Erdős Centennial, Budapest, Hungary 2013

Tree-width and Dimension

William T. Trotter
trotter@math.gatech.edu

Winkler on Erdős and Trotter (1998)

Paul Erdős called all (American) mathematicians by their last name … except Tom Trotter, whom he called "Bill."

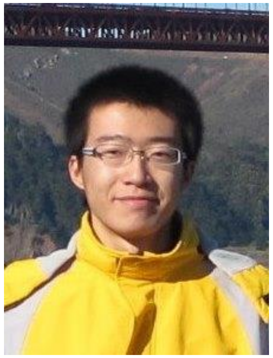
Co-authors

Gwenaël Joret

Piotr Micek

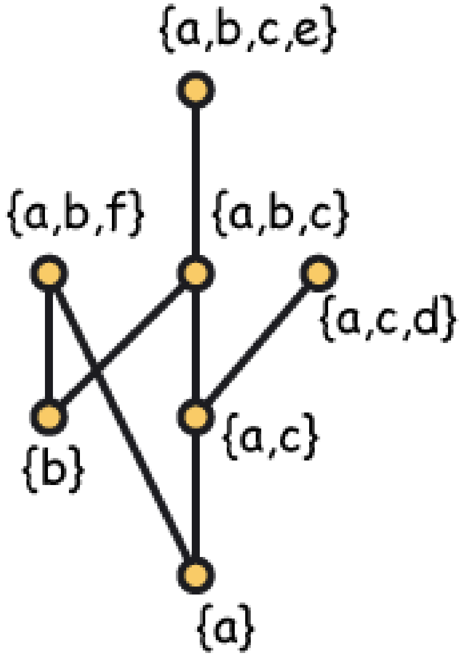
Kevin Milans

Bartosz Walczak

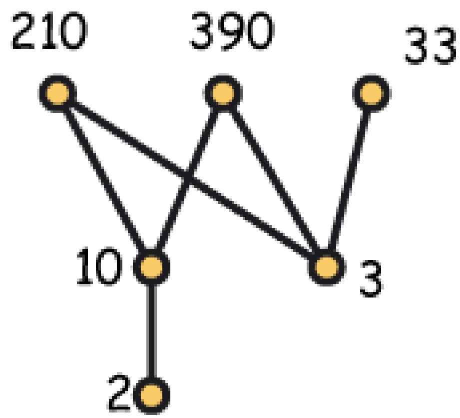


Ruidong Wang

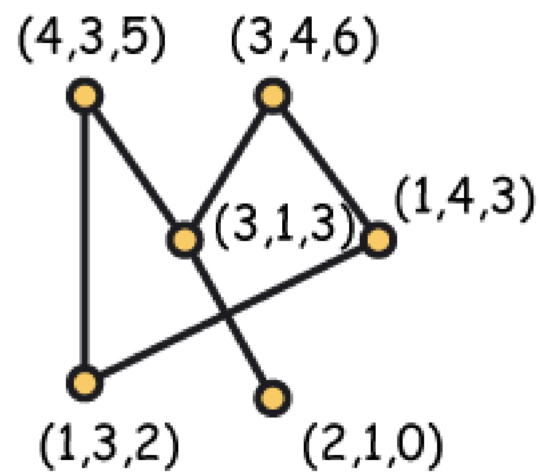
Examples of Partially Ordered Sets (Posets)



Inclusion

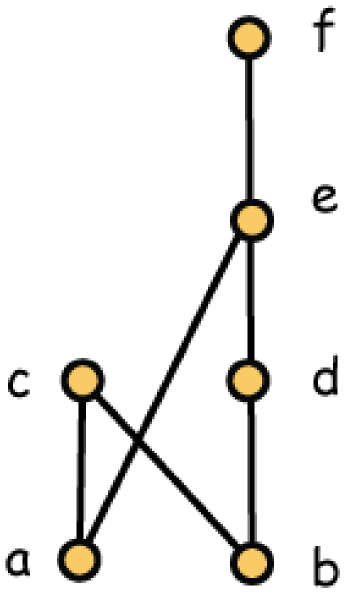


Division

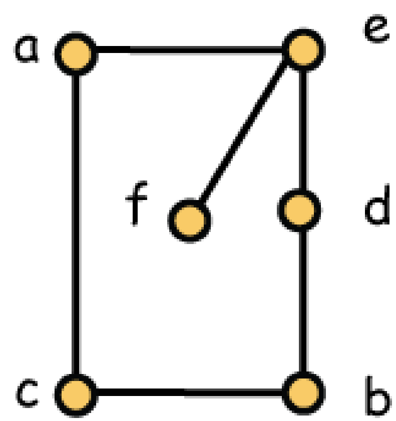


Embedding in $\mathbf{R}^3$

Order Diagrams and Cover Graphs

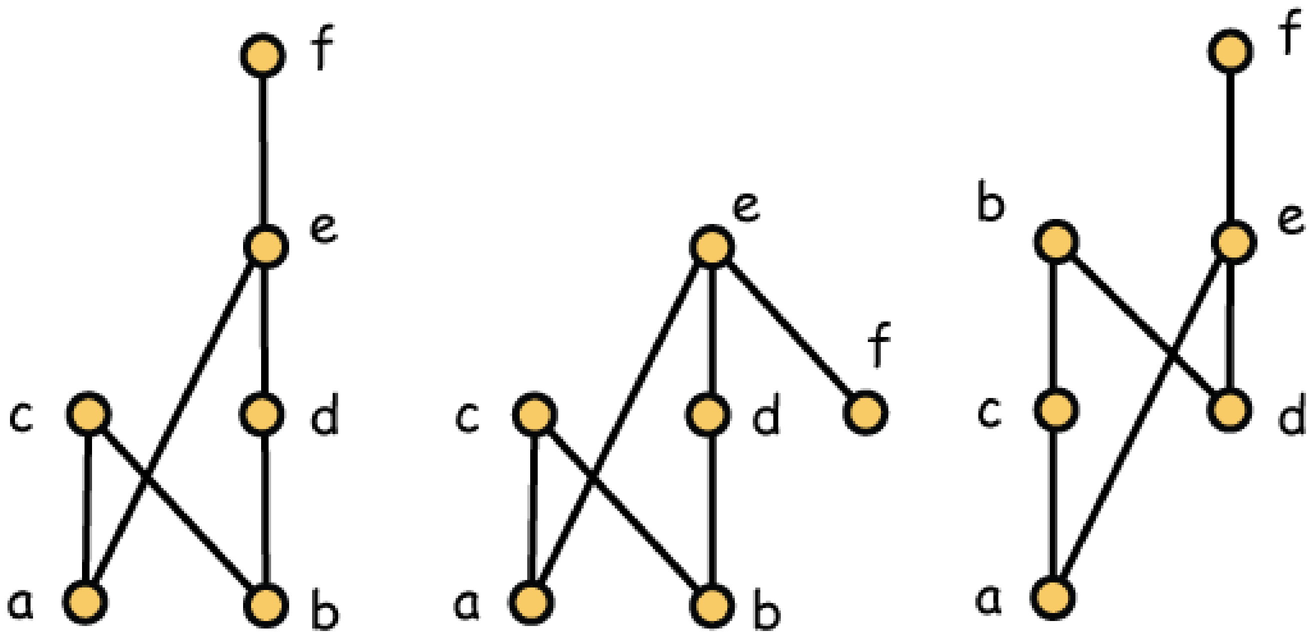


Order Diagram



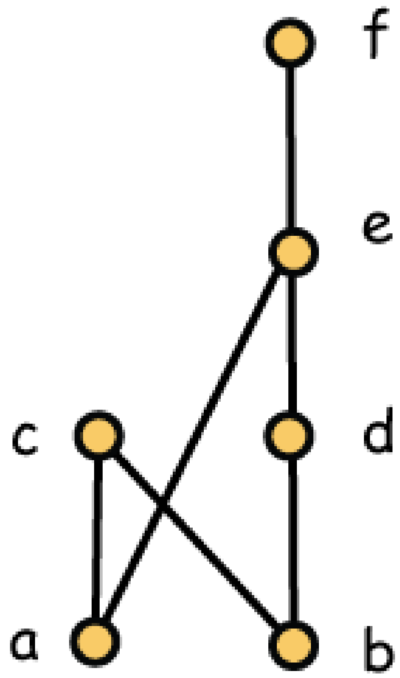
Cover Graph

Diagrams and Cover Graphs

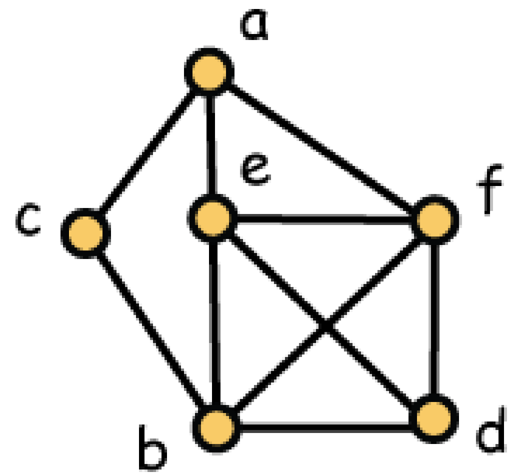


Three different posets with the same cover graph.

