Graph covering and coloring

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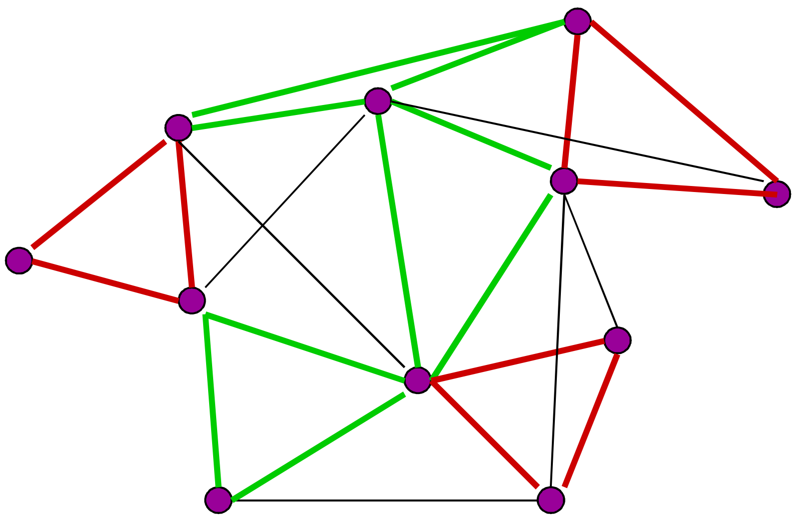
A covering problem

(not very young)

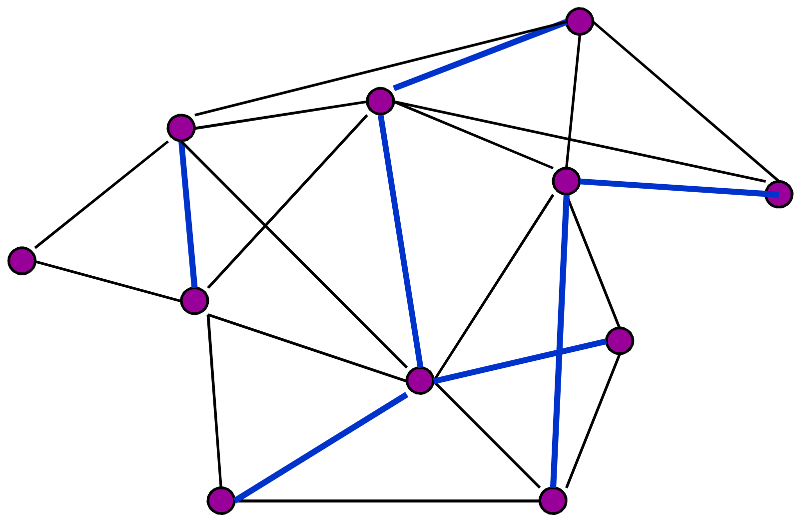
Packing / Covering of triangles

$\nu_\Delta(G)$: max # of edge-disjoint triangles in G

$\tau_\Delta(G)$: min # of edges meeting all triangles of G



$\nu_\Delta = 6$



$\tau_\Delta = 7$

Packing / Covering of triangles

$\nu_\Delta(G)$: max # of edge-disjoint triangles in G

$\tau_\Delta(G)$: min # of edges meeting all triangles of G

For every G ,

$$\nu_\Delta(G) \leq \tau_\Delta(G) \leq 3\nu_\Delta(G)$$

(any non-extendable packing meets all Δs)

Packing / Covering of triangles

$\nu_\Delta(G)$: max # of edge-disjoint triangles in G

$\tau_\Delta(G)$: min # of edges meeting all triangles of G

Conjecture.   For every G , $\tau_\Delta(G) \leq 2\nu_\Delta(G)$

[ZsT, Colloq. Math. Soc. J. Bolyai 1981]

would be tight:

K_4	–	$\nu_\Delta = 1$	$\tau_\Delta = 2$
K_5	–	$\nu_\Delta = 2$	$\tau_\Delta = 4$

Packing / Covering of triangles

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Observation.   It's a nice problem.

[Uncle Paul, 1981]

Packing / Covering of triangles

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[ZsT, Colloq. Math. Soc. J. Bolyai 1981]

digraphs, oriented graphs:

transitive triangles	harder
cyclic triangles	easier (?) $(2 - c)\nu_\Delta(G)$

Some works by...

