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Paul Erdős and Interpolation: Problems, Results, New Developments

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1. Introduction

Pál (Paul) Erdős was born 100 years ago (March 26, 1913 in Budapest). He died on September 20, 1996 in Warsaw, when he attended a conference. He wrote about 1500 papers mainly with coauthors including those more than 80 works which are closely connected with approximation theory (interpolation, mean convergence, orthogonal polynomials, a.s.o.).

The present lecture tries to give a short summary of some significant results proved by *Erdős* (and his coauthors) and their new developments in approximation theory, primarily in interpolation; in a way it is an updated version of my previous work [47].

2. Interpolation, Lagrange interpolation, Lebesgue function, Lebesgue constant, optimal Lebesgue constant

Interpolation theory has been one of the favorite subjects of the twentieth century's Hungarian approximators. The backbone (mainly of classical interpolation) is the theory developed by *Lipót Fejér, Ervin Feldheim, Géza Grünwald, Pál Turán* and, of course, by *Pál Erdős*.

2.1. Let us begin with some definitions and notation. Let $C = C(I)$ denote the space of continuous functions on the interval $I := [-1, 1]$, and let $\mathcal{P}_n$ denote the set of algebraic polynomials of degree at most n. $\|\cdot\|$ stands for the usual maximum norm on C. Let X be an *interpolatory matrix (array)*, i.e.,

$$X = \left\{x_{kn} = \cos\vartheta_{kn}; \qquad k = 1, \ldots, n; \qquad n = 0, 1, 2, \ldots\right\},$$

with

$$-1 \leq x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} \leq 1, \tag{2.1}$$

$0 \leq \vartheta_{kn} \leq \pi$, and consider the corresponding *Lagrange interpolation polynomial*

$$L_n(f, X, x) := \sum_{k=1}^{n} f(x_{kn})\ell_{kn}(X, x), \qquad n \in \mathbb{N}. \tag{2.2}$$

Here, for $n \in \mathbb{N}$,

$$\ell_{kn}(X, x) := \frac{\omega_n(X, x)}{\omega_n'(X, x_{kn})(x - x_{kn})}, \qquad 1 \leq k \leq n,$$

with

$$\omega_n(X, x) := \prod_{k=1}^{n}(x - x_{kn}),$$

are polynomials of exact degree $n-1$. They are called the *fundamental polynomials* associated with the *nodes* $\{x_{kn},\, k = 1, \ldots, n\}$ obeying the relations $\ell_{kn}(X, x_{jn}) = \delta_{kj}$, $1 \leq k, j \leq n$.

The main question is: For what choices of the interpolation array X we can expect that (uniformly, pointwise, etc.) $L_n(f, X) \to f \quad (n \to \infty)$?

By the classical Lebesgue estimate,

$$
\begin{aligned}
\tag{2.3} \big|L_n(f,X,x)-f(x)\big| &\le \big|L_n(f,X,x)-P_{n-1}(f,x)\big| + \big|P_{n-1}(f,x)-f(x)\big| \\
&\le \big|L_n(f-P_{n-1},X,x)\big| + E_{n-1}(f) \\
&\le \left(\sum_{k=1}^{n} \big|\ell_{k,n}(X,x)\big| + 1\right) E_{n-1}(f),
\end{aligned}
$$

therefore, with the notations

$$\lambda_n(X,x) := \sum_{k=1}^{n} \big|\ell_{kn}(X,x)\big|, \qquad n \in \mathbb{N}, \tag{2.4}$$

$$\Lambda_n(X) := \big\|\lambda_n(X,x)\big\|, \qquad n \in \mathbb{N}, \tag{2.5}$$

(*Lebesgue function* and *Lebesgue constant* (of Lagrange interpolation), respectively,) we have for $n \in \mathbb{N}$

$$\big|L_n(f,X,x)-f(x)\big| \le \big\{\lambda_n(X,x)+1\big\} E_{n-1}(f) \tag{2.6}$$

and

$$\big\|L_n(f,X)-f\big\| \le \big\{\Lambda_n(X)+1\big\} E_{n-1}(f). \tag{2.7}$$

Above, as usual

$$E_{n-1}(f) := \min_{P \in \mathcal{P}_{n-1}} \|f - P\|.$$

In 1914 Georg Faber proved the then rather surprising lower bound

$$\Lambda_n(X) \geq \frac{1}{12} \log n, \qquad n \geq 1, \tag{2.8}$$

for any interpolation array X. Based on this result he obtained

Theorem 2.1. *For any fixed interpolation array X there exists a function $f \in C$ for which*

$$\overline{\lim_{n\to\infty}} \big\|L_n(f, X)\big\| = \infty. \tag{2.9}$$

2.2. The preceding estimates underline the importance of the Lebesgue function, $\lambda_n(X, x)$, and the Lebesgue constant, $\Lambda_n(X)$.

Using an estimate of *L. Fejér*

$$\Lambda_n(T) = \frac{2}{\pi} \log n + O(1),$$

one can see that the order $\log n$ in (2.8) is best possible (here T is the Chebyshev matrix, i.e. $x_{kn} = \cos \frac{2k-1}{2n}\pi$).

A very natural problem, raised and answered in 1958 by *Erdős*, says that $\lambda_n(X,x)$ is "big" on a "large" set.

Theorem 2.2. (Erdős [4]) *For any fixed interpolation matrix* $X \subset [-1,1]$*, real* $\varepsilon > 0$*, and* $A > 0$*, there exists* $n_0 = n_0(A,\varepsilon)$ *so that the set*

$$\left\{x \in \mathbb{R},\ \lambda_n(X,x) \le A \quad \textit{for all} \quad n \ge n_0(A,\varepsilon)\right\}$$

has measure less than ε*.*

The proof of Theorem 2.2 is based on the following simple looking statement (cf. [4, Lemma 3]).

Let $x_1, x_2, \ldots, x_n$ *be any* n $(n > n_0)$ *distinct numbers in* $[-1,1]$ *not necessarily in increasing order. Then, for at least one* j $(1 \le j \le n)$*,*

$$\sum_{k=1}^{j-1} \frac{1}{|x_k - x_j|} > \frac{n \log n}{8}.$$

(The half–page proof is based on the inequality between the arithmetic and harmonic means.)

Let us mention a nice, relatively new, generalization of this statement. In his paper [31] *Ying Guang Shi* proved as follows:

Theorem A. *Let, for a fixed* p*,* $0 < p < \infty$*;* $\mathbf{x} = (x_1, \ldots, x_n)$*,*

$$f_j(p, \mathbf{x}) := \sum_{k=1}^{j-1} \frac{1}{|x_k - x_j|^p}, \quad j = 1, 2, \ldots, n; \quad n \geq 2.$$

Then

$$\frac{1}{n} \sum_{j=1}^{n-1} f_j(p, \mathbf{x}) \geq \begin{cases} \dfrac{n-1}{2^{1+p}}, & 0 < p < 1, \\ \dfrac{(n-1)\log n}{4}, & p = 1, \\ \dfrac{(n-1)^{1+p}}{2^p\, n}, & p > 1. \end{cases}$$

Moreover, the order is the best possible and it is attained by the equidistant nodes.