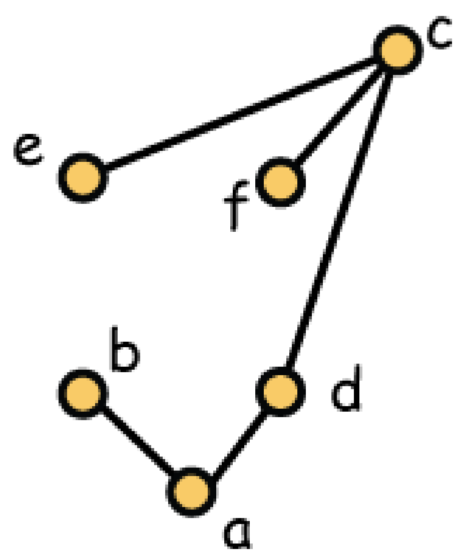
Comparability and Incomparability Graphs



Poset

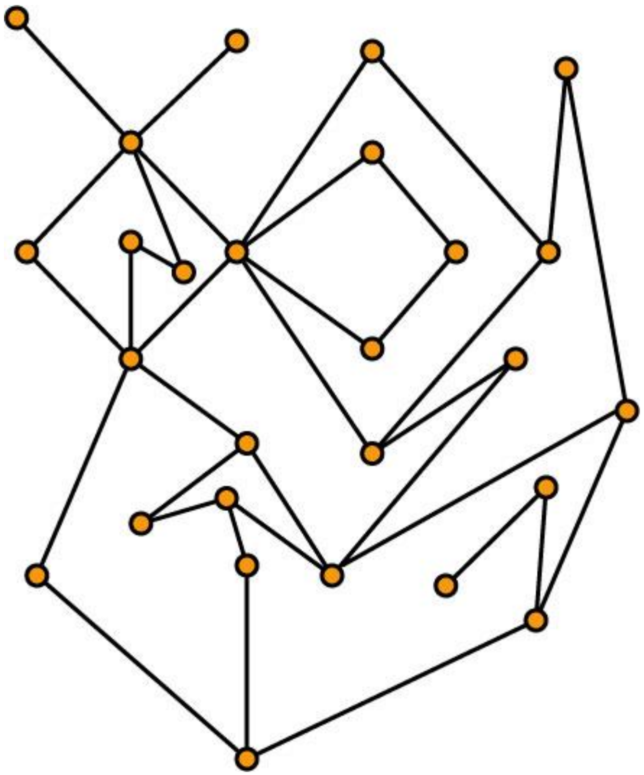


Comparability Graph



Incomparability Graph

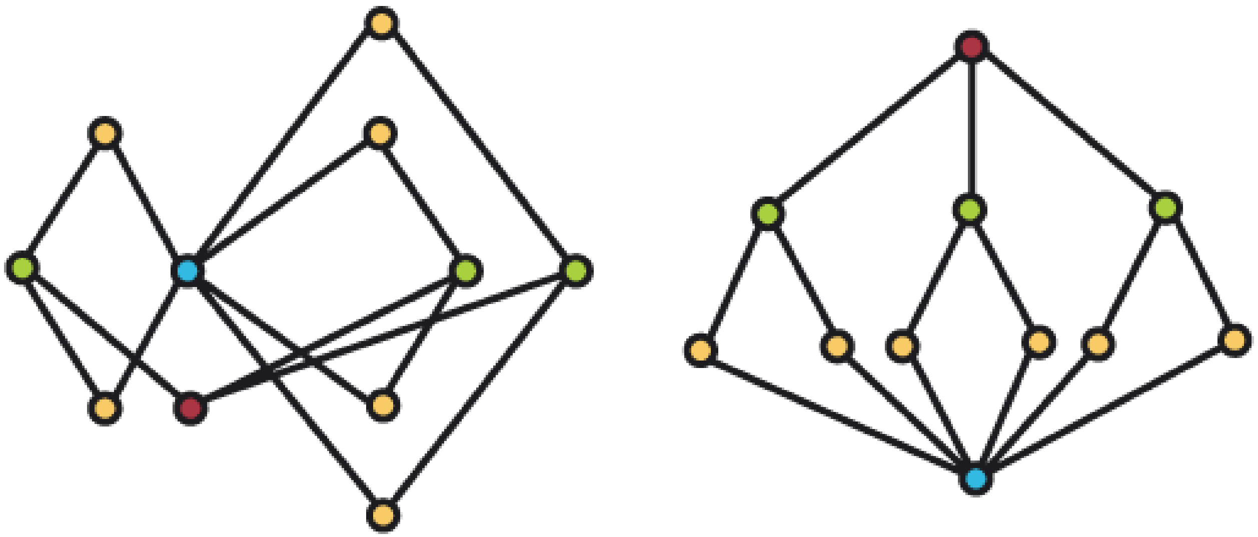
Planar Posets



Definition A poset P is planar when it has an order diagram with no edge crossings.

Fact If P is planar, then it has an order diagram with straight line edges and no crossings.

A Non-planar Poset



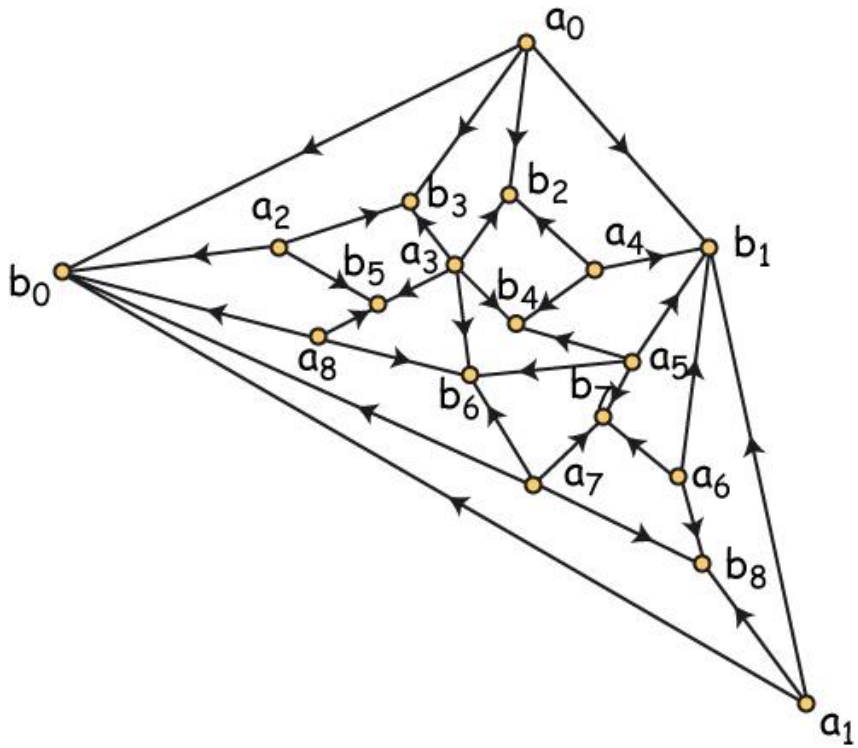
This height 3 non-planar poset has a planar cover graph.

Bipartite Planar Graphs

Theorem (Moore '75; Di Battista, Liu and Rival '90) If P is a poset of height 2 and the cover graph of P is planar, then P is planar, i.e., the order diagram of P is planar.

Note The result is best possible since there exist height 3 non-planar posets that have planar cover graphs.

Diagrams of Bipartite Planar Graphs



Why should it be possible to draw the order diagram of this height 2 poset without edge crossings?

