BÁLINT TÓTH
(TU Budapest and U of Bristol)

ER + FF = SOC

Erdős Centennial Conference, Budapest, 1-5 July 2013

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ERDŐS-RÉNYI RANDOM GRAPHS + FOREST FIRES = SELF-ORGANIZED CRITICALITY

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ERDŐS-RÉNYI RANDOM GRAPHS + FOREST FIRES = SELF-ORGANIZED CRITICALITY

based on joint work with Balázs Ráth (Budapest/Vancouver)
and (in progress) with Ed Crane and Nic Freeman (Bristol)

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Erdős-Rényi random graph, dynamical formulation:

$\Omega_N := \{0,1\}^{\binom{N}{2}}$, $\omega_{\{i,j\}}$ switches from 0 to 1 with rate N^{-1}

- $0 \le t < 1$ **subcritical phase**: small clusters

  $\mathbf{P}\Big(\text{size of the cluster of a randomly chosen site} > k\Big) < e^{-\gamma(t)k}$,

  Largest cluster $\sim \log N$.

- $1 < t < \infty$ **supercritical phase**:
  one giant of size $\sim \theta(t)N$ + small clusters.

- $t = 1$ **critical point**:

  $\mathbf{P}\Big(\text{size of the cluster of a randomly chosen site} > k\Big) \asymp k^{-1/2}$,

  $+$ some large clusters $\sim N^{2/3}$

Forest Fires:

Mechanism of instantaneous destruction of "large" clusters. (More precise definition soon.)

Self-Organized Criticality:

Dynamical phenomenon when by some competing mechanisms (e.g. coagulation + fragmentation) a large system is driven to a robust, persisting critical state, without fine tuning of parameters.

Examples (mostly from physics):

∘ various FF models [Drossel-Schwabl (1992), . . .];
∘ coagulation/fragmentation models [Smoluchowski (1916), . . .];
∘ sandpile models [Bak-Tang-Wiesenfeld (1986), . . .];

Main source of difficulty: **lack of monotonicity**.

ER + FF = coagulation + fragmentation

- Edges turn 0(off) $\longmapsto$ 1(on) with rate N^{-1}

- "Lightnings" hit vertices with rate $\lambda(N)$ (to be specified). When lightning hits a site edges in its connected cluster turn instantaneously 1 $\longmapsto$ 0. I.e. burn down.

- All Poisson flows independent.

$$V_k^{(N)}(t) := \#\{\text{vertices in clusters of size } k \text{ at time } t\}$$

$$\sum_{k \geq 1} V_k^{(N)}(t) \equiv N$$

This is a Markov process on its own.

$$\left(V_k^{(N)},\ V_l^{(N)},\ V_{k+l}^{(N)}\right) \longmapsto \left(V_k^{(N)} - k,\ V_l^{(N)} - l,\ V_{k+l}^{(N)} + (k+l)\right)$$

with rate $N^{-1} V_k^{(N)} V_l^{(N)}$

$$\left(V_k^{(N)},\ V_{2k}^{(N)}\right) \longmapsto \left(V_k^{(N)} - 2k,\ V_{2k}^{(N)} + 2k\right)$$

with rate $(2N)^{-1} V_k^{(N)} (V_k^{(N)} - 1)$

$$\left(V_1^{(N)},\ V_k^{(N)}\right) \longmapsto \left(V_1^{(N)} + k,\ V_k^{(N)} - k\right)$$

with rate $\lambda(N) V_k^{(N)}$

The empirical cluster size distribution:

$$v_k^{(N)}(t) := N^{-1} V_k^{(N)}(t), \qquad \sum_{k \geq 0} v_k^{(N)}(t) \equiv 1$$

??? $$\lim_{N \to \infty} v_k^{(N)}(t) =: v_k(t)$$ **???**

Does the limit exist? In what sense? How does it behave qualitatively?

$$\sum_{k \geq 1} v_k(t) \leq 1, \qquad \theta(t) := 1 - \sum_{k \geq 1} v_k(t), \text{ the gel.}$$

Regimes of the lightning rate:

I: $\lambda(N) \ll N^{-1}$: no lightning observed, good old ER

II: $\lambda(N) = \lambda N^{-1}$: "collapse of the giant", moderately interesting

III: $N^{-1} \ll \lambda(N) \ll 1$: this is the interesting regime

IV: $\lambda(N) = \lambda$: subcritical for ever, moderately interesting

The limits

I: $\lambda(N) \ll N^{-1}$: (no news)

$v_k^{(N)}(t) \to v_k(t)$ uniformly in $t \in [0, T]$, $k \in [0, K]$, as $N \to \infty$, where $v_k(t)$ is the unique solution of Smoluchowski's coag. eq.

