Sums of products of fixed primes

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On the influence of Paul Erdös on my work

A result from each decade

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My first paper: On a conjecture of Turán and Erdös, 1966

Conjecture of Erdös:

If for a system of n distinct nonzero complex numbers $z_1, \ldots, z_n$, $n \geq 2$, equality

$$\max_{\nu=m+1,\ldots,m+n-1} \left| \sum_{j=1}^{n} z_j^{\nu} \right| = 0$$

holds for *two* different values of m, then $z_1, z_2, \ldots, z_n$ are the solutions of the equation

$$z^n + a = 0$$

for a certain complex number a.

My first paper: On a conjecture of Turán and Erdös, 1966

Is it true that if $\max_{\nu=m+1,\ldots,m+n-1} \mid \sum_{j=1}^{n} z_j^{\nu} \mid = 0$ holds for *two* different values of m, then $z^n + a = 0$ for a certain complex number a?

Statements are true for odd n, but false for even n. All examples with even n can be explicitly given: Two full sets of $(n/2)$th roots of unity, each multiplied by a complex number in such a way that an extra zero sum arises.

Partially solved by D.G. Cantor, independently of me.

On integers with many small prime factors (1973).

Wintner communicated the following problem to Erdös orally.
Erdös published it in a survey paper in 1965.

Does there exist an infinite sequence of primes $p_1 < p_2 < \ldots$ such that if $n_1 < n_2 < \ldots$ are the integers composed of these primes, then

$$\lim_{j\to\infty} (n_{j+1} - n_j) = \infty?$$

On integers with many small prime factors (1973).

Does there exist an infinite sequence of primes $p_1 < p_2 < \ldots$ such that if $n_1 < n_2 < \ldots$ are the integers composed of these primes, then

$$\lim_{j\to\infty} (n_{j+1} - n_j) = \infty?$$

By using Baker's theory on linear forms I proved

Let $0 < \theta < 1$. Then there exists an infinite sequence of primes $p_1 < p_2 < \ldots$ such that if $n_1 < n_2 < \ldots$ are the integers composed of these primes, then

$$n_{j+1} - n_j > n_j^{1-\theta} \text{ for all } j.$$

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With K. Györy and C.L. Stewart,
On prime factors of sums of integers 1 (1986)

Let $\omega(n)$ be the number of distinct prime factors of n.

Erdös and Turán (1934) proved:
If A is a finite set of positive integers of cardinality k then, for $k \geq 2$, $\omega\left(\prod_{a,a' \in A}(a + a')\right) >> \log k$.

Erdös conjectured (1976):
For every n there is an N such that if A and B are finite sets of positive integers with cardinality $k > N$, then $\omega\left(\prod_{a \in A, b \in B}(a + b)\right) > n$.

With K. Györy and C.L. Stewart,
On prime factors of sums of integers 1 (1986)

Is there for every n an N such that if A and B are finite sets of positive integers with cardinality $k > N$, then $\omega\left(\prod_{a\in A, b\in B}(a+b)\right) > n$?

We proved, by Evertse:

Let A and B satisfy $|k| = A \geq B \geq 2$. Then

$$\omega\left(\prod_{a\in A, b\in B}(a+b)\right) >> \log k.$$

With Erdös and C.L. Stewart,
Some diophantine equations with many solutions (1988)

Let $|k| = A \geq B \geq 2$. Then

$$P\left(\prod_{a \in A, b \in B} (a+b)\right) >> \log k \log\log k.$$

Let $0 < \varepsilon < 1$. Let $k > k_0(\varepsilon), 2 \leq l \leq \sqrt{\log k}$.
Let $P(n) = \{\max_p : p|n\}$.
Then there are A with $|A| = k$ and B with $|B| = l$ s.t.

$$P\left(\prod_{i=1}^{k}\prod_{j=1}^{l}(a_i + b_j)\right) < \left((\frac{1}{2}+\varepsilon)\frac{\log k}{l}\log\log k\right)^l.$$

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$$P\left(\prod_{i=1}^{k}\prod_{j=1}^{l}(a_i+b_j)\right) < \left((\frac{1}{2}+\varepsilon)\frac{\log k}{l}\log\log k\right)^{l}.$$

Evertse (1984). Let $S=\{p_1,\ldots,p_s\}$ be a set of prime numbers. Then the equation (1) $x+y=z$ in coprime integers which are all composed of primes from S has at most $3\times 7^{2s+3}$ solutions.