Complexity Issues

Theorem (Garg and Tamassia, '94) The question "Does P have a planar order diagram?" is NP-complete.

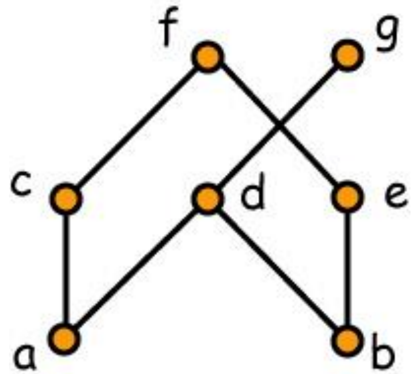
Theorem (Brightwell, '93) The question "Is G a cover graph?" is NP-complete.

Layout Issues

Fact When P is a planar poset on n vertices, it may take a super-polynomial size grid to lay out the order diagram of P so that the cover relations are straight lines and there are no crossings.

Realizers of Posets

A family **F** $= \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P is a realizer of P if $P = \cap$ **F**, i.e., whenever x is incomparable to y in P, there is some L_i in **F** with $x > y$ in L_i.



$L_1 = b < e < a < d < g < c < f$

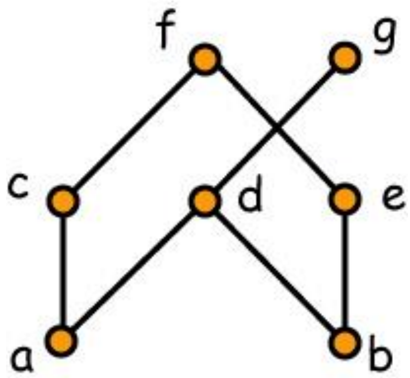
$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

$L_4 = b < e < a < c < f < d < g$

$L_5 = a < b < d < g < e < c < f$

The Dimension of a Poset



$L_1 = b < e < a < d < g < c < f$

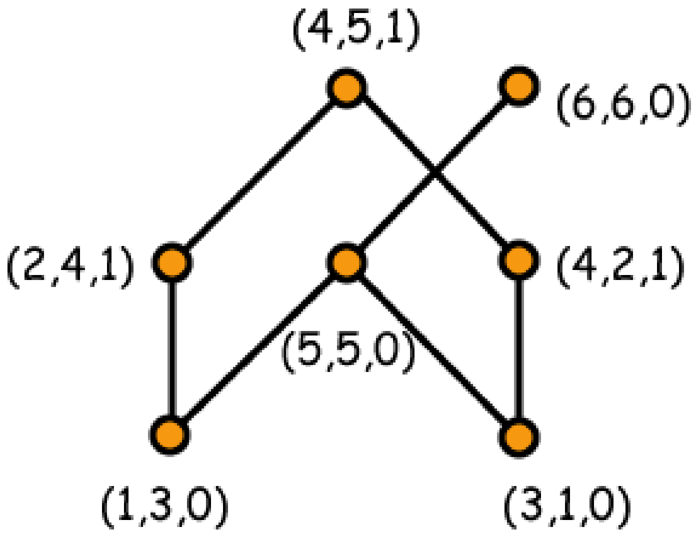
$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

The dimension of a poset is the minimum size of a realizer. This realizer shows $\dim(P) \leq 3$.
In fact,

$$\dim(P) = 3$$

Alternate Definition of Dimension

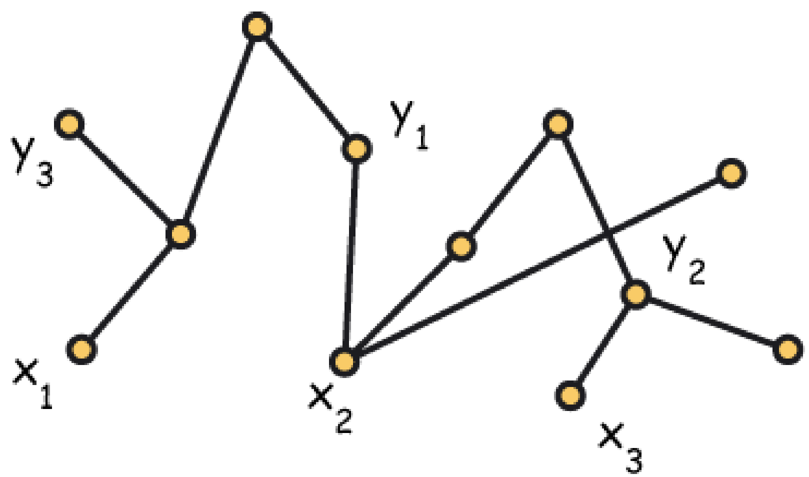


The dimension of a poset P is the least integer n for which P is a subposet of $\mathbf{R}^n$. This embedding shows that $\dim(P) \leq 3$. In fact,

$$\dim(P) = 3$$

Dimension is Coloring for Ordered Pairs

Restatement Computing the dimension of a poset is equivalent to finding the chromatic number of a hypergraph whose vertices are the set of all ordered pairs (x, y) where x and y are incomparable in P. In this poset, no linear extension can put x_i over y_i for all $i = 1, 2, 3$.

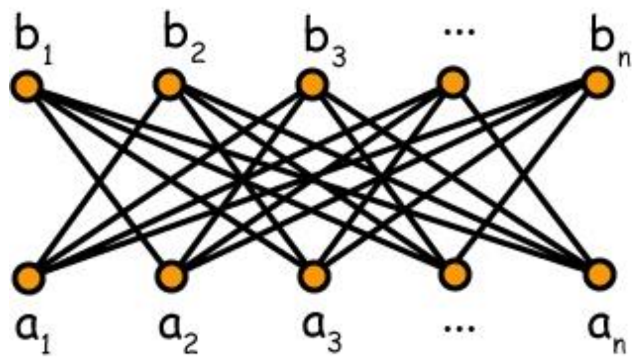


Complexity Issues for Dimension

Theorem (Yannakakis, '82) For fixed $t \geq 3$, the question $\dim(P) \leq t$? is NP-complete.

Theorem (Yannakakis, '82) For fixed $t \geq 4$, the question $\dim(P) \leq t$? is NP-complete, even when P has height 2.

Standard Examples



S_n

Fact For $n \geq 2$, the **standard example** S_n is a poset of dimension n.

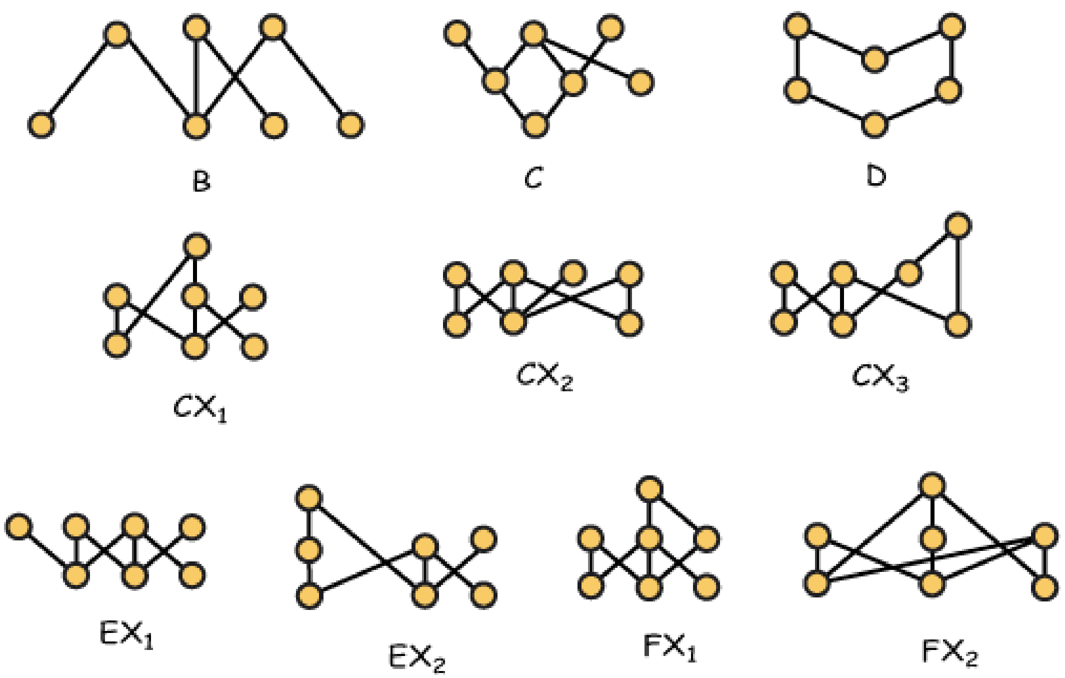
Note If L is a linear extension of S_n, there can only be one value of i for which $a_i > b_i$ in L.