- Aparna, Bujtás & ZsT
- Bacsó & ZsT
- Chapuy, DeVos, McDonald, Mohar, Scheide
- Cui, Haxell, Ma
- Haxell & Kohayakawa
- Haxell, Kostochka, Thomassé
- Krivelevich
- ZsT

More on triangles

(Erdős, Gallai & ZsT, Discr Math 1996)

Covering / Independence in triangles

$\alpha'_{\Delta}(G)$: max # of edges, ≤ 1 from any triangle

$\tau_{\Delta}(G)$: min # of edges, ≥ 1 from every triangle

For every G ,

$$\alpha'_{\Delta}(G) + \tau_{\Delta}(G) < |E(G)|$$

(unless G is triangle-free)

Covering / Independence in triangles

$\alpha'_\Delta(G)$: max # of edges, ≤ 1 from any triangle

$\tau_\Delta(G)$: min # of edges, ≥ 1 from every triangle

Theorem. Tight, valid bound for connected G :

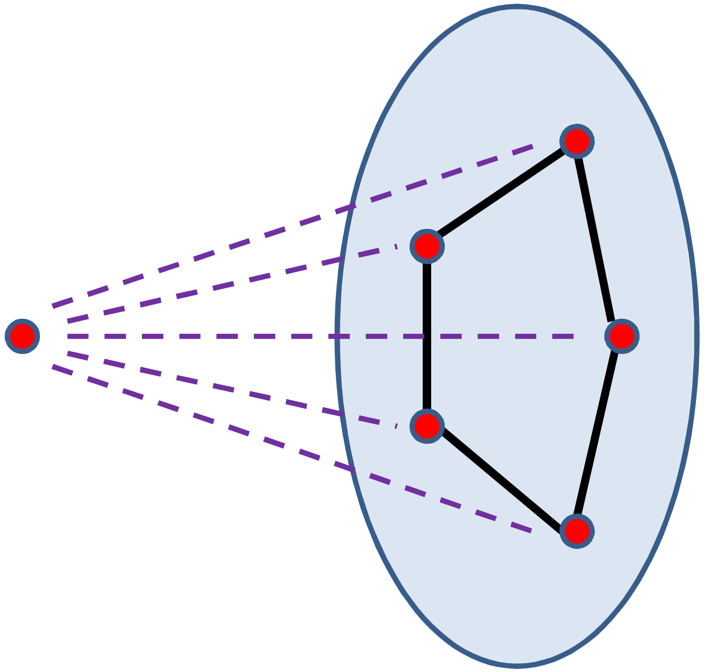
$$\alpha'_\Delta(G) + \tau_\Delta(G) \leq |E| - r(|V| - 1)$$

($r(|V| - 1)$: inverse of Ramsey function:

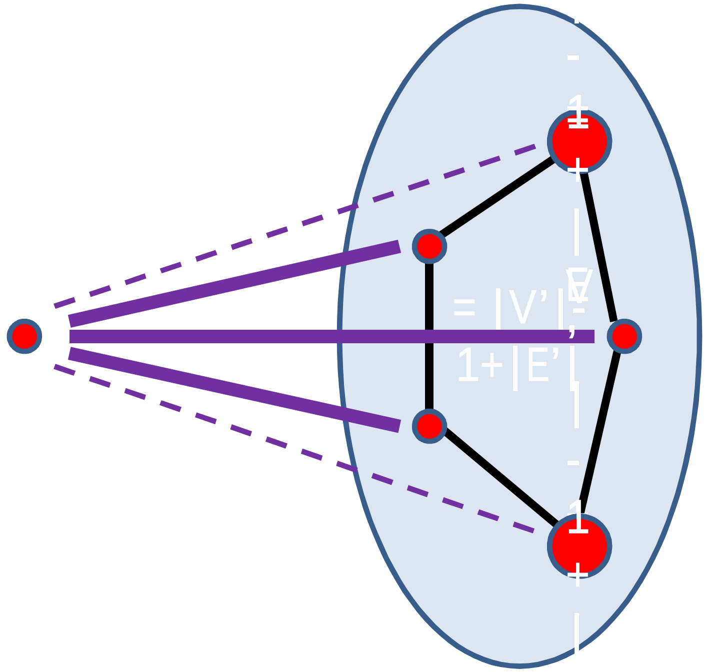
largest t s.t. $\forall$ triangle-free graph of order $|V| - 1$ has independence # at least t)

[Erdős, Gallai & ZsT, DM'96]