The next statement, the more or less complete pointwise estimation, is due to *P. Erdős* and *P. Vértesi* [5] from 1981.

Theorem 2.3. *Let $\varepsilon > 0$ be given. Then, for any fixed interpolation matrix $X \subset [-1, 1]$ there exist sets $H_n = H_n(\varepsilon, X)$ of measure $\leq \varepsilon$ and a number $\eta = \eta(\varepsilon) > 0$ such that*

$$\lambda_n(X, x) > \eta \log n \tag{2.10}$$

if $x \in [-1, 1] \setminus H_n$ and $n \geq 1$.

Closer investigation shows that (instead of the original $\eta = c\varepsilon^3$) $\eta = c\varepsilon$ can be attained. The behaviour of the Chebyshev matrix, T, shows that (2.10) is the best possible regarding the order $\log n$.

2.3. Let us say some words about the *optimal Lebesgue constant.* In 1961, *P. Erdős*, improving a previous result of *P. Turán* and himself (see [6]), proved that

$$\left| \Lambda_n^* - \frac{2}{\pi} \log n \right| \leq c, \tag{2.11}$$

where

$$\Lambda_n^* := \min_{X \subset I} \Lambda_n(X), \qquad n \geq 1,$$

is the *optimal Lebesgue constant.* As a consequence of this result, the closer investigation of Λ_n^* attracted the attention of many mathematicians.

In 1978, *Ted Kilgore, Carl de Boor* and *Alan Pinkus* proved the so–called Bernstein-Erdős conjectures concerning the optimal interpolation array X (cf. [7] and [8]).

To formulate the conjecture and the result, let X be *canonical* if $x_{1n} = -x_{nn} = 1$. An elementary argument shows that to obtain the value Λ_n^* it is enough to consider the *canonical* matrices only. Moreover, if

$$\mu_{kn}(X) = \max_{x_{kn} \le x \le x_{k-1,n}} \lambda_n(X, x), \quad 2 \le k \le n, \quad n \ge 3,$$

denote the $n-1$ unique local maximum values of $\lambda_n(X, x)$,[1] then we state

Theorem 2.4. *Let $n \ge 3$. We have*

(i) *there exists a unique optimal canonical X^* with*

(ii) $\mu_{kn}(X^*) = \mu_{\ell n}(X^*) \qquad 2 \le k,\, \ell \le n.$

Moreover, for arbitrary interpolatory X

(iii) $\displaystyle\min_{2 \le k \le n} \mu_{kn}(X) \le \Lambda_n^* \le \max_{2 \le k \le n} \mu_{kn}(X).$

[1] It is easy to see that for *arbitrary* interpolatory X, $\lambda_n(X, x)$ is a piecewise polynomial with $\lambda_n(X, x) \ge 1$ and $\lambda_n(X, x) = 1$ iff $x = x_{kn}$, $1 \le k \le n$. Between the consecutive nodes $\lambda_n(X, x)$ has a *single* maximum, and in $(-1, x_{nn})$ and $(x_{1n}, 1)$ it is convex and monotone (see [46, p. 95]).

Using this result, (2.11) can be considerably improved. Namely,

$$\Lambda_n^* = \frac{2}{\pi}\log n + \chi + o(1), \quad n \to \infty, \tag{2.12}$$

where $\chi = \frac{2}{\pi}\left(\gamma + \log\frac{4}{\pi}\right) = 0.521251\ldots$ and $\gamma = 0.577215\ldots$ is the Euler constant (cf. *P. Vértesi* [9]).

2.4. Let us quote two new developments (Parts **2.4** and **2.5**) which are closely connected to the Bernstein–Erdős conjecture. The first is the second part of the previously mentioned paper of Y. G. Shi [31].

Let (using the notations of **2.3**)

$$F_n(p, \mathbf{x}) := \max_{1\le j\le n-1} f_j(p, \mathbf{x}).$$

We sholud like to get the vector

$$\mathbf{y} \in \mathbf{X} := \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \;:\; -1, \le x_1 < x_2 < \cdots x_n \le 1\}$$

such that

$$F_n(p, \mathbf{y}) = \inf_{\mathbf{x}\in\mathbf{X}} F_n(p, \mathbf{x}). \tag{2.13}$$

We see that the solution admits the equioscillation characterization of Bernstein and Erdős. Namely, Shi gets as follows

Theorem B. *Let* $0 < p < \infty$ *and* $n \geq 2$. *Then the following statements are valid:*

(a) *there exits a unique vector* $\mathbf{y} \in \mathbf{X}$ *satisfying* (2.13);

(b) (2.13) *holds if and only if*

$$f_1(p, \mathbf{y}) = f_2(p, \mathbf{y}) = \cdots = f_{n-1}(p, \mathbf{y});$$

(c) *for any vector* $\mathbf{x} \in \mathbf{X} \setminus \{\mathbf{y}\}$

$$\min_{1 \leq i \leq n-1} f_i(p, \mathbf{x}) < F_n(p, \mathbf{y}) < \max_{\leq i \leq n-1} f_i(p, \mathbf{x}).$$

2.5. This part of the present lecture gives another (rational) interpolatory process which shows the Bernstein–Erdős equioscillation character.

Let us define the *classical barycentric interpolation* formula for $f \in C$:

$$B_n(f, X, x) =: \sum_{k=1}^{n} f(x_{kn}) y_{kn}(X, x), \qquad n \in \mathbb{N}, \tag{2.14}$$

where

$$y_{kn}(X,x) =: \frac{\omega_n(X,x)\dfrac{(-1)^k}{x-x_{kn}}}{\omega_n(X,x)\displaystyle\sum_{j=1}^{n}\frac{(-1)^j}{x-x_{jn}}} = \frac{\dfrac{(-1)^k}{x-x_{kn}}}{\displaystyle\sum_{j=1}^{n}\frac{(-1)^j}{x-x_{jn}}}. \tag{2.15}$$

The first equation shows that y_{kn} is a *rational function* of the form P_{kn}/Q_n, where

$$P_{kn}(X,x) = (-1)^k \left|\omega_n'(X,x_{kn})\right| \ell_{kn}(X,x), \qquad 1 \le k \le n, \tag{2.16}$$

$$Q_n(X,x) = \sum_{j=1}^{n} \left|\omega_n'(X,x_{jn})\right| \ell_{jn}(X,x), \qquad n \in \mathbb{N}. \tag{2.17}$$

Above, $P_{kn} \in \mathcal{P}_{n-1} \setminus \mathcal{P}_{n-2}$ and $Q_n \in \mathcal{P}_{n-1}$.