3-Critical Graphs

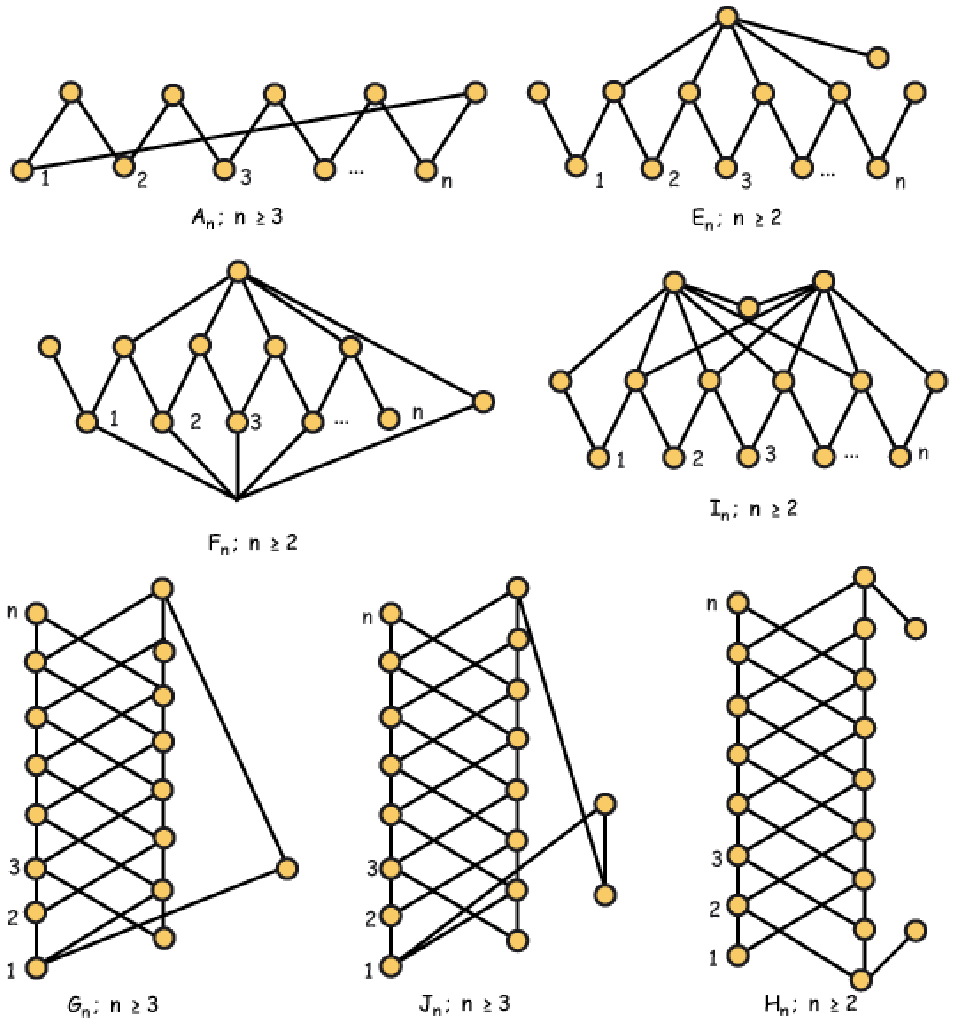
Fact If G is a graph, the chromatic number of G is at most 2 unless G contains an odd cycle.

3-Irreducible Posets – Sporadic Examples

Fact If P is a poset, the dimension of P is at most 2 unless P contains one of the posets shown on this slide and the next.



3-Irreducible Posets – Infinite Families



Gallai, Posets and Dimension

Remark If one knows and understands Gallai's forbidden subgraph characterization of comparability graphs, then the determination of the full list of 3-irreducible posets is an immediate corollary. Also, while it is trivial to see that height and width are comparability invariants, the fact that (a) dimension and (b) the number of linear extensions are comparability invariants follows easily from Gallai's work.

Maximum Degree and Chromatic Number

Definition Let $f(k)$ denote the maximum chromatic number among all graphs G with $\Delta(G) = k$.

Theorem (Brooks '41) $f(k) = k + 1$.

Furthermore, the chromatic number of a graph G with $\Delta(G) = k$ is $k + 1$ only when G is an odd cycle or a complete graph.

Maximum Degree and Dimension

Definition Let $f(k)$ denote the maximum dimension among all posets P with $\Delta(P) = k$ (in the comparability graph). Note that it is not immediately clear that $f(k)$ is well defined!

Observation The standard example S_{k+1} has maximum degree k and has dimension $k + 1$, so if $f(k)$ is well defined, we must have $f(k) \geq k + 1$.

Maximum Degree and Dimension

Theorem (Erdős, Kierstead, WTT '91; Füredi and Kahn '88) There are constants c_1 and c_2 so that

$$c_1 \; k \log k < f(k) < c_2 \, k \log^2 k$$

Observations The upper bound uses the Lovász Local Lemma. The lower bound results from an analysis of random posets of height 2.

Further Analogies

Observation There are posets with large dimension, not containing the standard example S_2. Such posets must have large height.

Observation For every pair (g, d), there is a height 2 poset P such that the girth of the comparability graph of P is at least g and the dimension of P is at least d. Such posets contain S_2 but not S_n when $n \geq 3$.

A General Perception

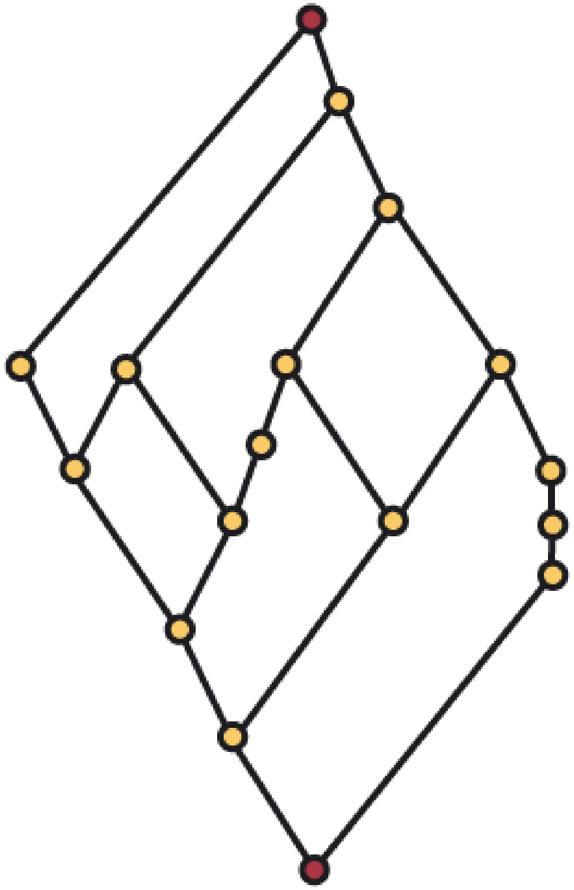
Observation Many invariants of a poset are determined entirely by its comparability graph, including height, width, dimension, and the number of linear extensions.

Observation **None** of these parameters are determined by the cover graph.