$$\dot{v}_k(t) = \frac{k}{2} \sum_{l=1}^{k-1} v_l(t) v_{k-l}(t) - k v_k(t) \text{ for } k \geq 1,$$

$$v_k(0) = \delta_{k,0}$$

Actually, decoupled and explicitly solved, one-by-one for $k = 1, 2, \ldots$:

$$v_k(t) = \frac{k^{k-1}}{k!} e^{-kt} t^{k-1}, \qquad \sum_{k \geq 1} v_k(t) \begin{cases} = 1 & \text{if } t \leq 1 \\ < 1 & \text{if } t > 1 \end{cases}$$

Remarks: (1) Let

$$V(t,x) := \sum_{k\geq 1} e^{-xk} v_k(t) - 1, \qquad x \geq 0$$

Then $V(t,x)$ is the solution of Burgers' equation in $x \geq 0$, $t > 0$:

$$\partial_t V(t,x) + \frac{1}{2}\partial_x V(t,x)^2 = 0, \qquad t > 0, \quad x \geq 0,$$

$$V(0,x) = e^{-x} - 1$$

☺☺☺ **The phase transition in ER is actually a** ☺☺☺
☺☺☺ **shock wave appearing in a hyperbolic PDE.** ☺☺☺

(2) Other initial conditions with $\sum_{k\geq 1} k^3 v_k(0)$ are equally good.

$$t_c = \left(\sum_{k\geq 1} k v_k(0)\right)^{-1}$$

II: $\lambda(N) = \lambda N^{-1}$: (moderately interesting)

very similar: limit exists, same differential equations (solvable one-by-one) + "collapse of the giant":

$v_1(t) \longmapsto v_1(t) + \theta(t)$, and $\theta(t) \longmapsto 0$ with rate $\lambda\theta(t)$.

IV: $\lambda(N) = \lambda$: (moderately interesting)

The limit exists. The system of differential equations is slightly different

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - (1+\lambda)kv_k(t) + \lambda\delta_{k,0}\sum_{l\geq 1} lv_l(t), \qquad k \geq 1,$$

$$v_k(0) = \delta_{k,0}$$

No explicit solution, but qualitative analysis not very difficult: Unique sln, with exponential decay in k. Subcritical forever.

III: $N^{-1} \ll \lambda(N) \ll 1$: the interesting case

The system:

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t) \text{ for } k \geq 2, \qquad \sum_{k\geq 1} v_k(t) = 1,$$

$$v_k(0) = \delta_{k,0}$$

No decoupling: infinite system with constraint.

The PDE: $V(t,x) := \sum_{k\geq 1} e^{-xk} v_k(t)$, $x \geq 0$

$$\partial_t V(t,x) + \frac{1}{2}\partial_x V(t,x)^2 = e^{-x}\varphi(t), \qquad V(t,0) = 0,$$

$$V(0,x) = e^{-x} - 1$$

Burgers control problem. (Looks like overdetermined . . .)

Theorem. *[B. Ráth, B. Tóth (2009)]*

(i) The Burgers control problem has got unique solution, with $t \mapsto \varphi(t)$ *Lipschitz-continuous, and* $0 < c_1 \leq \varphi(t) \leq c_2 < \infty$ *for* $t \geq t_c$.

(ii) $v_k^{(N)}(t) \to v_k(t)$ *uniformly in* $t \in [0, T]$, $k \in [0, K]$, *as* $N \to \infty$, *where* $v_k(t)$ *is the unique solution of the Burgers control problem.*

(iii) For $t \geq t_c$,

$$\sum_{l \geq k} v_l(t) \asymp \sqrt{\frac{2\varphi(t)}{\pi}} k^{-1/2}.$$

(iv)

$$\lim_{t \to \infty} v_k(t) = \frac{2}{n 4^n} \binom{2n-2}{n-1} \approx \frac{1}{\sqrt{4\pi}} k^{-3/2} =: \bar{v}_k.$$

Outlook: in progress with Ed Crane and Nic Freeman (Bristol)

All in the stationary regime.

Choose a site at random (uniformly) and follow the time-evolution of its connected cluster (Benjamini-Schramm limit): This is a random process $t \mapsto \gamma^{(N)}(t)$ with values in the space of finite rooted graphs.

??? $\gamma^{(N)}(\cdot) \Rightarrow \gamma(\cdot)$ **???**

- Description of the limit process $t \mapsto |\gamma(t)|$ in plain words:

  - $|\gamma(t)| \longmapsto |\gamma(t)| + \bar{v}_k$ with rate $|\gamma(t)|\,\bar{v}_k$.
  - At $\tau := \sup\{t : |\gamma(t)| < \infty\}$, $|\gamma(t)|$ jumps form ∞ to 1.
  - Go on like this . . . .

The Markov process $t \mapsto |\gamma(t)|$ is well defined on $\mathbb{N}$, in terms of the infinitesimal generator.

- Description of the limit process $t \mapsto \gamma(t)$ in plain words: . . .

Convergence (to be) proved using Trotter-Kurtz approach (cvg of the infin. gen. / resolvents)