A completeness theorem for general relativity

Judit Madarász and Gergely Székely

Rényi Institute of Mathematics, Budapest

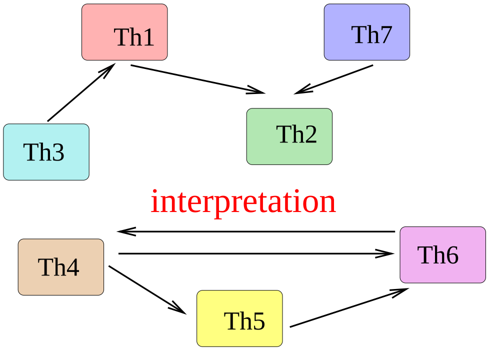
Logic, Relativity and Beyond, 2015

It was a dream of the Vienna Circle to build up scientific theories in mathematical logic

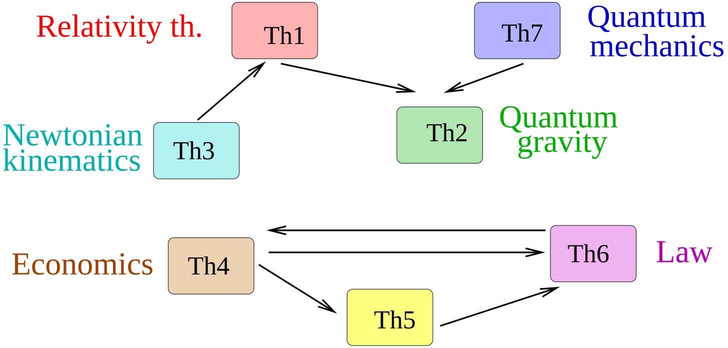
Benefits of building up a scientific theory (e.g. relativity theory) in mathematical logic:

- Uncover the tacit assumptions and make them explicit.
- Axiomatize theories (in the sense of math.logic).
- Analyze the relations between assumptions and consequences.
- Answer why type questions.
- Etc.

Breaking up a big theory into many smaller ones



CONNECT DIFFERENT THEORIES, AREAS



Relativity theory in first-order logic
Andréka–Németi School/Team

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Gergely Székely (after the coffee break) What structures can numbers have in relativity theory?

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István Németi and Hajnal Andréka Relativity theory via a network of logic theories

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Attila Molnár Some Expressive Temporal Logic of Minkowski Spacetimes

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Wednesday 5pm

Mike Stannett Using an Automated Theorem Prover to Support First Order Relativity Theory

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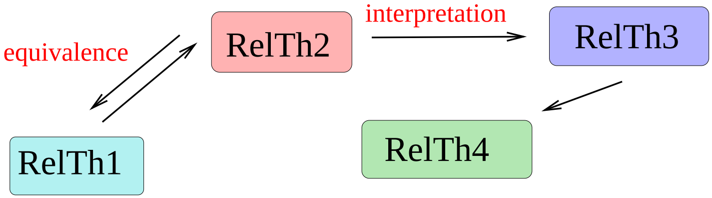


Axiomatization in general:

Axioms:

Ax.1.

Ax.2.

Ax.3.

Etc.

⟶

Theorems:

Thm.1.

Thm.2.

Thm.3.

Etc.

Economical

Streamlined

Transparent

Observational

Rich

Complex

Logic Language: { B, Q, }

B ⇝ Bodies (things that move)

Q ⇝ Quantities

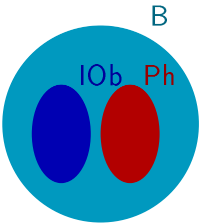
Logic Language: $\{B, IOb, Ph, Q, \qquad\qquad \}$

B ⇝ Bodies (things that move)

IOb ⇝ Inertial Observers   Ph ⇝ Photons (light signals)