The process B_n has the interpolatory property, i.e.

$$B_n(f,X,x_{kn}) = f(x_{kn}), \quad y_{kn}(X,x_{jn}) = \delta_{kj}, \qquad 1 \le k,j \le n;\ n \in \mathbb{N}.$$

Moreover it is not so difficult to prove the next fundamental relation valid for *arbitrary* matrix X:

$$Q_n(X,x) \ne 0 \quad \text{if} \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

One can verify that $\{y_{kn}(x),\ 1 \leq k \leq n\}$ is a Haar (Chebyshev (Tchebychev)) system (or briefly, T-system) for any fixed $n \in \mathbb{N}$. Actually, $T = T(\mathbf{x}_n)$ where $\mathbf{x}_n = (x_{1n}, x_{2n}, \ldots, x_{nn}) \in \mathbb{R}^n$.

Slightly modifying our previous notations we define for $f \in C$, $\mathbf{x}_n$ and $n \in \mathbb{N}$

$$L_n(f, \mathbf{x}_n, x) := \sum_{k=1}^{n} f(x_{kn}) y_{kn}(\mathbf{x}_n, x),$$

$$\lambda_n(\mathbf{x}_n, x) := \sum_{k=1}^{n} |y_{kn}(\mathbf{x}_n, x)|\,,$$

$$\Lambda_n(\mathbf{x}_n) := \|\lambda_n(\mathbf{x}_n, x)\|\,.$$

By definition, they are the Lagrange interpolatory *T-polynomials, T-Lebesgue functions and T-Lebesgue constants*, concerning the above defined T-system.

Using

$$1 = \sum_{k=1}^{n} y_{kn}(x) \leq \sum_{k=1}^{n} |y_{kn}(x)| = \lambda_n(\mathbf{x}_n, x),$$

the T-Lebesgue function $\lambda_n(\mathbf{x}_n, x) \geq 1$ with equality if $x \in \{x_{1n}, x_{2n}, \ldots, x_{nn}\}$. We define

$$\mu_{kn}(\mathbf{x}_n) := \max_{x_{k+1,n} \leq x \leq x_{kn}} \lambda_n(\mathbf{x}_n, x), \qquad 1 \leq k \leq n-1,\ n \geq 2.$$

We say that $\mathbf{x}_n$ is *canonical*, if $x_{1n} = -x_{nn} = 1$, $n \geq 2$ (if $\mathbf{x}_n$ is canonical, then obviously $\Lambda_n(\mathbf{x}_n) = \max_{1 \leq k \leq n-1} \mu_{kn}(\mathbf{x}_n)$).

Very recently we obtained the next statement which verifies the analogue of the Bernstein–Erdős conjecture for the above defined barycentric interpolation (cf. Theorem 2.4).

Theorem C. *Let $n \geq 3, fixed$. We have as follows*

(i) *There exists a unique optimal canonical* $\mathbf{x}_n^*$ *with*

(ii) $\mu_{kn}(\mathbf{x}_n^*) := \Lambda_n(\mathbf{x}_n^*), \quad 1 \leq k \leq n-1.$

Moreover for arbitrary interpolatory $\mathbf{x}_n$

(iii) $\min_{1 \leq k \leq n-1} \mu_{kn}(\mathbf{x}_n) \leq \Lambda_n(\mathbf{x}_n^*) \leq \max_{1 \leq k \leq n-1} \mu_{kn}(\mathbf{x}_n).$

In their paper B. A. Ibrahimoglu and A. Cuyt proved that for the equidistant nodes, $\mathbf{e}_n = \{-1 + 2j/(n-1);\ j = 0, 1, \ldots, n-1\}$

$$\Lambda(\mathbf{e}_n) = \frac{2}{\pi}\left(\log n + \log 2 + \gamma\right) + O\left(\frac{1}{n}\right).$$

Then by our Theorem C (iii), we immediately obtain

$$\Lambda_n(\mathbf{x}_n^*) = \frac{2}{\pi}(\log n + \log 2 + \gamma) + O\left(\frac{1}{n}\right) = \Lambda(\mathbf{e}_n) + O\left(\frac{1}{n}\right).$$

Let us remark that by (2.12)

$$\Lambda_n(\mathbf{x}_n^*) - \Lambda_n^* = \frac{2}{\pi}\log\frac{\pi}{2} + o(1) > 0.$$

The main ingredient of the proof of Theorem C is the next special case of a general statement proved by Y. G. Shi in 1998. Namely, using [12, Theorem 1], we can state, by obvious short notations, as follows.

Let $n \geq 3$ be fixed. Further, let $f_i(\mathbf{x}) \geq 0$, $i = 1, 2, \ldots, n-1$, be continuously differentiable functions on $\mathbf{X} = \{\mathbf{x} = (x_1, x_2, \ldots, x_n)\ :\ -1 = x_n < x_{n-1} < \cdots < x_2 < x_1 = 1\}$. Denote

$$f(\mathbf{x}) := \max_{1\leq k\leq n-1} f_k(\mathbf{x}).$$

With $\delta_k = x_k - x_{k+1}$ and $\delta = \min \delta_k$ $(1 \le k \le n-1)$, we state

Theorem D. *Suppose that the functions $f_k(\mathbf{x})$ satisfy the conditions*

(A) $$\lim_{\delta \to 0} \left(\max_{1 \le k \le n-1} |f_{k+1}(\mathbf{x}) - f_k(\mathbf{x})| \right) = \infty$$

and

(B) $$D_k(\mathbf{x}) := \det \left(\frac{\partial f_i(\mathbf{x})}{\partial x_j} \right)_{i=1, j=2, i \ne k}^{n-1, n-1} \ne 0, \quad \mathbf{x} \in \mathbf{X}, \quad 1 \le k \le n-1.$$

Then we have as follows.

(a) *There exists a unique vector $\mathbf{x}^* \in \mathbf{X}$ with*

$$f(\mathbf{x}^*) = \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x});$$

(b) *the relation* (a) *holds if and only if*

$$f_1(\mathbf{x}^*) = f_2(\mathbf{x}^*) = \cdots = f_{n-1}(\mathbf{x}^*);$$

(c) *for any other $\mathbf{x} \in \mathbf{X}$*

$$\min_{1 \le k \le n-1} f_k(\mathbf{x}) < f(\mathbf{x}^*) < \max_{1 \le k \le n-1} f_k(\mathbf{x})$$

2.6. As we know the Lagrange interpolation can be very bad even for the good matrix $T = \left\{ \cos \frac{2k-1}{2n}\pi \right\}$.

Theorem 2.5 (Grünwald–Marcinkiewicz)**.** *There exists a function $f \in C$ for which*

$$\overline{\lim_{n\to\infty}} \left| L_n(f, T, x) \right| = \infty$$

for every $x \in [-1, 1]$.