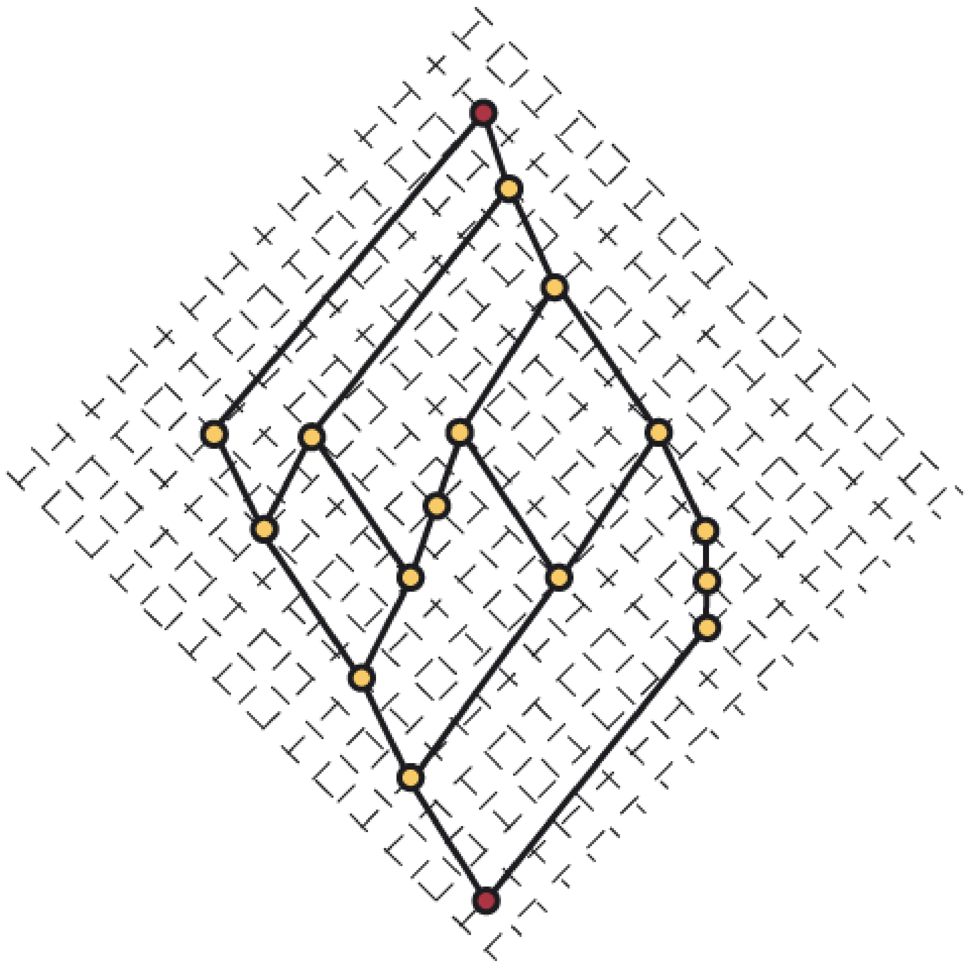
Planar Posets with Zero and One

Theorem (Baker, Fishburn and Roberts, '71 + Folklore)

If P has both a 0 and a 1, then P is planar if and only if it is a lattice and has dimension at most 2.

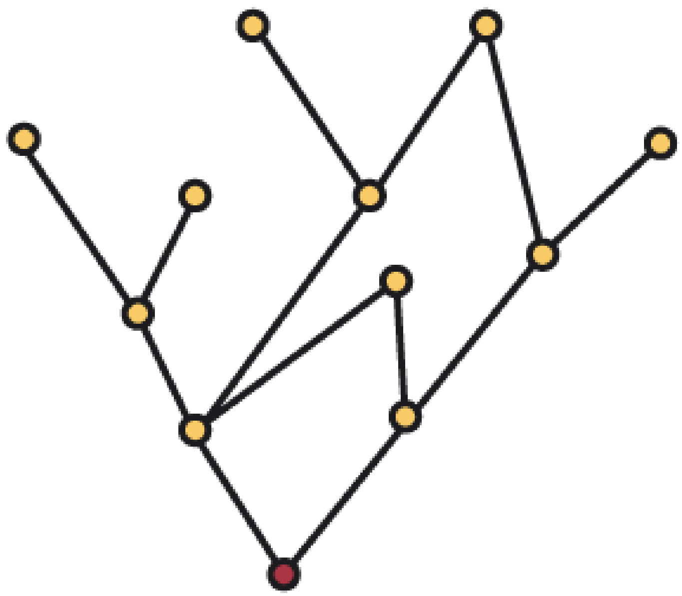


Explicit Embedding on the Integer Grid



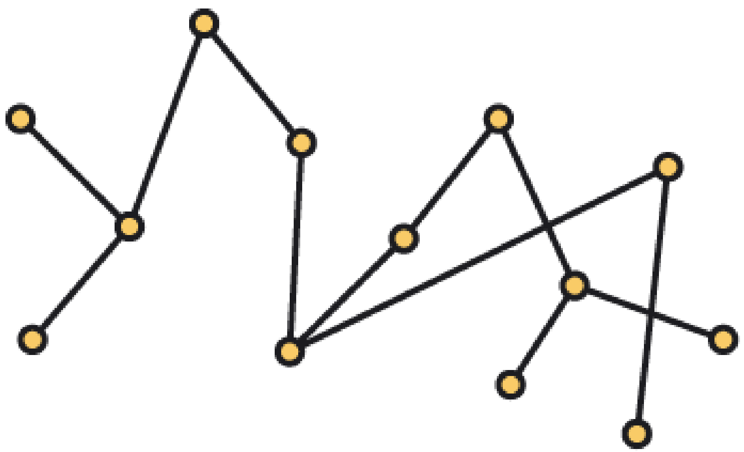
Dimension of Planar Poset with a Zero

Theorem (WTT and Moore, '77) If P has a 0 and the diagram of P is planar, then $\dim(P) \leq 3$.



The Dimension of a Tree

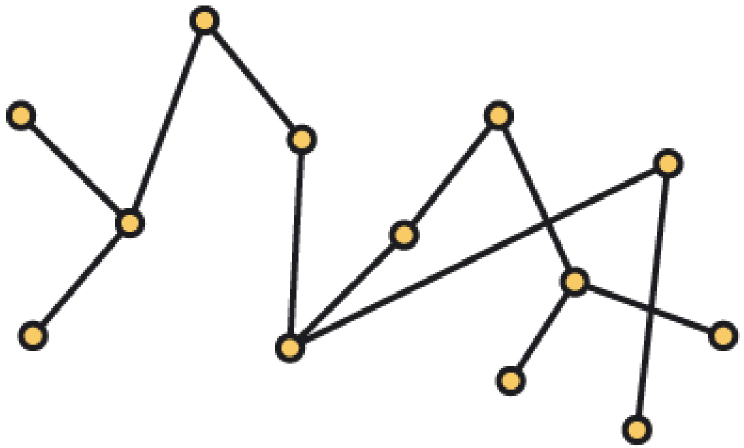
Corollary (WTT and Moore, '77) If the cover graph of P is a tree, then dim(P) ≤ 3.



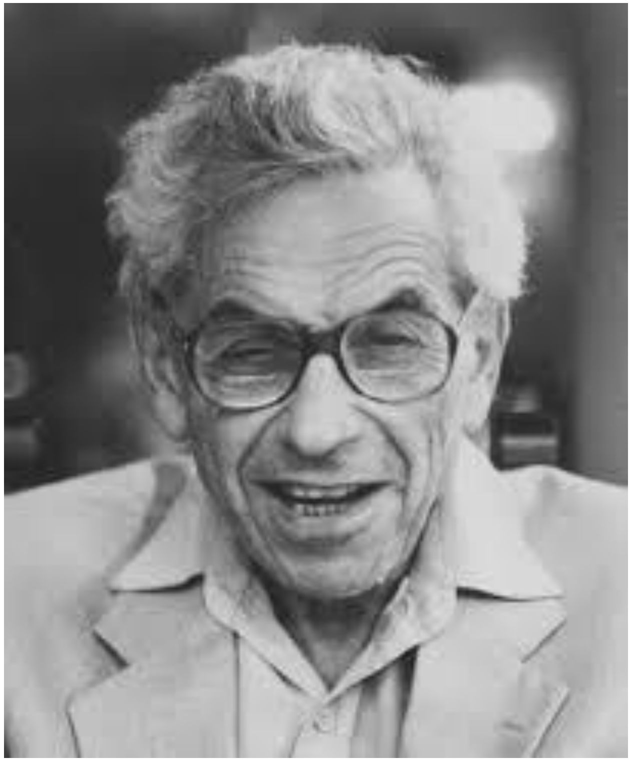
Remark Of course, the corollary follows by showing that the poset obtained by adding a zero to a tree is planar.

A Restatement – With Hindsight

Corollary (WTT and Moore, '77) If the cover graph of P has tree-width 1, then $\dim(P) \leq 3$.