Q ⇝ Quantities

Logic Language: $\{\,B, IOb, Ph, Q, +, \cdot, \leq, \quad\}$

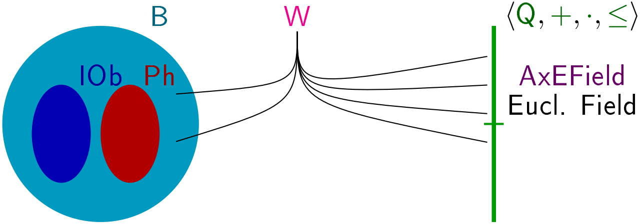


B ⇝ Bodies (things that move)

IOb ⇝ Inertial Observers   Ph ⇝ Photons (light signals)

Q ⇝ Quantities $+$, $\cdot$ and $\leq$ ⇝ field operations and ordering

Logic Language: $\{ \mathsf{B}, \mathsf{IOb}, \mathsf{Ph}, \mathsf{Q}, +, \cdot, \leq, \mathsf{W} \}$



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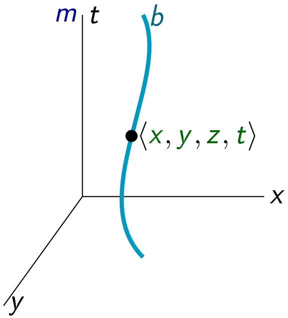
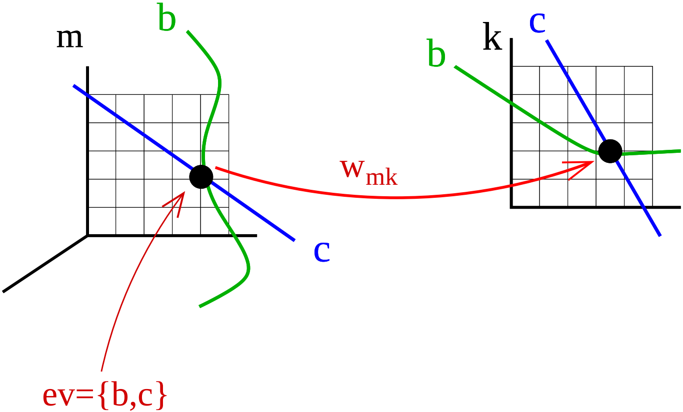
W ⇝ Worldview (a 6-ary relation of type BBQQQQ)

$\mathsf{W}(m, b, x, y, z, t) \leftrightsquigarrow$ "observer m coordinatizes body b at spacetime location $\langle x, y, z, t \rangle$."



Worldline of body b according to observer m

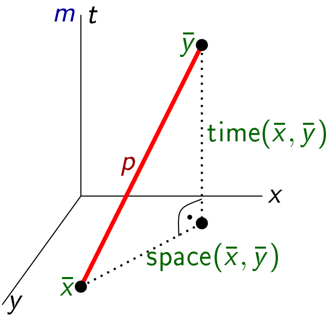
$$wline_m(b) = \{\langle x, y, z, t\rangle \in \mathsf{Q}^4 \ : \ \mathsf{W}(m, b, x, y, z, t)\}$$



m
b
c
k
b
c
w_{mk}
ev={b,c}

AxPh :

For any inertial observer, the speed of light is 1*. Furthermore, it is possible to send out a light signal in any direction.*



$$\forall m\Big(\mathsf{IOb}(m) \rightarrow \ \forall \bar{x}\bar{y}\ \Big(\exists p\big[\mathsf{Ph}(p) \wedge \mathsf{W}(m,p,\bar{x}) \ \wedge \mathsf{W}(m,p,\bar{y})\big] \leftrightarrow$$

$$\leftrightarrow \ \mathsf{space}(\bar{x},\bar{y}) = \mathsf{time}(\bar{x},\bar{y})\Big)\Big)$$

AxEv :

Inertial observers coordinatize the same events.

AxSelf :

Every Inertial observer is stationary according to himself.