In their third joint paper, [10] *Erdős* and *Grünwald* claimed to prove the existence of an $f \in C$ for which

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \left| \sum_{k=1}^{n} L_k(f, T, x) \right| = \infty,$$

for all $x \in [-1, 1]$. However, as it was discovered later by *Erdős* himself, there had been an oversight in the proof and the method only gives the result with the modulus sign *inside* the summation.

Only in [11], where *Erdős* and *Gábor Halász* (who was born four years after the Erdős–Grünwald paper) were able to complete the proof and obtained the following.

Theorem 2.6. *Given a positive sequence* $\{\varepsilon_n\}$ *converging to zero however slowly, one can construct a function* $f \in C$ *such that for almost all* $x \in [-1, 1]$

$$\frac{1}{n}\left|\sum_{k=1}^{n} L_k(f, T, x)\right| \geq \varepsilon_n \log\log n$$

for infinitely many n.

The right-hand side is optimal, for in the paper [12] *Erdős* proved

Theorem 2.7.

$$\frac{1}{n}\left|\sum_{k=1}^{n} L_k(f, T, x)\right| = o(\log\log n)$$

for almost all x*, whenever* $f \in C$.

The proof of Theorem 2.7 is an ingenious combination of ideas from number theory, probability and interpolation.

2.7. After the result of *Grünwald* and *Marcinkiewicz* a natural problem was to obtain an analogous result for an *arbitrary* array X. In [4, p. 384], *Erdős* wrote: "In a subsequent paper I hope to prove the following result:

Let $X \subset [-1,1]$ be any point group [interpolatory array]. Then there exists a continuous function $f(x)$ so that for almost all x

$$\overline{\lim_{n\to\infty}} \left|L_n(f,X,x)\right| = \infty.\text{''}$$

After 4 years of work, *Erdős* and *P. Vértesi* proved the above result ([14]–[15]). *Erdős* writes in [13]: "[Here we prove the above] statement in full detail. The detailed proof turns out to be quite complicated and several unexpected difficulties had to be overcome."[2]

2.8. Another significant contribution of the Hungarian approximators to interpolation is the so called "fine and rough theory" (the name was coined by *Erdős* and *Turán* in their basic joint paper [16] dedicated to *L. Fejér* on his 75th birthday in 1955).

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[2] In a personal letter *Erdős* wrote about the *main idea of the proof*: [First] "we should prove that for every fixed A and $\eta > 0$ there exists an M ($M = M(A,\eta)$) such that if we divide the interval $[-1,1]$ into M equal parts $I_1, \dots, I_M$ then

$$\sum\nolimits_k' |\ell_{k,n}(X,x)| > A, \qquad x \in I_r,$$

apart from a set of measure $\le \eta$. Here $\sum'$ means that k takes those values for which $x \notin I_r$".

In the class $\mathrm{Lip}\,\alpha$ ($0 < \alpha < 1$; we use the natural setting) a natural error estimate for Lagrange interpolation is

$$\left\|L_n(f, X) - f\right\| \leq cn^{-\alpha}\Lambda_n(X)$$

(cf. (2.7)). *Erdős* and *Turán* raised the obvious question: *How sharp is this estimate in terms of the order of the Lebesgue constant as $n \to \infty$?* They themselves considered interpolatory arrays X where

$$\Lambda_n(X) \sim n^{\beta} \qquad (\beta > 0).$$

In the above paper [6] they prove essentially

Theorem 2.8. *Let X be as above. If $\alpha > \beta$, then we have uniform convergence in $\mathrm{Lip}\,\alpha$. If $\alpha \leq \beta/(\beta + 2)$, then for some $f \in \mathrm{Lip}\,\alpha$, Lagrange interpolation is divergent.*

These two cases comprise what is called the "rough theory", since *solely on the basis of the order* of $\Lambda_n(X)$ one can decide the convergence-divergence behavior. However,

Theorem 2.9. *If $\beta/(\beta+2) < \alpha \le \beta$ then anything can happen. That is, there is an interpolatory array Y_1 with $\Lambda_n(Y_1) \sim n^\beta$ and a function $f_1 \in \operatorname{Lip}\alpha$ such that $\overline{\lim}_{n\to\infty} \left\| L_n(f_1, Y_1) \right\| = \infty$, and another interpolation array Y_2 with $\Lambda_n(Y_2) \sim n^\beta$, such that $\lim_{n\to\infty} \left\| L_n(f, Y_2) - f \right\| = 0$ for every $f \in \operatorname{Lip}\alpha$.*

That is, to decide the convergence-divergence behavior *we need more information than just the order of the Lebesgue constant.* The corresponding situation is called "fine theory".

This paper of *Erdős* and *Turán* has been very influential. It left open a number of problems and attracted the attention not only of the Hungarian school of interpolation (*Géza Freud, Ottó Kis, Melánia Sallay, József Szabados, P. Vértesi*), but also of others (including *R.J. Nessel, W. Dickmeis, E. van Wickeren*).

2.9. The Faber-theorem is a special case of a general statement proved by *S.M. Losinskii* and *F.I. Harsiladze* on *(linear) projection operators* (p.o.). (That means $\mathcal{L}_n : C \to \mathcal{P}_{n-1}$ is a linear bounded operator and $\mathcal{L}_n(f) \equiv f$ iff $f \in \mathcal{P}_{n-1}$). Namely, they proved that if

$$|||\mathcal{L}_n||| := \sup_{\|f\|\le 1} \left\| \mathcal{L}_n(f, x) \right\|, \qquad f \in C,$$

then

$$|||\mathcal{L}_n||| \geq \frac{\log n}{8\sqrt{\pi}} \tag{2.18}$$

($\mathcal{L}_n$ is a p.o.). If $\mathcal{L}_n = L_n(X)$ (Lagrange interpolation), then, obviously $\Lambda_n(X) = |||\mathcal{L}_n|||$.

In his paper [17], *G. Halász* formulated some results on

$$\mathcal{L}_n(x) := \sup_{\|f\|\leq 1} \big|\mathcal{L}_n(f,x)\big|, \qquad f \in C$$

(it generalizes the Lebesgue function $\lambda_n(X,x)$). Among others he states

Theorem E. *For any sequence of projections $\mathcal{L}_n$*

(i) $\overline{\lim\limits_{n\to\infty}}\, \mathcal{L}_n(x) = \infty$ *on a set of positive measure in* $[-1,1]$*;*

(ii) $\lim\limits_{n\to\infty} \int\limits_{-1}^{1} h\big(\log \mathcal{L}_n(x)\big) \log \mathcal{L}_n(x)\, dx = \infty$ *whenever*

$$I := \int\limits_{2}^{\infty} \frac{h(x)}{x \log x}\, dx = \infty.$$

(iii) *If* $I < \infty$ *then there exists a sequence* $\mathcal{L}_n$ *such that*

$$\sup_n \int\limits_{-1}^{1} h\big(\log \mathcal{L}_n(x)\big) \log \mathcal{L}_n(x)\, dx < \infty.$$

2.10. Here we mention some recent developments of the previous results. First, let us see the multidimensional analogon of the estimation (2.18).