We proved: The number of coprime solutions of (1) can be as large as $\exp\left((4-\epsilon)\sqrt{\frac{s}{\log s}}\right)$.

With T.N. Shorey, Some methods of Erdös applied to finite arithmetic progressions (1996)

Erdös and Selfridge (1976). The equation

$$x(x+1)(x+2)\dots(x+k-1) = y^l$$

has no solutions in integers $x > 0, k > 1, l > 1, y > 1$.

Erdös conjectured: The equation

$$x(x+d)(x+2d)\dots(x+(k-1)d) = y^l$$

has no solutions in integers
$x > 0, d > 0, k > 3, l > 1, y > 1$ subject to $(x, d) = 1$.

With T.N. Shorey, Some methods of Erdös applied to finite arithmetic progressions (1996)

Does the equation

$$x(x+d)(x+2d)\dots(x+(k-1)d) = y^l$$

have no solutions in $x > 0, d > 0, k > 3, l > 1, y > 1$ with $(x, d) = 1$?

We (and others) obtained several partial results. E.g.
Let l be prime and d_1 the maximal divisor of d such that all the prime factors of d_1 are $\equiv 1 \pmod{l}$.
Then, for k large,

$$d_1 >> k^{l-2}.$$

With J. Hancl, On the irrationality of Cantor series (2004)

Erdös mentioned more than once (1958,1974) that he could not prove:

Let $p_1, p_2, \ldots$ be the sequence of primes in increasing order. Then

$$\sum_{n=1}^{\infty} \frac{p_n}{2^n}$$

is irrational.

With J. Hancl, On the irrationality of Cantor series (2004)

Is $\sum_{n=1}^{\infty} \frac{p_n}{2^n}$ irrational?

We proved:

Suppose $\{a_n\}_{n=1}^{\infty}$ is a monotonic sequence of positive integers such that $\lim_{n\to\infty} \frac{a_n}{\log n} \to \infty$. Then $\sum_{n=1}^{\infty} \frac{p_n}{a_1 \dots a_n}$ is rational if and only if $\frac{p_n}{a_n - 1}$ is constant for $n > n_0$.

Erdös' problem is still open.

At the Erdös conference Schlage-Puchta told me that he has improved the bound $\log n$ to the order $\exp(\sqrt{\log \log n})$.

With L. Hajdu, Representations as sums of power products (2012)

We use the theorem of Erdös, Pomerance and Schmutz (1991):

Let $\lambda(m)$ be the least positive integer for which $b^{\lambda(m)} \equiv 1 \pmod{m}$ for all $b \in \mathbb{Z}$ with $\gcd(b, m) = 1$.

Then for any increasing sequence $(n_i)_{i=1}^{\infty}$ of positive integers, and any constant $0 < C_1 < 1/\log 2$,

$$\lambda(n_i) > (\log n_i)^{C_1 \log\log\log n_i}.$$

But, there exist a strictly increasing sequence $(n_i)_{i=1}^{\infty}$ and a constant $C_2 > 0$, such that, for every i,

$$\lambda(n_i) < (\log n_i)^{C_2 \log\log\log n_i}.$$

Nathanson's theorem

Let $P = \{p_1, \ldots, p_t\}$ be a nonempty finite set of at least two prime numbers.
Let A be the set of positive integers that are products of powers of primes from P. Put $A_{\pm} = A \cup (-A)$.

Nathanson (2011) proved as a refinement of earlier results:

For every positive integer k there exist infinitely many integers n such that k is the smallest value of l for which n can be written as

$$n = a_1 + a_2 + \cdots + a_l \;\; (a_1, a_2, \ldots, a_l \in A_{\pm}).$$

Nathanson's problems

Let $F(k)$ be the smallest positive integer which cannot be represented as a sum of less than k terms of A. Let $F_{\pm}(k)$ the smallest positive integer which cannot be presented as a sum of less than k terms of $A_{\pm}$.

$F_{\pm}(k) \to \infty$ as $k \to \infty$.

Nathanson posed as problems:

Give estimations for $F(k)$ and for $F_{\pm}(k)$.

Our results

1. There exist computable numbers $c = c(P)$ and $C = C(\varepsilon)$ such that

$$k^{ck} < F(k) < C(kt)^{(1+\varepsilon)kt}.$$

2. There exist computable numbers $c = c(P)$ and $C_{\pm}$ such that

$$k^{ck} < F_{\pm}(k) < \exp((kt)^{C_{\pm}}).$$

3. For some numbers $c^* = c^*(P, k)$ and ε, we have

$$F_{\pm}(k) \leq (kt)^{(1+\varepsilon)kt} \text{ for } k > c^*.$$

Used methods for result 1: $k^{ck} < F(k) < C(kt)^{(1+\varepsilon)kt}$.