AxSym :

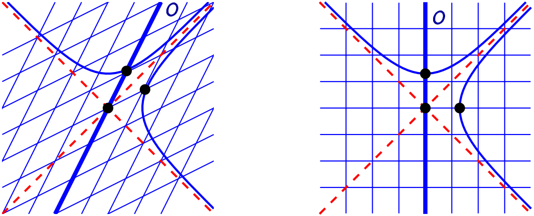
Inertial observers use the same units of measurements.

$$\mathsf{SpecRel} = \mathsf{AxPh} + \mathsf{AxEField} + \mathsf{AxEv} + \mathsf{AxSelf} + \mathsf{AxSym}$$

SpecRel = AxPh + AxEField + AxEv + AxSelf + AxSym

Theorem:

SpecRel $\models$ "The worldview transformations between inertial observers are Poincaré transformations."

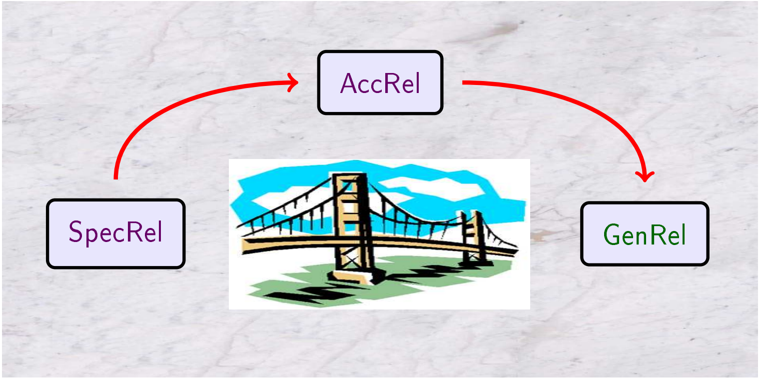


worldview of o' worldview of o

Theorem:

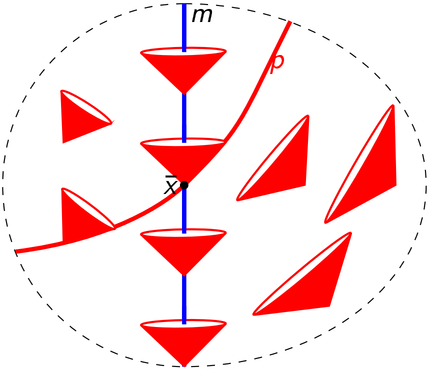
SpecRel $\Rightarrow$ {
"Relatively moving clocks slow down.",
"Relatively moving spaceships shrink."
etc.

General Relativity



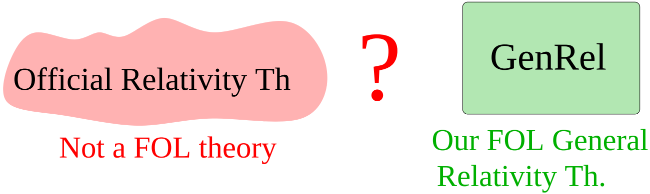
AxPh^{-} :

If an observer sends out a light signal, then the speed of the light-signal is 1 *according to the observer, and it is possible to send out a light signal in any direction.*