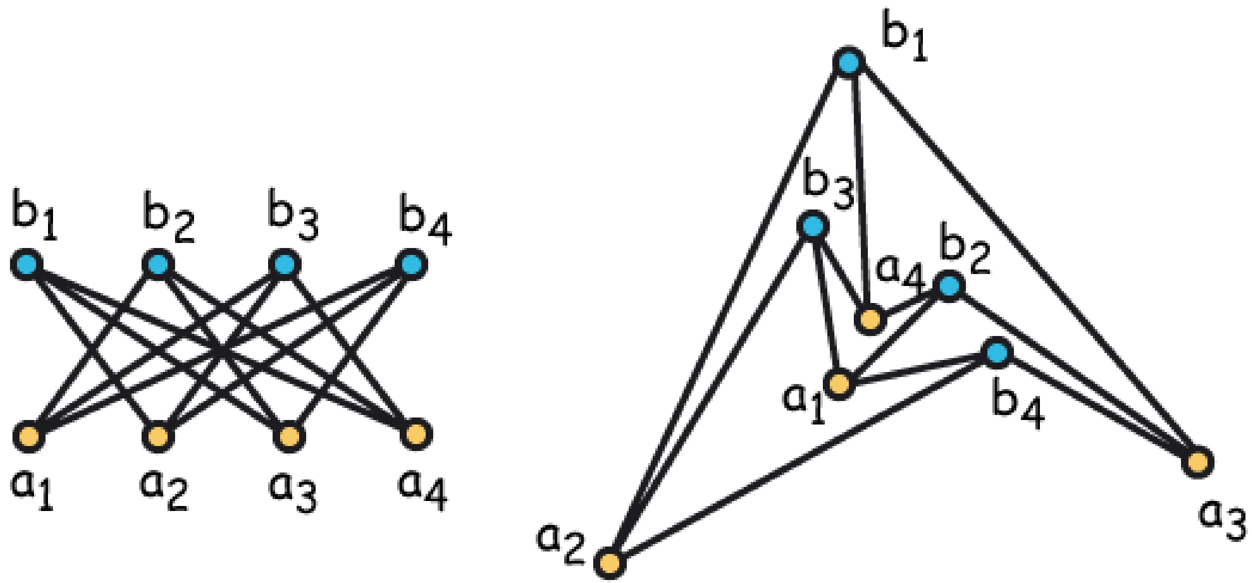


Paul Erdős: Is your Brain Open?



A 4-dimensional planar poset

Fact The standard example S_4 is planar!



Wishful Thinking: If Frogs Had Wings …

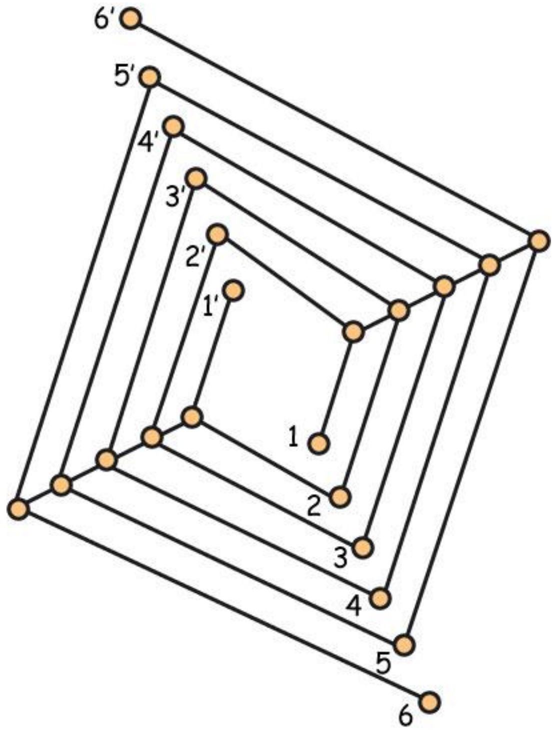
Question Could it possibly be true that $\dim(P) \le 4$ for every planar poset P?

We observe that

- $\dim(P) \le 2$ when P has a zero and a one.
- $\dim(P) \le 3$ when P has a zero or a one.
- So why not $\dim(P) \le 4$ in the general case?

No … by Kelly's Construction

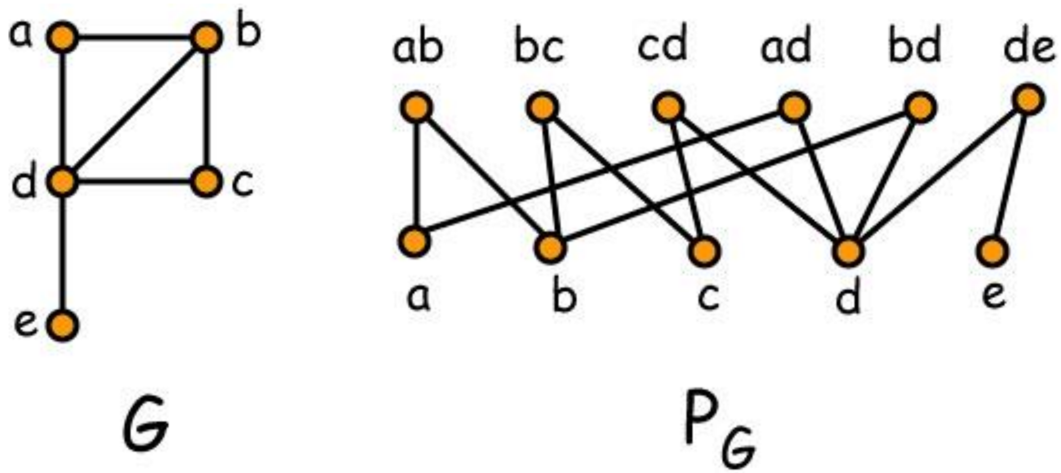
Theorem (Kelly, '81) For every $n \geq 5$, the standard example S_n is non-planar but it is a subposet of a planar poset.



Eight Years of Silence

Kelly's construction more or less killed the subject, at least for the time being.

The Vertex-Edge Poset of a Graph



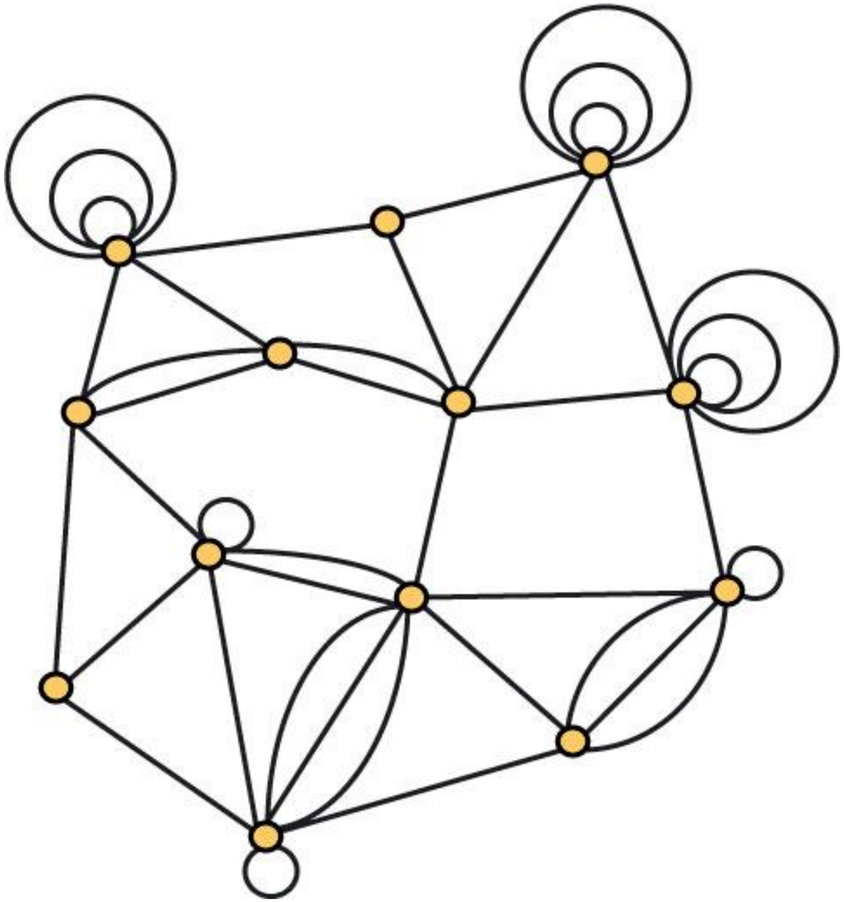
The vertex-edge poset of a graph is also called the incidence poset of the graph.