$|E| = |V'| + |E'|$ $\alpha'_{\Delta}(G) = |E'|$ $\tau_{\Delta}(G) = |V'| - \alpha(G')$



$G' = (V', E')$



$\alpha'_{\Delta}(G) + \tau_{\Delta}(G) = |E| - \alpha(G')$

Covering / Independence in triangles

$\alpha'_{\Delta}(G)$: max # of edges, ≤ 1 from any triangle

$\tau_{\Delta}(G)$: min # of edges, ≥ 1 from every triangle

Theorem. Tight, valid bound for connected G :

$$\alpha'_{\Delta}(G) + \tau_{\Delta}(G) \leq |E| - r(|V| - 1)$$

($r(|V| - 1)$: inverse of Ramsey function)

Open: Prove or disprove for disconnected G .

Covering / Independence in triangles

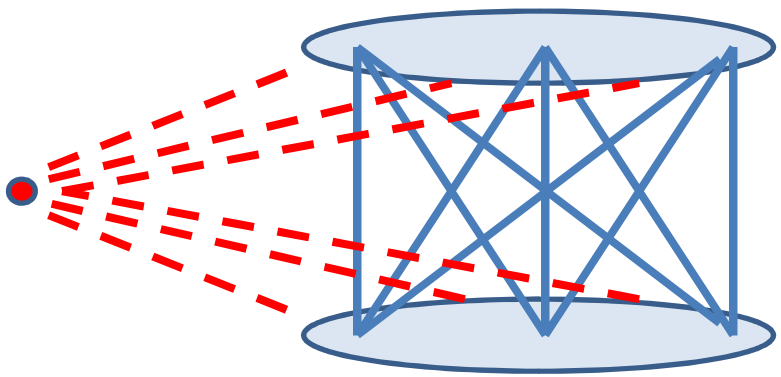
$\alpha'_{\Delta}(G)$: max # of edges, ≤ 1 from any triangle

$\tau_{\Delta}(G)$: min # of edges, ≥ 1 from every triangle

Theorem. If every edge is in some Δ of G ,

$$\alpha'_{\Delta}(G) \leq (|V| - 1)^2 / 4$$

[Erdős, Gallai & ZsT, DM'96]



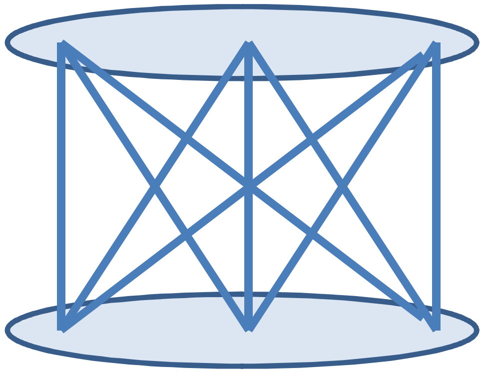
Covering / Independence in triangles

$\alpha'_\Delta(G)$: max # of edges, ≤ 1 from any triangle

$\tau_\Delta(G)$: min # of edges, ≥ 1 from every triangle

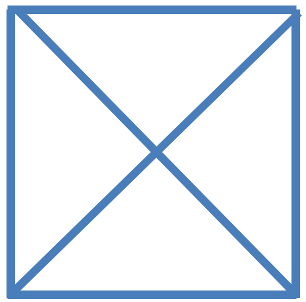
Conjecture. In every G,

$$\alpha'_\Delta(G) + \tau_\Delta(G) \leq |V|^2 / 4$$



$\tau_\Delta = 0$

$\alpha'_\Delta = n^2/4$



$\alpha'_\Delta = n/2$

$\tau_\Delta = n^2/4 - n/2$

Covering / Independence in triangles

$\alpha'_\Delta(G)$: max # of edges, ≤ 1 from any triangle

$\tau_\Delta(G)$: min # of edges, ≥ 1 from every triangle

Conjecture. In every G,

$$\alpha'_\Delta(G) + \tau_\Delta(G) \leq |V|^2 / 4$$

Problem. Determine

$$\liminf_{|E| \to \infty} \left(\alpha'_\Delta(G) + \tau_\Delta(G) \right) / |E|^{2/3}$$

(known: $\geq 6^{-1/3}$ and $\leq 4^{1/3}$)