Let $\mathbb{R}^d$ (direct product) be the Euclidean d-dimensional space ($d \geq 1$, fixed) and let $\mathbb{T}^d = \mathbb{R}^d \pmod{2\pi\mathbb{Z}^d}$ denote the d-dimensional torus, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$.

Further, let $C(\mathbb{T}^d)$ denote the space of (complex valued) continuous functions on $\mathbb{T}^d$. By definition they are 2π-periodic in each variable.

For $g \in C(\mathbb{T}^d)$ we define its Fourier series by

$$g(\boldsymbol{\vartheta}) \sim \sum_{\mathbf{k}} \hat{g}(\mathbf{k}) e^{i\mathbf{k}\cdot\boldsymbol{\vartheta}}, \qquad \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} g(\mathbf{t}) e^{-i\mathbf{k}\cdot\mathbf{t}}\, d\mathbf{t},$$

where $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \dots, \vartheta_d) \in \mathbb{T}^d$, $\mathbf{k} = (k_1, k_1, \dots, k_d) \in \mathbb{Z}^d$ and $\mathbf{k} \cdot \boldsymbol{\vartheta} = \sum_{l=1}^d k_l \vartheta_l$ (scalar product).

The *rectangular* n-th partial sum of the Fourier series is defined by

$$S_{nd}^{[r]}(g,\boldsymbol{\vartheta}) := \sum_{|\mathbf{k}|_\infty \le n} \hat{g}(\mathbf{k}) e^{i\mathbf{k}\cdot\boldsymbol{\vartheta}} \qquad (n \in \mathbb{N}_0 = \{0,1,2,\dots\});$$

the *triangular* one is

$$S_{nd}(g,\boldsymbol{\vartheta}) := \sum_{|\mathbf{k}|_1 \le n} \hat{g}(\mathbf{k}) e^{i\mathbf{k}\cdot\boldsymbol{\vartheta}} \qquad (n \in \mathbb{N}_0).$$

Above, $|\mathbf{k}|_\infty = \max\limits_{1\le l\le d} |k_l|$ and $|\mathbf{k}|_1 = \sum\limits_{k=1}^{d} |k_l|$ (they are the l_p norms of the multiindex $\mathbf{k}$ for $p=\infty$ and $p=1$). The names "rectangular" and "triangular" refer to the shape of the corresponding indices of terms when $d=2$ and $0 \le k_1, k_2$, $|\mathbf{k}|_\infty \le n$, $|\mathbf{k}|_1 \le n$ respectively.

In a way the investigation of the $S_{nd}^{[r]}$ is apparent: in many cases in essence it is a one variable problem (see [42] and [41]).

However there are only relatively few works dealing with the triangular (or l_1) summability (cf. [43] and [44]).

Introducing the notations

$$D_{nd}(\boldsymbol{\vartheta}) = \sum_{|\mathbf{k}|_1 \le n} e^{i\mathbf{k}\cdot\boldsymbol{\vartheta}} \qquad (n \ge 1),$$

where $\mathbf{k} \in \mathbb{Z}^d$, one can see that

$$\begin{aligned} S_{nd}(g,\boldsymbol{\vartheta}) = \big(g * D_{nd}\big)(\boldsymbol{\vartheta}) : &= \frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} g(\boldsymbol{\vartheta} - \mathbf{t}) D_{nd}(\mathbf{t})\, d\mathbf{t} = \\ &= \frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} g(\boldsymbol{\vartheta} + \mathbf{t}) D_{nd}(\mathbf{t})\, d\mathbf{t}, \end{aligned}$$

where as before, $g \in C(\mathbb{T}^d)$, $\boldsymbol{\vartheta}, \mathbf{t} \in \mathbb{T}^d$.

Let $\|g\| := \max\limits_{\boldsymbol{\vartheta}\in\mathbb{T}^d} |g(\boldsymbol{\vartheta})|$,

$$\|S_{nd}\| := \max_{\substack{g \in C(\mathbb{T}^d) \\ \|g\| \le 1}} \big\|S_{nd}(g,\boldsymbol{\vartheta})\big\| \qquad (n \ge 1)$$

and

$$\|g\|_p := \left(\int_{\mathbb{T}^d} |g(\boldsymbol{\vartheta})|^p \, d\boldsymbol{\vartheta} \right)^{1/p}$$

if $g \in L^p :=$ {the set of all measurable 2π periodic (in each variable) functions on $\mathbb{T}^d$}, $1 \leq p < \infty$.

We state

Theorem F. *We have, for any fixed* $d \geq 1$,

$$\|D_{nd}\|_1 = \|S_{nd}\| \sim (\log n)^d \qquad (n \geq 2).^{3}$$

One of the most characteristic properties of the Fourier series in one dimension is the so called Faber–Marcinkiewicz–Berman theorem, namely that the operator S_n has the smallest norm among all projection operators (cf. [45, p. 281] for other details). This part extends the above statement for S_{nd}, $d \geq 1$.

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3 Here and later $a_n \sim b_n$ means that $0 < c_1 \leq a_n b_n^{-1} \leq c_2$ where $c, c_1, c_2, \ldots$ are positive constants, not depending on n; they may denote different values in different formulae.

Let $\mathcal{T}_{nd}$ be the space of trigonometric polynomials of form

$$\sum_{|\mathbf{k}|_1 \le n} \big(a_{\mathbf{k}} \cos(\mathbf{k}\cdot\boldsymbol{\vartheta}) + b_{\mathbf{k}} \sin(\mathbf{k}\cdot\boldsymbol{\vartheta})\big),$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_d)$ and $k_1, \ldots, k_d \ge 0$, arbitrary real numbers. Moreover, let T_{nd} be a linear trigonometric projection operator on $C(\mathbb{T}^d)$, i.e. $T_{nd}(g, \boldsymbol{\vartheta}) = g(\boldsymbol{\vartheta})$ for $g \in \mathcal{T}_{nd}$ and $T_{nd}(g, \boldsymbol{\vartheta}) \in \mathcal{T}_{nd}$ for other $g \in C(\mathbb{T}^d)$.

Theorem G. *For any linear trigonometric projection operator T_{nd}, one has*

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} T_{nd}(g_{\mathbf{t}}, \boldsymbol{\vartheta} - \mathbf{t})\, d\mathbf{t} = S_{nd}(g, \boldsymbol{\vartheta}) \qquad (g \in C(\mathbb{T}^d)),$$

$$\|T_{nd}\| \ge \|S_{nd}\|,$$

where $g_{\mathbf{t}}(\boldsymbol{\vartheta}) = g(\boldsymbol{\vartheta} + \mathbf{t})$ is the $\mathbf{t}$-translation operator.