Schnyder's Theorem

Theorem (Schnyder, '89) *A graph is planar if and only if the dimension of its vertex-edge poset is at most 3.*

Note Testing graph planarity is linear in the number of edges while testing for dimension at most 3 is NP-complete!!!

Planar Multigraphs

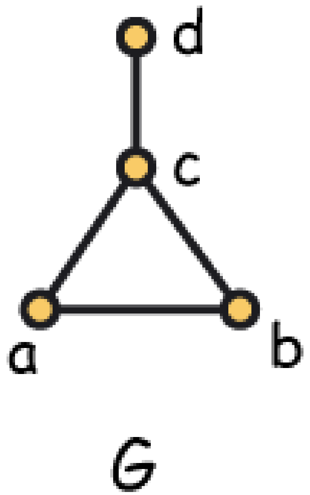


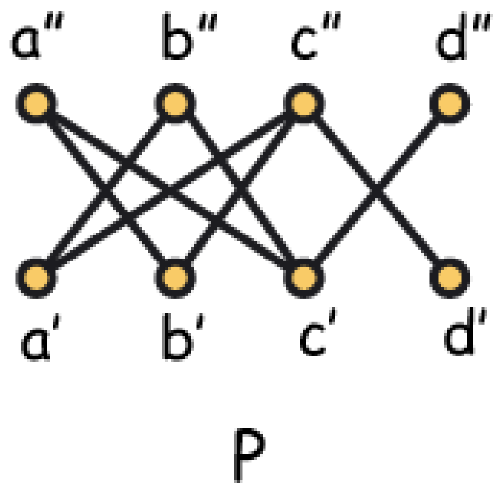
Planar Multigraphs and Dimension

Theorem (Brightwell and WTT, '96, '93): Let D be a non-crossing drawing of a planar multigraph G, and let P be the vertex-edge-face poset determined by D. Then $\dim(P) \leq 4$. Furthermore, if G is a simple 3-connected graph, then the subposet of P determined by the vertices and faces is 4-irreducible.

Adjacency Posets

The adjacency poset P of a graph $G = (V, E)$ is a height 2 poset with minimal elements $\{x' : x \in V\}$, maximal elements $\{x'' : x \in V\}$, and ordering: $x' < y''$ if and only if $xy \in E$.





Adjacency Posets

Observation Let P be the adjacency poset of a graph G. Then $\dim(P) \geq \chi(G)$.

Observation The standard example S_n is the adjacency poset of the complete graph K_n.

Observation If G is the subdivision of K_n, then $\chi(G) = 2$ but the dimension of the adjacency poset of G goes to infinity like $\lg \lg n$.

Adjacency Posets of Planar Graphs

Theorem (Felsner, Li, WTT, '10) Let G be a graph and let P be its adjacency poset.

1. If G is planar, then $\dim(P) \leq 8$.
2. If G is outerplanar, then $\dim(P) \leq 5$.

Observation The proofs use the machinery from Schnyder's theorem.

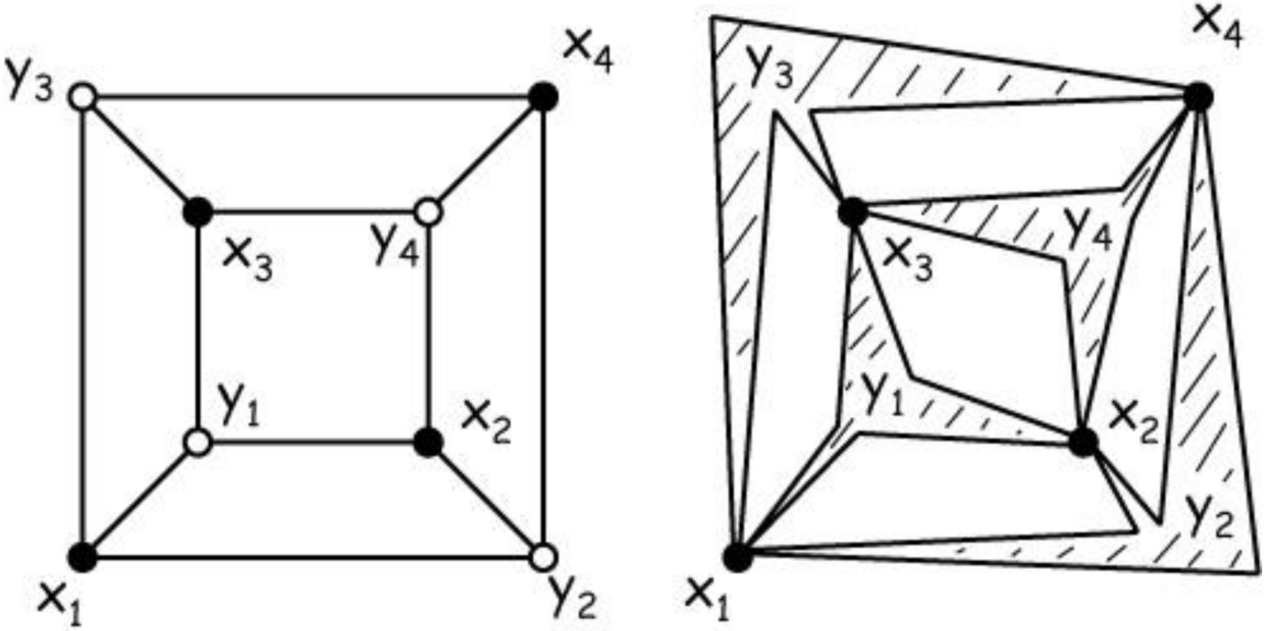
Bipartite Planar Graphs

Theorem (Felsner, Li, WTT, '10) If P is the adjacency poset of a bipartite planar graph, then $\dim(P) \leq 4$.

Corollary If P has height 2 and the cover graph of P is planar, then $\dim(P) \leq 4$.

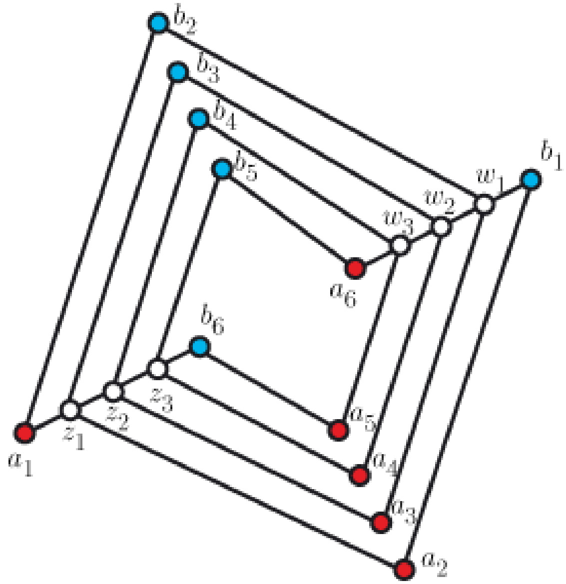
Fact Both results are best possible as evidenced by S_4.

Maximal Elements as Faces



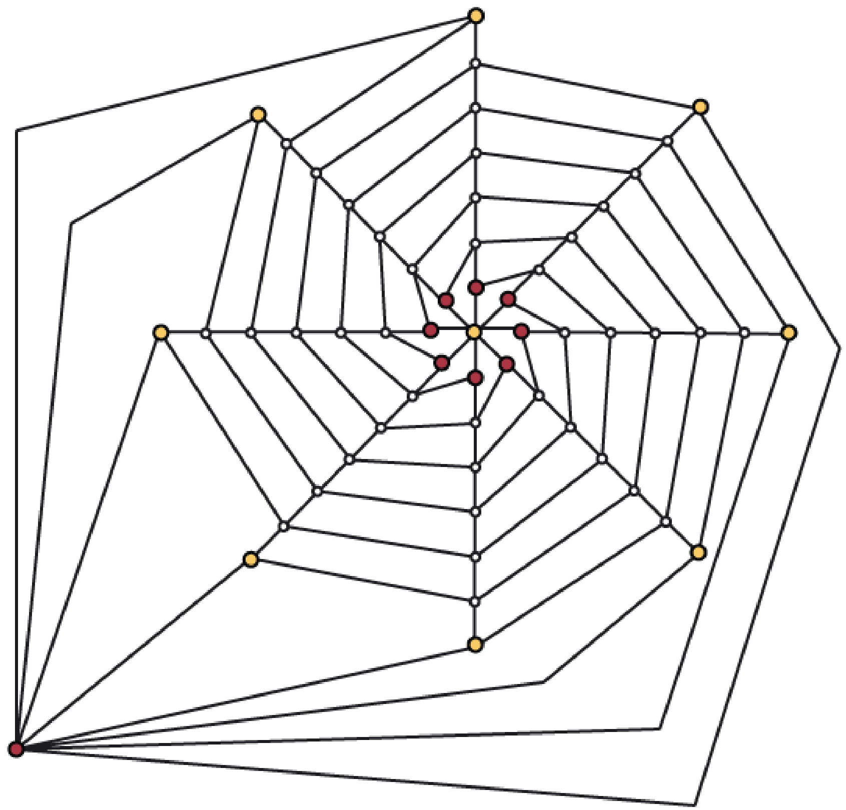
Kelly's Construction Revisited

Observation For every h ≥ 3, there is a planar poset with height h and dimension h+1.