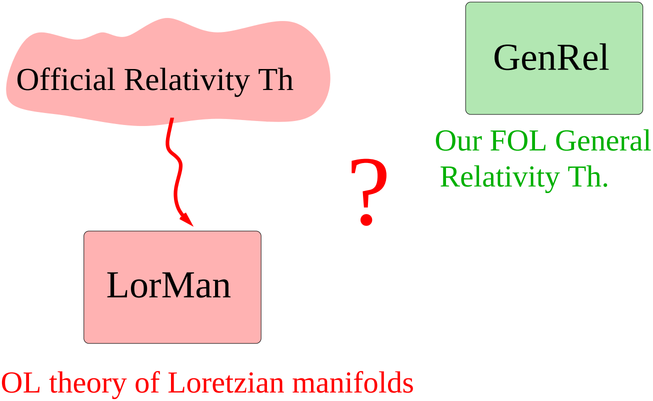
General Relativity

GenRel $=$**AxPh**$^{-}$$+$AxSm^{-}$+\dots$



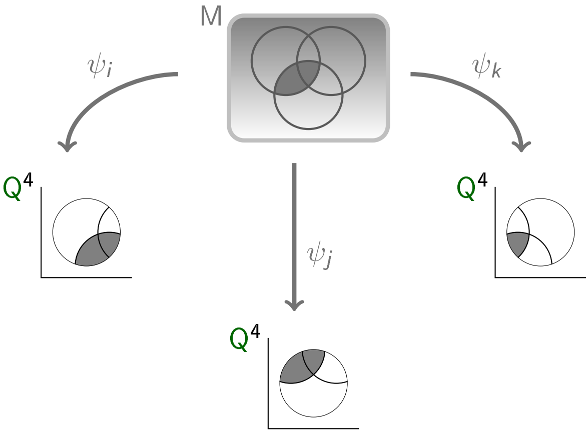
General Relativity

$$\text{GenRel} = \mathbf{AxPh}^{-} + \text{AxSm}^{-} + \ldots$$



Lorentzian Manifolds

(M, g), M is a 4 dim. Man. and $g_p : TM_p \times TM_p \to \mathrm{Q}$ Lorentzian metric



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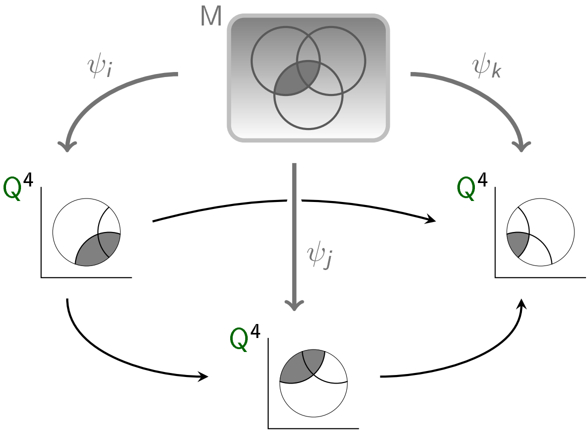

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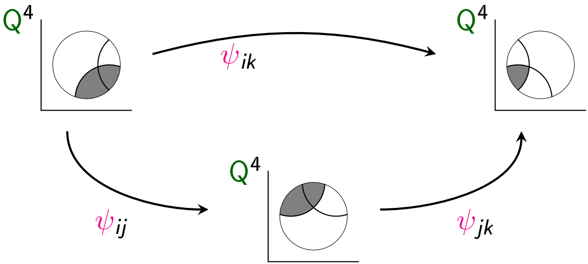

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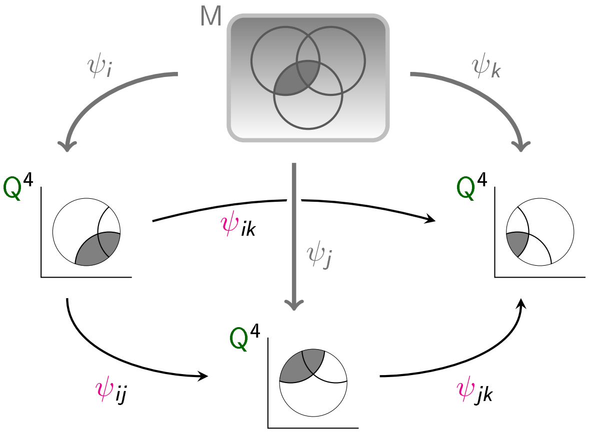


FOL language for Lorentzian manifolds

$$\{\, I, Q, \qquad\qquad\qquad \}$$

$I \leftrightsquigarrow$ (Indexes) of charts $\quad Q \leftrightsquigarrow$ Quantities, FOL language for Lorentzian manifolds

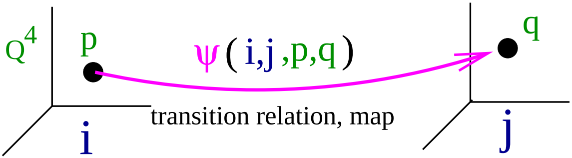
$$\{\, \mathsf{I}, \mathsf{Q}, +, \cdot, \leq, \qquad \}$$

$\mathsf{I} \leftrightsquigarrow$ (Indexes) of charts $\quad \mathsf{Q} \leftrightsquigarrow$ Quantities, $\quad +, \ \cdot, \ \leq$