Now we formulate a generalization of (2.15).

Theorem H. *If $\mathcal{L}_{nd}$ is a projection of $C(I^d)$ onto $\mathcal{P}_{nd}$ then*

$$\|\mathcal{L}_{nd}\| \ge \frac{1}{2}\|S_{nd}\|.$$

Above, $\mathcal{L}_{nd}$ is a projection of $C(I^d)$ (:= the set of continuous functions of d-variables on $I^d = [-1,1]^d$) onto $\mathcal{P}_{nd}$ iff it is linear, $\mathcal{L}_{nd}(p) = p$ if $p \in \mathcal{P}_{nd}$ and $\mathcal{L}_{nd}(f) \in \mathcal{P}_{nd}$ for any $f \in C(I^d)$.

Proofs, further statements, references and some historical remarks about Part 2.8 are in the paper *László Szili* and *P. Vértesi* [19].

2.11. In this part we give an application of Theorem H.

In one variable, the zeros of Chebishev polynomials are optimal if we consider Gaussian quadrature with respect to the Chebishev weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ in $I := [-1,1]$.

Considering several variables many new problems arise. First to ensure that the interpolation is well-posed, we need a proper subspace of polynomials. Another natural question is to get a point system which is suitable for Gaussian quadrature. H.M. Möller gave a lower bound which says that the number of points, R, for two-dimensional Gaussian quadrature must fulfill the relation $R = R(n) \geq \dim(\Pi_{n-1,2}) + [n/2] := M$, where Π_{s2} denotes the set of polynomials of two variables of total degree $\leq s$. (It is easy to see that $\dim(\Pi_{s2}) \equiv \binom{s+2}{2}$).

In his paper Yuan Xu [48] introduced a set of Chebishev-like points for the two-dimensional Lagrange interpolation in the square I^2 (direct product) as follows.

Let

$$z_r = z_{r,n} = \cos\frac{r\pi}{n}, \qquad 0 \le r \le n,$$

$$\begin{gathered}
\mathbf{x}_{2i,2j+1} = \mathbf{x}_{2i,2j+1,n} = (z_{2i}, z_{2j+1}),\\
0 \le i \le m, \;\; 0 \le j \le m-1\\
\mathbf{x}_{2i+1,2j} = \mathbf{x}_{2i+1,2j,n} = (z_{2i+1}, z_{2j}),\\
0 \le i \le m-1, \;\; 0 \le j \le m,\\
\text{if } n = 2m,\\
\mathbf{x}_{2i,2j} = \mathbf{x}_{2i,2j,n} = (z_{2i}, z_{2j}),\\
0 \le i, j \le m\\
\mathbf{x}_{n-2i,n-2j} = \mathbf{x}_{n-2i,n-2j,n} = (z_{n-2i}, z_{n-2j}),\\
0 \le i, j \le m,\\
\text{if } n = 2m+1
\end{gathered}$$

and let with $\mathbb{T}_n = \{\mathbf{x}_{ks}\}$, $|\,\mathbb{T}_n| = N$,

$$L_{n2}(f, \mathbb{T}_n, \mathbf{x}) = \sum_{\mathbf{x}_{ks} \in \mathbb{T}_n} f(\mathbf{x}_{k,s}) \ell_{ks}(\mathbf{x}),$$

where $f(\mathbf{x}) = f(x, y) \in C(I^2) :=$ {the set of continuous functions on I^2} and ℓ_{ks} are the fundamental functions of Lagrange interpolation, $\ell_{ks} \in V_n$ (a certain subspace of Π_{n2}).

Xu proved the uniqueness of the Lagrange interpolation in V_n and the property $\Pi_{n-1,2} \subset V_n \subset \Pi_{n2}$. Moreover he proved the relation $\dim(V_n) = N = \dim(\Pi_{n-1,2}) + [\frac{n+1}{2}]$. In the same paper Xu proved that the above points $\mathbb{T}_n$ are suitable for the Gaussian quadrature. Considering the number of points, N, we may say that $\mathbb{T}_n$ is "almost" optimal because $N - M \leq 1$ (actually, if $n = 2m$, then $N = M$).

Generally, to prove the existence and the uniqueness is a difficult problem!

In their paper [49] Bos, De Marchi and Vianello proved that

$$\Lambda_{n2}(\mathbb{T}_n) \leq C(\log n)^2, \quad n \geq 2,$$

where,as above, the constants $C, C_1, \ldots,$ are different positive constants which may assume different values in different formulas,

$$\Lambda_{n2}(\mathbb{T}_n) = \max_{\substack{\|f\| \leq 1 \\ f \in C(I^2)}} \|L_{n2}(f, \mathbb{T}_n, \mathbf{x})\|,$$

and

$$\|f\| = \max_{\mathbf{x} \in [-1,1]^2} |f(\mathbf{x})|, \ f \in C(I^2).$$

Caliari, De Marchi and Vianello [50] investigated the so-called Padua points $\mathbb{P}_n$, which are a set of points analogous to $\mathbb{T}_n$. (Let's remark that the cardinality of $\mathbb{P}_n$ is equal to the dimension of Π_{n2}). In the same paper the authors proved that

$$\Lambda_{n2}(\mathbb{P}_n) \leq C(\log n)^2,$$

when $\Lambda_{n2}(\mathbb{P}_n)$ is the corresponding Lebesgue constant.

Our aim is to give lower estimations for the two above mentioned Lebesgue constants.

For the point system $\mathbb{T}_n$ we state

Theorem I. *We have*

$$\Lambda_{n2}(\mathbb{T}_n) \sim (\log n)^2.$$

Similar estimation is valid considering the node-matrix $\mathbb{P}_n$, that means

Theorem J. *For the Padua points $\mathbb{P}_n$ we have*

$$\Lambda_{n2}(\mathbb{P}_n) \sim (\log n)^2.$$

The above Theorems I and J were proved in the paper [51] B. Della Vechia, G. Mastroianni and P. Vértesi.

3. Mean convergence of interpolation

3.1. As it has turned out the estimation of the Lebesgue function

$$\lambda_n(X, x) = \sum_{k=1}^{n} |\ell_{kn}(X, x)|$$

is fundamental in getting "negative" (divergence)-type results for the Lagrange interpolation using the uniform (or maximum) norm.

These facts resulted that the attention turned to the *mean convergence* of interpolation. The first such result is due to *P. Erdős* and *P. Turán* [20] from 1937.

Theorem 3.1. *For an arbitrary weight w and $f \in C$,*

$$\lim_{n\to\infty} \int_{-1}^{1} \{L_n(f, w, x) - f(x)\}^2 w(x) dx = 0.$$

Here and later w is a weight if $w \geq 0$ and $0 < \int_{-1}^{1} w < \infty$; $L_n(f, w)$ is the Lagrange interpolation with nodes at on the roots of the corresponding orthonormal polynomials (ONP) $p_n(w)$.

