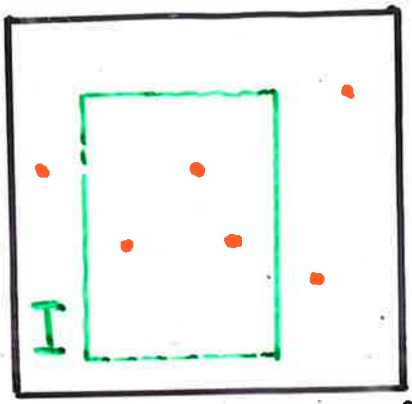
# METRIC DISCREPANCY THEORY

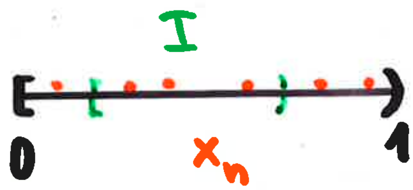
$$D_N(x_n) = \sup_{I \subset U^s} \left| \frac{1}{N} \#\{n \le N : x_n \in I\} - \lambda_s(I) \right|$$

$s=2$




$\lambda_2 =$ area $I$ $\qquad U^2 = [0,1)^2$

$s=1$




$x_n$

$N=6$ $\qquad U^1$

$\lambda_1 = |I|$

infinite sequence $(x_n)$ u.d. mod 1 $\iff$ $D_N(x_n) \to 0$ (as $N \to \infty$)

<u>Examples</u> of u.d. sequences:

$(\{n\alpha\})_{n \ge 1}$ $\quad \alpha \notin \mathbb{Q}$ $\qquad D_N(n\alpha) \asymp \frac{\log N}{N}$

reduced mod 1 $\qquad$ for quadratic irrationals

$(\{\sqrt{n}\})_{n \ge 1}$

$(\{n\underline{\alpha}\})_{n \ge 1}$ $\quad \underline{\alpha} = (\alpha_1, \dots, \alpha_s)$ with $1, \alpha_1, \dots, \alpha_s$ $\mathbb{Z}$-lin. indep.

<u>Counterexample</u> $\quad (\{\log n\})_{n \ge 1}$ not u.d but dense in $[0,1)$

$\boxed{\text{J.v. NEUMANN:}}$ Every dense sequence in $[0,1)$ can be rearranged to a u.d. sequence

Hardy, Littlewood
H. WEYL (1916)

exponential sums

$(x_n)_{n\geq 1}$ u.d. $\Leftrightarrow \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \langle \underline{h}, x_n \rangle} = 0$

$\forall \underline{h} \in \mathbb{Z}^s \setminus \{0\}$

$\leadsto$ EXPONENTIAL SUMS

ERDÖS-TURÁN $(s=1)$

$$D_N(x_n) \leq c \left( \frac{1}{H} + \sum_{h=1}^{H} \frac{1}{h} \sum_{n=1}^{N} e^{2\pi i h x_n} \right)$$

SÜSZ-KOKSMA $s>1$ $(H \in \mathbb{N})$

H. WEYL (1916) Let $(a_n)$ be a sequence of distinct integers. Then

$(a_n x)$ is u.d a.e.

(for almost all $x$)

$\Rightarrow$ $a_n = b^n$ : almost all numbers $x \in \mathbb{R}$ are normal in base $b$.

ERDÖS – GÁL – KOKSMA (1950's)

For $m \neq n$: $a_m'(x) - a_n'(x)$ monotone on $[\alpha, \beta]$

$|a_m'(x) - a_n'(x)| \geq \delta > 0$ on $[\alpha, \beta]$

$$\Rightarrow \quad D_N(a_n(x)) \ll N^{-\frac{1}{2}} (\log N)^{5/2 + \varepsilon}$$

$(\varepsilon > 0)$

REMARK: "Random sequences" satisfy LIL

QUESTION of ERDŐS (1962)

LIL for discrepancies of lacunary sequences $\frac{a_{n+1}}{a_n} \geq q > 1$

($a_n$ integers)

W. PHILIPP (1975):

$$C_1 \leq \underbrace{\limsup_{N\to\infty} \frac{N D_N(a_n x)}{\sqrt{N \log\log N}}} \leq C_2 \quad \text{a.e.}$$

$= \frac{1}{2}$ for random sequences $x_n$

CHUNG-SMIRNOV-LIL

- ULTRA-LACUNARY SEQUENCES $\frac{a_{n+1}}{a_n} \to \infty$

  LIL : $\limsup_{N\to\infty}$ —"— $= \frac{1}{2}$

- PROPERLY LACUNARY : $\frac{a_{n+1}}{a_n} \to \alpha$

  surprising results

- SUBLACUNARY (LIL maybe true or false)