A Modest Improvement – Streib and WTT

Fact For every $h \geq 2$, there is a poset of height h and dimension $h + 2$ with a planar cover graph.



Planar Cover Graphs, Dimension and Height

Conjecture (Felsner, Li, WTT, '10) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

Observation The conjecture holds trivially for $h = 1$ and $c_1 = 2$. Although non-trivial, the conjecture also holds for $h = 2$, and $c_2 = 4$.

Fact The wheel construction shows that c_h - if it exists - must be at least $h + 2$.

Planar Cover Graph Conjecture Resolved

Theorem (Streib and WTT, 2012) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

Observation The proof uses Ramsey theory at several key places and the bound we obtain is **very** large in terms of h.

A Key Detail

Observation The cover graph of a poset can be planar and have arbitrarily large tree-width, even when the poset has small height, e.g., consider an $n \times n$ grid.

However The argument used by Streib and WTT used a reduction to the case where the diameter of the cover graph is bounded as a function of the height.

Fact The tree-width of a planar graph of bounded diameter is bounded.

Planar Cover/Comparability Graphs

Theorem (Felsner, WTT, Wiechert, 2011) Let P be a poset.

1. If the cover graph of P is outerplanar, then $\dim(P) \leq 4$.
2. If the cover graph of P is outerplanar and P has height 2, then $\dim(P) \leq 3$.
3. If the comparability graph of P is outerplanar, then $\dim(P) \leq 4$.

Observation Outerplanar graphs have tree-width at most 2.

Summary of the Evidence

Observations Let P be a poset and let G be the cover graph of P. Then $\dim(P)$ is bounded if any of the following statements hold:

1. The tree-width of G is 1.
2. G is outerplanar (and therefore has tree-width at most 2).
3. G is planar and has diameter bounded in terms of its height (and therefore bounded tree-width).

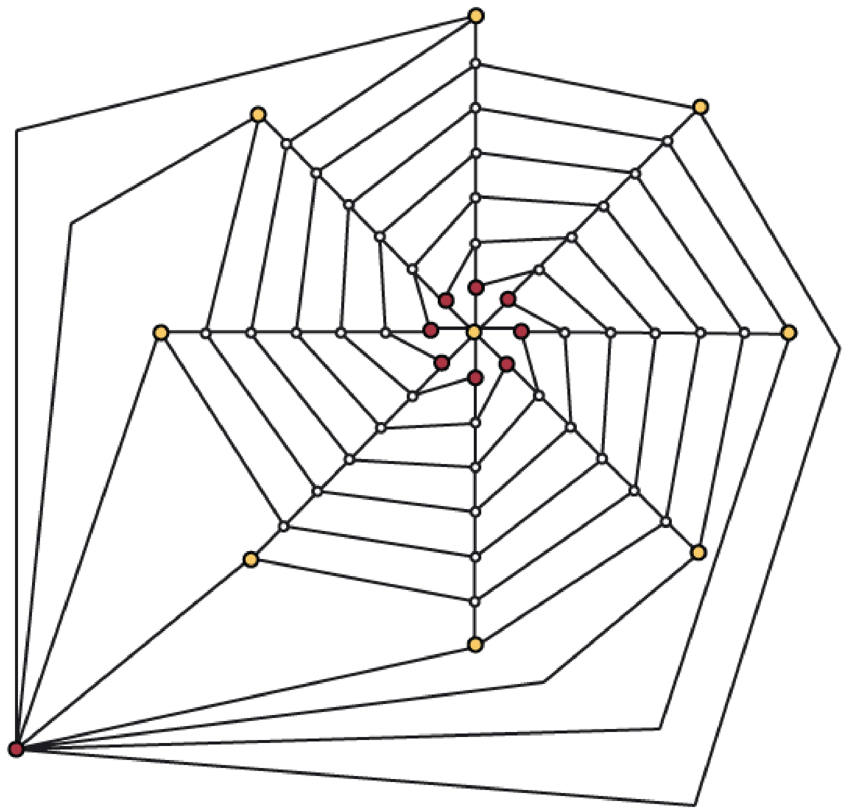
Observation If the tree-width of G is bounded and $\dim(P)$ is large, then it seems the height of P must also be large.

Joret's Conjecture

Conjecture The dimension of a poset is bounded in terms of its height and the tree-width of its cover graph. Formally, for every pair (t, h), there is a constant $d = d(t, h)$ so that if P is a poset of height at most h and the tree-width of the cover graph of P is at most t, then $\dim(P) \leq d$.

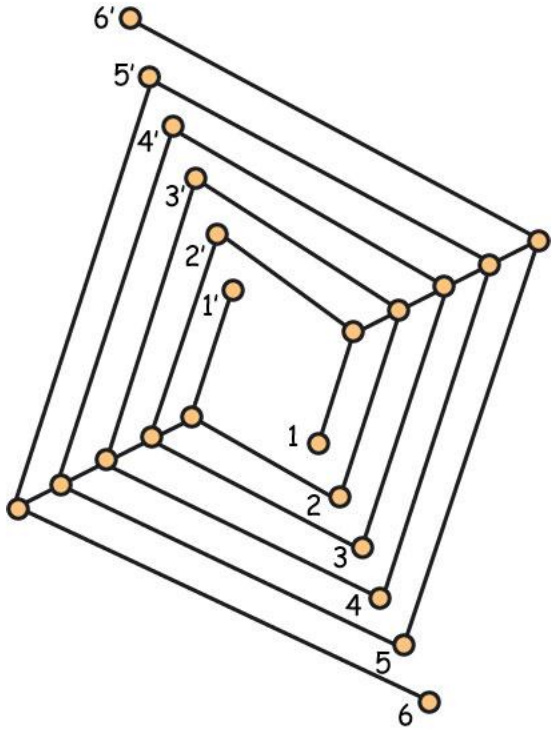
A Momentary Hiccup

Observation The cover graphs in the wheel construction have large tree-width, as they contain large grids.



Kelly's Construction and Tree-width

Observation The cover graphs in Kelly's construction have tree-width 3. In fact, they have path-width 3.



The Resolution of Joret's Conjecture

Theorem (Joret, Micek, Milans, WTT, Walczak, Wang, 2012) The dimension of a poset is bounded in terms of its height and the tree-width of its cover graph. Formally, for every pair (t, h), there is a constant $d = d(t, h)$ so that if P is a poset of height at most h and the tree-width of the cover graph of P is at most t, then $\dim(P) \leq d$.

Open Problems

Is the dimension of a poset bounded when the tree-width of its cover graph is 2?

Remark Biró, Keller and Young (2013+) have just proved that the answer is yes when the path-width of the cover graph is 2.

More Open Problems

1. Must planar posets of large dimension contain large standard examples?
2. If the tree-width of the cover graph is bounded and the dimension is large, must the poset contain a large standard example?
3. For what other minor closed classes is there a bound on the dimension of a poset (perhaps as a function of height) when the cover graph does not contain a graph from the class as a minor?

WTT on Erdős (1998)

Paul Erdős was one of those very special geniuses, the kind who comes along only once in a very long while, yet he chose, quite consciously I am sure, to share mathematics with mere mortals--like me. And for this, I will always be grateful to him. I will miss the times he prowled my hallways at 4:00 A.M. and came to my bed to ask whether my "brain is open." I will miss the problems and conjectures and the stimulating conversations about anything and everything. But most of all, I will just miss Paul, the human. I loved him dearly.