During the years 1936–1939, *P. Erdős* and *P. Turán* wrote 3 fundamental papers "On interpolation I, II, III" [20], [32], [33]; they appeared in 1937, 1938 and 1940). This survey will quote many problems and theorems of them. We strongly suggest to read these papers to the interested readers.

Using the Chebyshev roots, *P. Erdős* and *Ervin Feldheim* proved much more [21]:

Theorem 3.2. *Let $f \in C$ and $p > 1$. Then*

$$\lim_{n=\to\infty} \int_{-1}^{1} |f(x) - L_n(f, T, x)|^p \frac{1}{\sqrt{1-x^2}} dx = 0.$$

3.2. Theorem 3.1 is a reasonable motivation of the problem (cf. *P. Erdős, Géza Freud, P. Turán* [23, Problem VIII], [24], [25]).

Does there exists a weight w and $f \in C$ such that

$$\overline{\lim_{n\to\infty}} \|f - L_n(f, w)\|_{p,w} = \infty$$

for every $p > 2$?

(Above $\|g\|_{p,w}$ stands for $\|gw^{1/p}\|_p$.)

After a lot of results proved by *Richard Askey*, *Paul Nevai* and others *Y.G. Shi* came to a new general idea where the nodes x_{kn} are *not necessarily* the roots of an orthogonal system $p_n(w)$. Namely he realized the surprising fact that for the mean convergence the expressions

$$\gamma_1(X,x) := \sum_{k=1}^{n} |x - x_{kn}(X,x)|\,|\ell_{kn}(X,x)|\,, \qquad n \geq 1, \tag{3.1}$$

are fundamentals (instead of $\lambda_n(X,x) = \sum_{k=1}^{n} |\ell_{kn}(X,x)|$). Using many basic ideas of the proof of Theorem 2.3, he proves (among others)

Theorem K. *Let $\varepsilon > 0$ be given. Then for any fixed interpolatory matrix $X \subset [-1,1]$, there exists sets $H_n = H_n(\varepsilon, X)$ of measure $\leq \varepsilon$ such that*

$$\gamma_1(X,x) \geq \frac{\varepsilon}{24}, \qquad n \geq 1, \tag{3.2}$$

whenever $x \in [-1,1] \setminus H_n$. (cf. Theorem 2.3).

The above statement is a special case of [26, Theorem 1], the latter one uses the proof of the generalization of Theorem 2.3 (cf. *P. Vértesi* [27]).

Now, by [26, Theorem 1], *Y.G. Shi* ([26, Corollary 14]) obtains.

Theorem L. *Let u and w be weights. If with a fixed $p_0 \geq 2$*

$$\left\| \frac{1}{\sqrt{w\sqrt{1-x^2}}} \right\|_{p,u} = \infty \qquad \text{for every} \qquad p > p_0,$$

then there exists an $f \in C$ satisfying

$$\overline{\lim_{n\to\infty}} \|L_n(f,w)\|_{p,u} = \infty \qquad \text{whenever} \qquad p > p_0.$$

This theorem obviously answers the (generalization of the) question raised at the beginning of Part 3.2. For other similar problems the reader may consult with [26].

4. Convergence by raising the degree

4.1. Motivated by *Lipót Fejér's* classical results (i.e, if the degree of the interpolation polynomial is about two times bigger than the number of interpolation points, then we can get convergence (cf. [28, Theorem XI])), *Erdős* raised the following question. Given $\varepsilon > 0$, suppose we interpolate at n nodes, but allow polynomials of degree at most $n(1+\varepsilon)$. Under what conditions will they converge for all continuous function?

The first answer was given by himself in [29]. Namely, he proved:

Theorem 4.1. *If the absolute values of the fundamental polynomials $\ell_{kn}(X, x)$ are uniformly bounded in $x \in [-1, 1]$, k $(1 \le k \le n)$ and $n \in \mathbb{N}$, then for every $\varepsilon > 0$ and $f \in C$ there exists a sequence of polynomials $\varphi_n = \varphi_n(x) = \varphi_n(f, \varepsilon, x)$ with*

(i) $\deg \varphi_n \le n(1+\varepsilon)$,

(ii) $\varphi_n(x_{kn}) = f(x_{kn})$, $1 \le k \le n$, $n \in \mathbb{N}$,

(iii) $\lim\limits_{n\to\infty} \|\varphi_n - f\| = 0$.

The complete answer for a more general system is in the paper of *Erdős, András Kroó* and *Szabados* [30].

Theorem 4.2. *For every $f \in C$ and $\varepsilon > 0$, there exists a sequence of polynomials $p_n(f)$ of degree at most $n(1+\varepsilon)$ such that*

$$p_n(f, x_{k,n}) = f(x_{k,n}), \qquad 1 \le k \le n,$$

and that

$$\|f - p_n(f)\| \le cE_{[n(1+\varepsilon)]}(f)$$

holds for some $c > 0$, if and only if

$$\limsup_{n\to\infty} \frac{N_n(I_n)}{n|I_n|} \le \frac{1}{\pi} \tag{4.1}$$

whenever I_n is a sequence of subintervals of I such that $\lim_{n\to\infty} n|I_n| = \infty$ and

$$\liminf_{n\to\infty} \left(n \min_{1\le k\le n-1} (\vartheta_{k+1,n} - \vartheta_{n,k})\right) > 0. \tag{4.2}$$

Here $N_n(I_n)$ is the number of the $\vartheta_{k,n}$ in $I_n \subset I$. Condition (4.1) ensures that the nodes are not too dense, and condition (4.2) says that adjacent nodes should not be too close.

5. Weighted Lagrange interpolation, weighted Lebesgue function, weighted Lebesgue constant

5.1. Let f be a continuous function. If, instead of the interval $[-1, 1]$, we try to approximate it on $\mathbb{R}$, we have to deal with the obvious fact that polynomials (of degree ≥ 1) tend to infinity if $|x| \to \infty$. So to get a suitable approximation tool, we may try to *moderate their growth applying proper weights*.