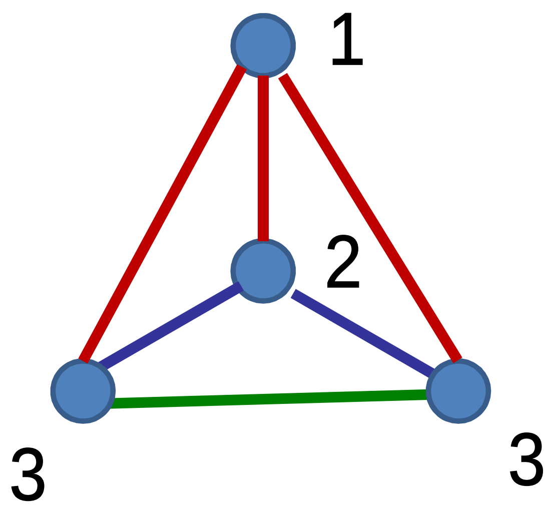
Edge colorings of K_n – rainbow subgraphs

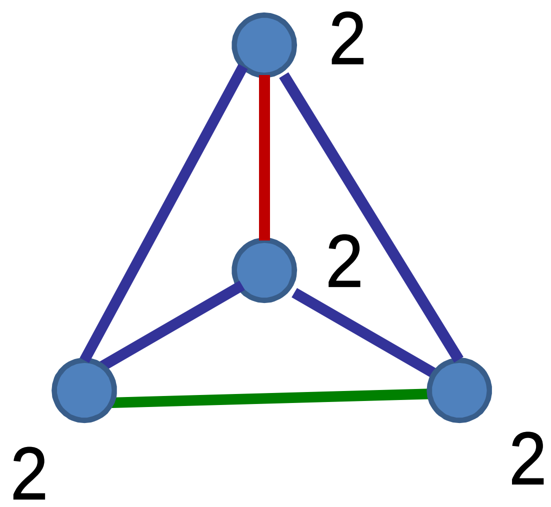
(Erdős & ZsT, Ann Discr Math 1993)

Rainbow subgraphs

rainbow subgraph $F \subset G$: each edge of F has a distinct color (i.e. $|E(F)|$ colors in a copy of F)

$k(i)$: # of colors at vertex v_i





Rainbow subgraphs

rainbow subgraph $F \subset G$: each edge of F has a distinct color (i.e. $|E(F)|$ colors in a copy of F)

$k(i)$: # of colors at vertex v_i

Theorem. In a complete graph, if

$$\sum_i 2^{-k(i)} < 1$$

then a rainbow K_3 occurs.

[Erdős & ZsT, AnnDM'93]

Problem. Find analogues for other subgraphs.

Rainbow subgraphs

Problem. If the edges of K_n are k-colored, how large degrees (in each color) imply a rainbow copy of F ?

Theorem. For K_3 and any k , degrees $\sim n/2^{k-1}$ suffice. (e.g. 3 colors – degrees $n/4$)

Theorem. For C_4 and 4 colors: $n/4 - cn$ $(c > 0)$.

Problem. For C_5 and 5 colors: $n/5 - cn$ $(c > 0)$?

Problem. For any F and $k > |E(F)|$ colors: degrees $n/k - cn$ $(c > 0)$ imply rainbow F ???

Covering the edges

(Erdős & ZsT, Proc Kalamazoo 1993 -
Graph Th Combin & Appl 1995)

Covering the edge set

$G = (V, E)$

$\tau(G)$ – transversal number, min # of vertices meeting all edges of G

Proposition. If G is connected, then

$\tau(G) \leq 2/7\ (|V|+|E|+1)$

[Erdős & ZsT, Proc GTCA'93-95]

(without connectivity: $1/3\ (|V|+|E|)$)

Hypergraph covering

$\mathcal{H} = (X, \mathcal{E})$ – set system $\mathcal{E}$ over X

$\tau(\mathcal{H})$ – transversal number, min # of vertices meeting all edges $E \in \mathcal{E}$

r-uniform: $|E| = r, \quad \forall E \in \mathcal{E}$

General problem. Find tight valid bounds

$$\tau(\mathcal{H}) \leq a|X| + b|\mathcal{E}|$$

E.g., $a = b = ¼$ for $r = 3$

[ZsT, DM'90 – Chvátal & McDiarmid, CCA'92]

Hypergraph covering

d-regular: $\forall\, x \in X$ in exactly d edges $E \in \mathrm{E}$

Conjecture. H is 6-uniform and 3-regular $\Rightarrow$

$$\tau(\mathrm{H}) \leq |X| / 4$$

[ZsT & Vestergaard, DMGT'02]

References for more problems and results

References (1)

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