For the **lower bound** we used the greedy algorithm.
By an application of Baker's linear form estimates (T, 1974) we know that A has no big gaps.
Hence each time a substantial part is subtracted and the number of subtractions is as expected.

For the **upper bound** we count the number of integers at most n that can be represented as sum of less than k elements from A.
As soon as this number is less than n we know that there is an integer at most n which requires at least k terms of A to be represented.

Used method for upper bound of result 2: $F_{\pm}(k) < \exp((kt)^{C_{\pm}})$.

We used the following localized form of the result of Erdös, Pomerance and Schmutz.

There exist positive constants C_1, C_2 such that for every large integer i there is an integer m with

$$\log m \in [\log i, (\log i)^{C_1}] \text{ and } \lambda(m) < (\log m)^{C_2 \log\log\log m}.$$

Used method for upper bound of result 2: $F_{\pm}(k) < \exp((kt)^{C_{\pm}})$.

$$\log m \in [\log i, (\log i)^{C_1}] \text{ and } \lambda(m) < (\log m)^{C_2 \log\log\log m}.$$

We used this to derive the following refinement of a result of Ádám, Hajdu and Luca:

Put $H_{P,k} = \{n \in \mathbb{Z} : n = \sum_{i=1}^{l} a_i \text{ with } l \leq k\}$ where $a_i \in A \ (i = 1, 2, \ldots, k)$.

For $m \in \mathbb{Z}, m \geq 2$, we write $H_{P,k} (\text{mod } m) = \{i : 0 \leq i < m, h \equiv i \pmod m \text{ for some } h \in H_{P,k}\}$.

There is a constant C_3 such that for every large i there is an m with $\log m \in [\log i, (\log i)^{C_1}]$ and

$$| H_{P,k} \ (\text{mod } m) | < (\log m)^{C_3 kt \log\log\log m}.$$

Method for result 3: $F_{\pm}(k) \leq (kt)^{(1+\varepsilon)kt}$ for $k > c^*$.

We applied the following result of Evertse (1984).
(p-adic value $|x|_p$ is defined as $|x|p^{-r}$ where $p^r || x$.)

Let c, d be constants with $c > 0, 0 \leq d < 1$. Let S_0 be a finite set of primes and let l be a positive integer. Then there are only finitely many coprime tuples $(x_0, x_1, \dots, x_l)$ of rational integers such that

$$x_0 + x_1 + \cdots + x_l = 0; x_{i_0} + x_{i_1} + \dots x_{i_s} \neq 0$$

for each nontrivial subset $\{i_0, i_1, \dots, i_s\}$ of $\{0, 1, \dots, l\}$;

$$\prod_{j=0}^{l} \left(|x_j| \prod_{p \in S_0} |x_j|_p \right) \leq c \left(\max_{0 \leq j \leq l} |x_j| \right)^d .$$

Used method for result 3: $F_{\pm}(k) \le (kt)^{(1+\varepsilon)kt}$ for $k > c^*$.

Let n be an integer not divisible by any prime from P.
Suppose $n = a_1 + a_2 + \cdots + a_l$ with $a_j \in A_{\pm}$ and $l \le k$.
We may assume that l is minimal.
Moreover, we know that $\gcd(a_1, a_2, \ldots, a_l) = 1$.
We apply Evertse with $c = 1, d = 1/2, S_0 = P$
to $a_0 + a_1 + \cdots + a_l = 0$ with $a_0 = -n$.
Thus there are only finitely many tuples with

$$n \le \sqrt{\max_{1 \le j \le l} |a_j|}.$$

Let N_0 be the maximum of $|n|$ for all such tuples.

Used method for result 3: $F_{\pm}(k) \leq (kt)^{(1+\varepsilon)kt}$ for $k > c^*$.
Apart from finitely many tuples we have $n > \sqrt{\max_{1\leq j\leq l} |a_j|}$.

Next consider positive integers $n > N_0$ which are not divisible by any prime from P.
Then, for any representation $n = a_1 + a_2 + \cdots + a_l$ we have $|a_j| < n^2$ for $j = 1, 2, \ldots, l$.
Now we can apply the argument used for the upper bound of result 2.

Thank you for your attention

My papers up to 2011 can be found at www.math.leidenuniv.nl/ ˜ tijdeman/ ,
from 2011 on they can be found in the arXiv.