In general: $(a_n)$ sequence of distinct integers

R. BAKER (1990's) $D_N(a_n x) \ll N^{-\frac{1}{2}} (\log N)^{\frac{3}{2}+\varepsilon}$ a.e.

(tool: inequality of Carleson-Hunt)

LIL remains true for sublacunary sequences satisfying ERDÖS gap condition

$$\frac{a_{n+1}}{a_n} \geq 1 + \frac{c}{n^\gamma} \quad \text{with} \quad \gamma < \frac{1}{2}$$

LIL false for $\gamma = \frac{1}{2}$ (BERKES- PHILIPP)

W. PHILIPP (1990's) LIL holds for "HARDY-LITTLEWOOD- POLYA" sequences $a_n$ [TOOL: Theory of S-unit equations]

$S = \{ a \in \mathbb{N} : a = q_1^{k_1} \cdots q_t^{k_t}, \ k_j \geq 0 \}$

$q_1, \dots, q_t$ coprime positive integers

$a_n \dots$ sequence of numbers $a \in S$ in increasing order

$(a_n)$ SUBLACUNARY for $t \geq 2$.

$\rightsquigarrow$ H. FURSTENBERG (1960's)

- PROPERLY LACUNARY SEQUENCES

$a_n = a^n$, $a$ integer $\geq 2$ (Fukuyama 2010)

$$\Sigma_a = \limsup_{N\to\infty} \frac{N D_N(a_n x)}{\sqrt{2N \log\log N}} \quad \text{a.e.}$$

with $\Sigma_a = \frac{\sqrt{42}}{9}$ for $a=2$

$$\Sigma_a = \frac{\sqrt{a(a+1)(a-2)}}{2\sqrt{(a-1)^3}} \quad a \geq 4 \text{ even}$$

$$\Sigma_a = \frac{\sqrt{a+1}}{2\sqrt{a-1}} \quad a \geq 3 \text{ odd}$$

HLP-sequences $(a_n)$, $q_i$ odd

$$\Sigma = \frac{1}{2}\left(\prod_{i=1}^{t} \frac{q_i+1}{q_i-1}\right)^{1/2}$$

- LIL depends on arithmetic properties of underlying sequence $(a_n)$

Counterexample of Erdős - Fortet $2^n - 1$

- LIL is <u>not</u> permutation invariant in general

# SYMMETRIC LIMIT LAWS (Aistleitner, Berkes–Tichy)

- $(a_k)$ lacunary
- diophantine condition: # solutions of
  $$b a_k + c a_l = d \qquad 1 \le k, l \le N$$
  is bounded by constant $K(b,c)$

Then:

LIL $$\limsup_{N\to\infty} \frac{N D_N(a_{\sigma(k)}x)}{\sqrt{2N \log\log N}} = \frac{1}{2} \quad \text{a.e.}$$

(for any permutation $\sigma: \mathbb{N} \to \mathbb{N}$)

CLT $$D_N(a_{\sigma(k)}x) \xrightarrow{\mathcal{D}} N(0,1)$$

REMARK. •) diophantine condition automatically satisfied for
$$a_{k+1}/a_k \to \infty$$
(ultralacunary)

•) results remain true for sums
$$\frac{1}{N}\sum_{k=1}^{N} f(a_{\sigma(k)}x) \quad \text{with 1-periodic functions } f, \text{ bounded variation}$$

and mean-value $= 0$

LIL $\rightsquigarrow$ $\|f\|_2$

•) results remain true for general weights:

$p_k > 0$, $P(N) = \sum_{k=1}^{N} p_k$

CLT: $\frac{1}{P(N)} \sum_{k=1}^{N} f(a_{\sigma(k)} x) \xrightarrow[\mathcal{D}]{} N(0,1)$

EXAMPLE $a_k = b^k$ $b \in \mathbb{N} \setminus \{1\}$

$(x a_k)$ u.d $\Leftrightarrow$ $x$ normal in base $b$

Constructions: $b = 10$

$x = 0.1234567891011\ldots$

$$D_N(x b^k) \asymp \frac{1}{\log N}$$

Extensions: (Erdős-Davenport) $x = 0.[f(1)]_b [f(2)]_b \ldots$

$f$ ... Polynomial

$$D_N \asymp \frac{1}{\log N}$$