FOL language for Lorentzian manifolds

$$\{\, \mathsf{I}, \mathsf{Q}, +, \cdot, \leq, \psi, \quad \}$$

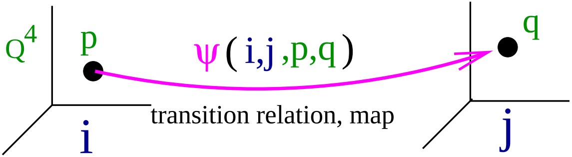
I ⇝ (Indexes) of charts Q ⇝ Quantities, $+, \cdot, \leq$

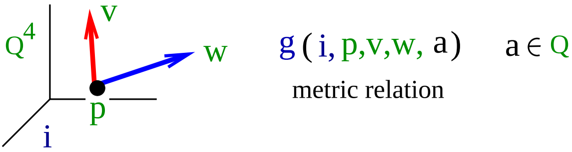


FOL language for Lorentzian manifolds

$$\{\, \mathsf{I}, \mathsf{Q}, +, \cdot, \leq, \psi, g \,\}$$

I ⇝ (Indexes) of charts Q ⇝ Quantities, $+, \cdot, \leq$





LorMan is a FOL theory of Lorentzian Manifolds.

LorMan is a FOL theory of Lorentzian Manifolds.

$$\mathcal{G} \models \text{GenRel}$$

$$\mathcal{G} \stackrel{TR}{\longmapsto} \mathcal{M} \qquad \models \text{LorMan}$$

$$tr : Fm(\text{LorMan}) \longrightarrow Fm(\text{GenRel})$$

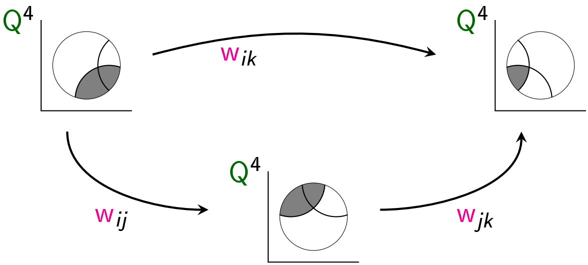
Theorem:

GenRel is complete with respect to the "standard models of GR", i.e., Lorentzian manifolds.

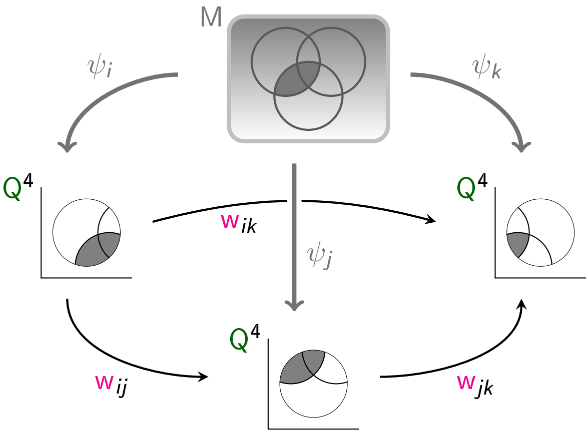
$$\text{GenRel} \models \; tr(\text{LorMan})$$

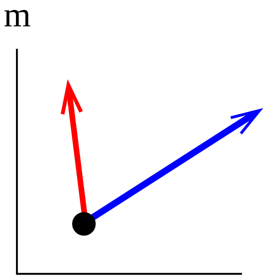
.

