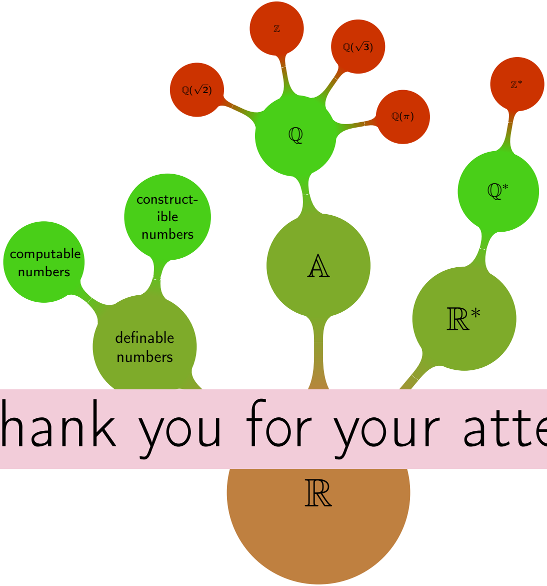
Question:

$$\mathsf{Numbers}_n(\mathsf{AccRel} + \mathsf{COMPR}) = ???$$
$$\mathsf{Numbers}_n(\mathsf{GenRel} + \mathsf{COMPR}) = ???$$

???differentially closed fields??? (Abraham Robinson)


$\mathbb{Z}$
$\mathbb{Q}(\sqrt{3})$
$\mathbb{Q}(\sqrt{2})$
$\mathbb{Q}(\pi)$
$\mathbb{Z}^*$
$\mathbb{Q}$
$\mathbb{Q}^*$
construct-
ible
numbers
computable
numbers
$\mathbb{A}$
$\mathbb{R}^*$
definable
numbers
$\mathbb{R}$



Thank you for your attention!