If the weight $w(x) = e^{-Q(x)}$, $x \in \mathbb{R}$, satisfies

$$\lim_{|x|\to\infty} \frac{Q(x)}{\log |x|} = \infty,$$

as well as some other mild restrictions and the Akhiezer–Babenko–Carleson–Dzrbasjan relation

$$\int_{-\infty}^{\infty} \frac{Q(x)}{1+x^2} dx = \infty,$$

then for $f \in C(w,\mathbb{R})$, where

$$C(w,\mathbb{R}) := \{f;\ f \text{ is continuous on } \mathbb{R} \text{ and } \lim_{|x|\to\infty} f(x)w(x) = 0\},$$

we have, if $\|\cdot\|$ denotes now the supnorm on $\mathbb{R}$,

$$E_n(f,w) := \inf_{p\in\mathcal{P}_n} \|(f-p)w\| \equiv \inf_{p\in\mathcal{P}_n} \|fw - pw\| \to 0 \qquad \text{as } n \to \infty.$$

So, instead of approximating $f \in C$ by $L_n(f,X)$ on $[-1,1]$, *we may estimate* $\{f(x)w(x) - L_n(f,w,X,x)\}$ *on the real line* $\mathbb{R}$ *for* $f \in C(w,\mathbb{R})$. Here $X \subset \mathbb{R}$,

$$t_k(x) := t_{kn}(w,X,x) := \frac{w(x)\omega_n(X,x)}{w(x_k)\omega_n'(X,x_k)(x-x_k)}, \qquad 1 \le k \le n,$$

and

$$L_n(f,w,X,x) := \sum_{k=1}^{n} \{f(x_k)w(x_k)\} t_k(x), \qquad n \in \mathbb{N}.$$

The Lebesgue estimate now has the form

$$\text{(5.1)} \qquad \big|L_n(f,w,X,x) - f(x)w(x)\big| \le \big\{\lambda_n(w,X,x) + 1\big\} E_{n-1}(f,w)$$

where the *(weighted) Lebesgue function* is defined by

$$\text{(5.2)} \qquad \lambda_n(w,X,x) := \sum_{k=1}^{n} \big|t_k(w,X,x)\big|,\ x \in \mathbb{R},\ n \in \mathbb{N};$$

the existence of $r_{n-1}(f,w)$ for which $E_{n-1}(f,w) = \|(f-r_{n-1})w\|$ is well-known. Formula (5.2) implies the natural definition of the *(weighted) Lebesgue constant*

$$\Lambda_n(w,X) := \|\lambda_n(w,X,x)\|, \qquad n \in \mathbb{N}. \tag{5.3}$$

Estimation (5.1) and its immediate consequence

$$\|L_n(f,w,X) - fw\| \le \big\{\Lambda_n(w,X)+1\big\}E_{n-1}(f,w), \qquad n \in \mathbb{N},$$

show that, analogously to the classical case, the investigation of $\lambda_n(w,X,x)$ and $\Lambda_n(w,X)$ is of fundamental importance to get convergence-divergence results for the weighted Lagrange interpolation (cf. Part 2.1).

To expect reasonable estimations, as it turns out, we need a considerable knowledge about the weight $w(x)$ and on the behaviour of the ONP $p_n(w^2,x)$ corresponding to the weight w^2.

5.2. As *P. Nevai* writes in his instructive monograph [34, Part 4.15], about 40 years ago there was a great amount of information on orthogonal polynomials on infinite intervals, however as *Géza Freud* realized in the sixties, there had been a complete lack of *systematic treatment* of the general theory; the results were of mostly ad hoc nature. And *G. Freud*, in the last 10 years of his life, laid down the basic tools of the systematic investigation.

During the years a great number from the approximators and/or orthogonalists joined *G. Freud* and his work, including many Hungarians. As a result, today our knowledge is more comprehensive and more solid than before.

Now we introduce the so called Mhaskar–Rakmanov–Saff number, denoted by $a_n(w)$. $a_n(w)$ is a generalization of the number $q_n(w)$ defined by *G. Freud*. Instead of the definition we show a useful property of $a_n(w)$ and give an example (cf. [35]).

$$\begin{cases} \|r_n w\| = \max\limits_{|x| \le a_n(w)} |r_n(x)w(x)|, \\ \|r_n w\| > |r_n(x)w(x)| \qquad \text{for } |x| > a_n(w) \end{cases} \tag{5.4}$$

if $r_n \in \mathcal{P}_n$ ($r_n \not\equiv 0$; $\|\cdot\|$ is the supnorm on $\mathbb{R}$) and that asymptotically (as $n \to \infty$) $a_n(w)$ is the smallest such number. Relation (5.4) may be formulated such that $r_n w$ "lives" on $[-a_n, a_n]$.

As *an example*, let $Q(x) = |x|^\alpha$. Then

$$q_n(w) \sim n^{1/\alpha} \quad \text{and} \quad a_n(w) = c(\alpha)\, n^{1/\alpha}, \qquad \alpha > 1.$$

In 1972, *P. Erdős* defined (as today called) the Erdős weights. The prototype of $w \in \mathcal{E}$ ($\mathcal{E}$ is the collection of the Erdős weights) is the case when $Q(x) = Q_{k,\alpha} = \exp_k(|x|^\alpha)$ ($k \ge 1$, $\alpha > 1$, $\exp_k := \exp(\exp(\ldots))$, the kth iterated exponential); for other details on $\mathcal{E}$ see [36], [37] and [35]. As an interesting and maybe surprising fact that generalizing the method and ideas of our common paper with *Erdős*, one can prove a statement on the *weighted* Lebesgue function $\lambda_n(w, X, x)$ (see *P. Vértesi* [38]).

Theorem M. *Let $w \in \mathcal{E}$. If $\varepsilon > 0$ is an arbitrary fixed number, then for any interpolatory matrix $X \subset \mathbb{R}$ there exist sets $H_n = H_n(w, \varepsilon, X)$ with $|H_n| \leq 2a_n(w)\varepsilon$ such that*

$$\lambda_n(w, X, x) \geq \frac{\varepsilon}{3840} \log n$$

if $x \in [-a_n(w), a_n(w)] \setminus H_n$, $n \geq n_1(\varepsilon)$.

This statement is a complete analogue of Theorem 2.4. Roughly speaking, it says that the weighted Lebesgue function is at least $c \log n$ on a "big part" of $[-a_n, a_n]$ for arbitrary fixed $X \subset (-\infty, \infty)$ and $w \in \mathcal{E}$.

Without going into the details we remark that the previous consideration and statement can be developed for other weights (cf. [38]).

To finish this survey we quote another theorem on weighted approximation which corresponds to the result of *Erdős* from 1943 (see Theorem 4.1). Namely we have

Theorem N. *Let* $w \in \mathcal{E}$. *If* $|t_{kn}(w, X, x)| \leq A$ *uniformly in* $x \in \mathbb{R}$, k *and* n, *then for every* $\varepsilon > 0$ *and to every* $f \in C(w^{1+\varepsilon}, \mathbb{R})$, *there exists a sequence of polynomials* $\varphi_\Delta(x) = \varphi_\Delta(f, \varepsilon, x) \in \mathcal{P}_\Delta$ *such that*

(i) $\Delta \leq n(1 + \varepsilon + c\,\varepsilon n^{-2/3})$,

(ii) $\varphi_\Delta(x_{kn}) = f(x_{kn})$, $1 \leq k \leq n$, $n \in \mathbb{N}$,

(iii) $\|w^{1+\varepsilon}(f - \varphi_\Delta)\| \leq cE_\Delta(f, w^{1+\varepsilon})$.

The proof and similar results using other exponents $Q(x)$ are in *L. Szili* and *P. Vértesi* [39] and [40].

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