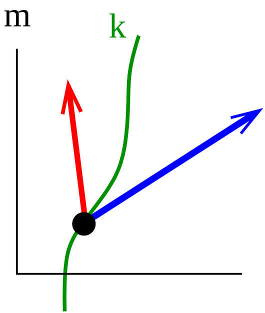
GenRel Models

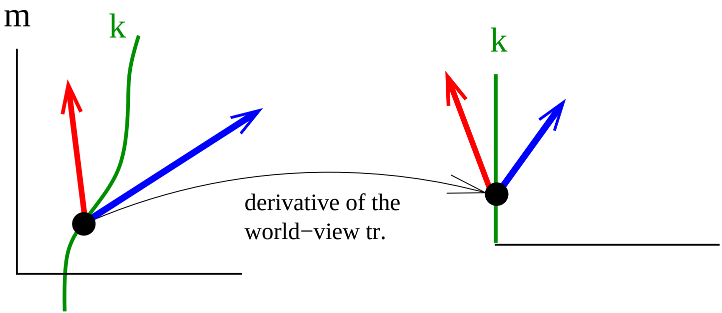


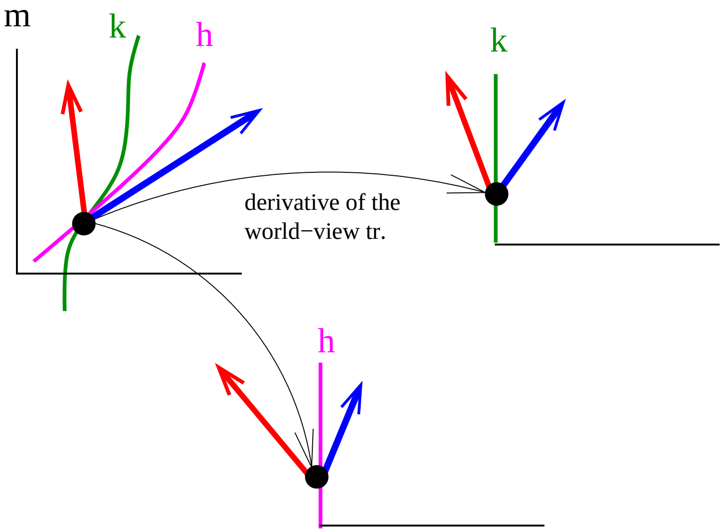
GenRel Models

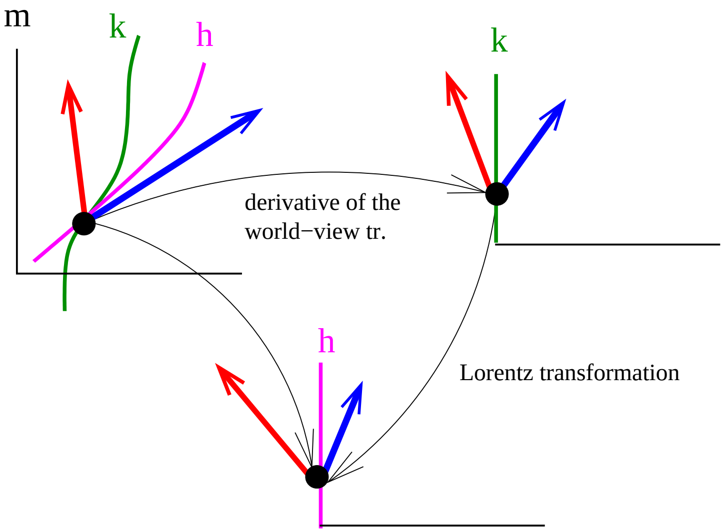



m

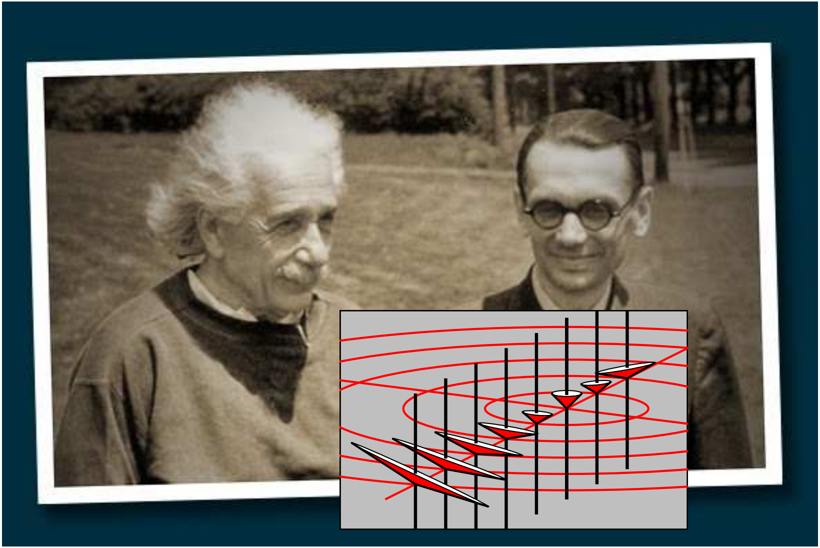

m
k


m
k
derivative of the
world−view tr.
k


m
k
h
k
derivative of the
world−view tr.
h


m
k
h
k
derivative of the
world−view tr.
h
Lorentz transformation

What about the other direction???



Einstein and Gödel (Princeton, 1948)

