Erdős on polynomials

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- Interpolation

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- Polynomials with integer coefficients

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- Interpolation (Lubinsky's and Vértesi's surveys)
- Discrepancy theorems for zeros
- Inequalities (Erdélyi's survey)
- Growth of polynomials
- Geometric problems for lemniscates (Eremenko-Hayman's and Borwein's surveys)
- Orthogonal polynomials
- Spacing of zeros
- Geometry of zeros of derivatives
- Polynomials with integer coefficients (Borwein's and Erdélyi's surveys)

Chebyshev polynomials

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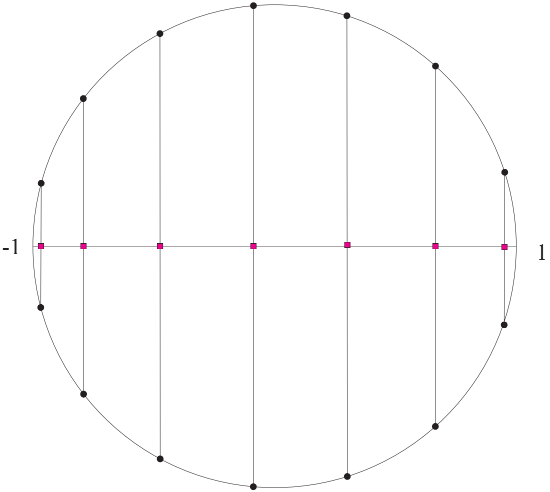
$$T_n(z) = \frac{1}{2^{n-1}} \cos(n \arccos x)$$

Uniform spacing of zeros



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Examples:

- $$d\mu_{[-1,1]}(t) = \frac{1}{\pi\sqrt{1-t^2}} dt$$
- If C_1 is the unit circle, then
$$d\mu_{C_1}(e^{it}) = \frac{1}{2\pi} dt$$
is the normalized arc measure

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$$\|P_n\|_K \geq \mathrm{cap}(K)^n$$

Transfinite diameter, Chebyshev constants

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Fekete, Zygmund, Szegő:

$$\mathrm{cap}(K) = \delta(K) = t(K)$$

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A general discrepancy theorem

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Recall: if $P_n(z) = z^n + \cdots$, then $\|P_n\|_K \geq \mathrm{cap}(K)^n$

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$$|\nu_n(J^*) - \mu_K(J^*)| \leq C\sqrt{\frac{\log A_n}{n}}.$$

In particular, if $\|P_n\|_K^{1/n} \to \mathrm{cap}(K)$, then $\nu_n \to \mu_K$

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Jentsch, 1918: If the radius of convergence of $\sum_{j=0}^{\infty} a_j z^j$ is 1, then the zeros of the partial sums $\sum_0^n a_j z^j$, $n = 1, 2, \ldots$ are dense at the unit circle

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Szegő, 1922: There is a sequence $n_1 < n_2 < \cdots$ such that if $z_{j,n} = r_{j,n} e^{i\theta_{j,n}}$, $1 \le j \le n$ are the zeros of $\sum_0^n a_j z^j$, then $\{\theta_{j,n_k}\}_0^{n_k}$ is asymptotically uniformly distributed (and $r_{j,n_k} \approx 1$ for most j)

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Th. (Erdős-Turán, 1950) For any $J \subset [0, 2\pi]$

$$\left| \frac{\#\{\theta_j \in J\}}{n} - \frac{|J|}{2\pi} \right| \leq 16 \sqrt{\frac{\log \|P_n\|_{C_1} / \sqrt{|a_0 a_n|}}{n}}$$

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Note

$$\|P_n\|_{C_1} \leq \sum_j |a_j|$$

Number of real zeros of a polynomial

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Better than previous estimates of Bloch, Pólya, Schmidt, basically a theorem of Schur, Szegő

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General discrepancy theorems

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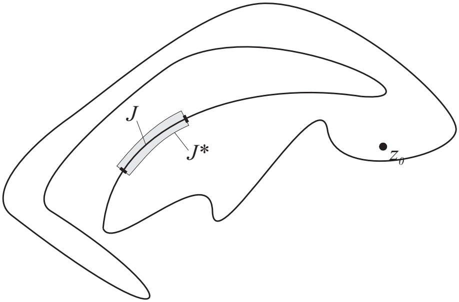