$\mathcal{M}$ is a Lorentzian Manifold over the Reals

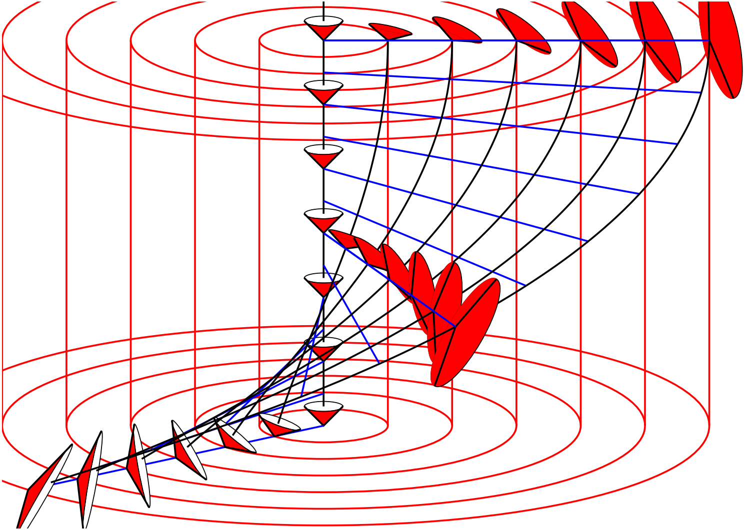
$$\mathcal{M} \overset{TR'}{\longmapsto} \mathcal{G} \qquad \models \text{GenRel}$$

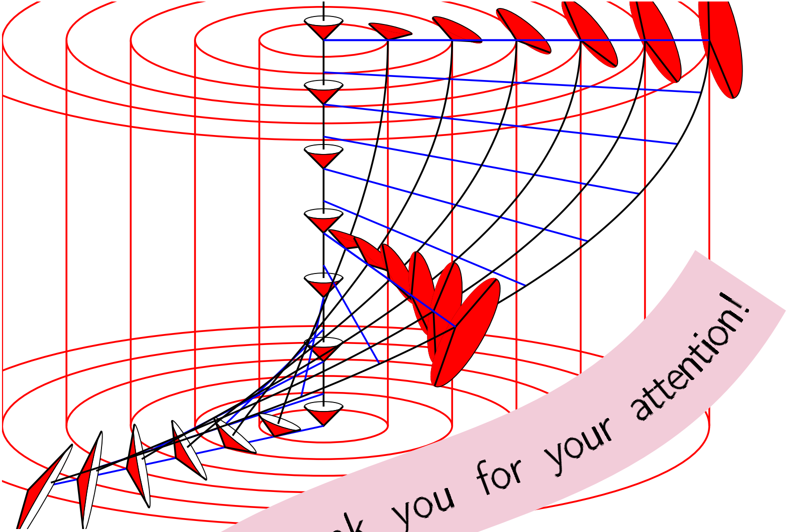
Theorem:

For any Lorentzian Manifold $\mathcal{M}$ over the Reals, if

$$\mathcal{M} \overset{TR'}{\longmapsto} \mathcal{G} \overset{TR}{\longmapsto} \mathcal{N}$$

then $\mathcal{M}$ and $\mathcal{N}$ are izometric, thus they reprezent the same spacetime.





Thank you for your attention!