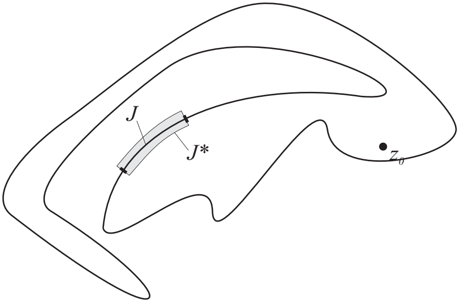

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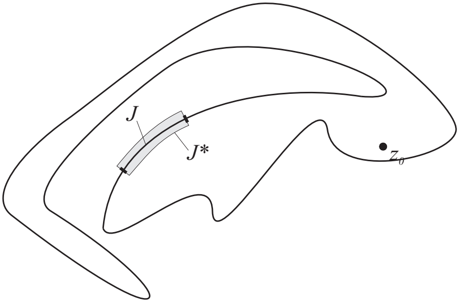

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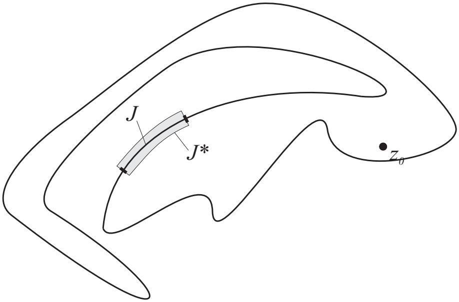

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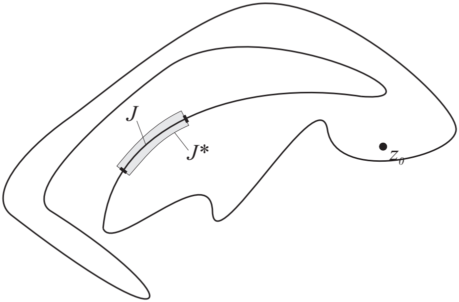

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It is true for a family of Jordan curves

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$p_n(z) = \gamma_n z^n + \cdots$ orthonormal polynomials with respect to ρ:

$$\int p_n \overline{p_m} d\rho = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

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Th. (Erdős-Turán, 1940) If the support of ρ is $[-1, 1]$, $d\rho(x) = w(x)dx$ and $w > 0$ almost everywhere,

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Orthogonal polynomials

ρ a measure with compact support

$p_n(z) = \gamma_n z^n + \cdots$ orthonormal polynomials with respect to ρ:

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$$\frac{1}{\mathrm{cap}(S)} \leq \liminf \gamma_n^{1/n}$$

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In this case we say $\rho \in$ **Reg**

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Sufficiency follows from some strong regularity criteria

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ρ is doubling if $\rho(2I) \leq C\rho(I)$ for $I \subset [-1, 1]$

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Levin-Lubinsky: under these conditions (2) is true

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Is (3)/universality true (say on $[-1,1]$) a.e. solely under the Erdős-Turán condition $w > 0$ a.e.?

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Many other conjectures (by Schoenberg, Katroprinakis, ...) on the relation of the z_k's and ξ_j's

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Th. (Malamud, Pereira, 2003) There is a doubly stochastic matrix $\mathbf{A}$ such that $\Xi = \mathbf{AZ}$

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2) If all zeros lie in the upper-half plane, then

$$\left(\prod_1^n \Im z_k\right)^{1/n} \le \left(\prod_1^{n-1} \Im \xi_j\right)^{1/(n-1)}$$

Gauss-Lucas, de Brjuin-Springer etc. are all immediate consequences:

$\xi_j = \sum_k a_{jk} z_k$

$$\frac{1}{n-1}\sum_j \varphi(\xi_j) \le \frac{1}{n-1}\sum_j \sum_k a_{jk}\varphi(z_k)$$

$$= \frac{1}{n-1}\sum_k \varphi(z_k)\sum_j a_{jk} = \frac{1}{n}\sum_k \varphi(z_k)$$

Examples: **1)**

$$\frac{1}{n-1}\sum_1^{n-1} |\Re \xi_j|^m \le \frac{1}{n}\sum_1^{n} |\Re z_k|^m$$

2) If all zeros lie in the upper-half plane, then

$$\left(\prod_1^n \Im z_k\right)^{1/n} \le \left(\prod_1^{n-1} \Im \xi_j\right)^{1/(n-1)}$$